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Short Communication

Vibration tailoring of heterogeneous beams and annular plates

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Abstract

In this study, it is demonstrated that clamped–clamped heterogeneous Bernoulli–Euler beams and the Kirchhoff–Love annular plate that is clamped along both inner and outer perimeters possess the common fundamental mode shape that is a fourth-order polynomial. This remarkable finding leads to the possibility of vibration tailoring, namely, the analytical design of annular heterogeneous plate with a pre-specified natural frequency.

1. Introduction

The vibration of circular and annular plates was investigated in much detail by Leissa [1]. The researches conducted during about 25 years were analyzed in Refs. [2–4]. In some papers the solution is conducted in exact terms. Exact solution of homogeneous uniform plates is found in terms of Bessel functions. Hypergeometric functions were utilized by Lizarev and Kuzmentsov [5]. Prasad et al. [6] investigated plates with linearly varying thickness, whereas Barakat and Baumann [7] considered the case of parabolic variation. Kirkhope and Wilson [8] utilized the finite element method. The closed-form solution for the annular plate having a parabolic thickness variation was derived by Lenox and Conway [9]. This pioneering solution is remarkable in the sense that the closed-form solution was derived for polynomially varying flexural rigidity.

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In a recent monograph [10] vibration and buckling of heterogeneous structures were studied in a closed form. It turned out that the heterogeneous beams or plates may possess mode shapes that are represented as fourth or higher-order polynomial functions or constitute rational expressions. Likewise, as Ref. [11] proved, the exact mode shape may also be represented as a trigonometric function. To the best of our knowledge, there are no homogeneous structures, which possess polynomial or rational mode shapes. The closed-form solutions were derived in Ref. [10] for various structures with variable modulus of elasticity. This finding opens new avenues of producing tailored structures, i.e. those systems that meet specific pre-set criteria. These criteria may consist in the demand that the static displacement under specified load to be less than indicated; or natural frequency not to exceed a selected value; or buckling load to be not less than a chosen load, etc. In all these cases, one can come up with an *analytical* expression of the variation of the modulus of elasticity that meets either of the above requirements. This remarkable feature of heterogeneous structures, uncovered in monograph [10] and some references cited therein, is generalized in the present study, to prove the statement in its title.

The free vibration of the heterogeneous Bernoulli–Euler terms that are clamped at both ends was studied in Ref. [10]. It was shown that the simple polynomial function

$$Y(\xi) = \xi^2 - 2\xi^3 + \xi^4 \tag{1}$$

may serve as an exact mode shape of an heterogeneous Bernoulli–Euler beam, $\xi = x/L$ being the non-dimensional axial coordinate, x is the axial coordinate and L is the length of the beam. The corresponding variation of the modulus of elasticity is

$$E(\xi) = -(\tfrac{11}{18} + 2\xi/3 + \xi^2/3 - 2\xi^3 + \xi^4)b_4, \tag{2}$$

where b_4 is a negative real number.

Due to arbitrariness of negative coefficient b_4, we conclude that there are infinite amount of beams that have the mode shape in Eq. (1).

In this study we demonstrate that a heterogeneous *annular plate* may possess the suitably transformed mode shape given in Eq. (1). This remarkable phenomenon does not occur in homogeneous and uniform structures, to the best of our knowledge; it is possible due to the *heterogeneity* of the annular plate and the corresponding beam. Note that the transformation of the coordinates was also utilized by Ramiah and Vijayakumar [12], but not in the context of exact solution.

2. Basic equations

The differential equation that governs the axisymmetric vibrations of annular inhomogeneous plates reads:

$$\begin{aligned} &D(r)r^3\nabla^2\nabla^2 w + (\mathrm{d}D/\mathrm{d}r)[2r^3\,\mathrm{d}^3w/\mathrm{d}r^3 + r^2(2+\nu)\mathrm{d}^2w/\mathrm{d}r^2 - r\,\mathrm{d}w/\mathrm{d}r] \\ &\quad + (r^3\mathrm{d}^2w/\mathrm{d}r^2 + \nu r^2\,\mathrm{d}w/\mathrm{d}r)\mathrm{d}^2D/\mathrm{d}r^2 - \rho h\omega^2 r^2 w = 0, \end{aligned} \tag{3}$$

where ∇^2 is the Laplacian operator in polar coordinates:

$$\nabla^2 = \mathrm{d}^2/\mathrm{d}r^2 + (1/r)\,\mathrm{d}/\mathrm{d}r, \tag{4}$$

D is the flexural rigidity, which varies along the radial coordinate r

$$D = D(r) = E(r)h^3/[12(1-v^2)], \tag{5}$$

where h is the thickness, v the Poisson ratio, ρ the material density, r the radial coordinate, and $w(r)$ is the mode shape. Poisson's ratio is assumed to be a constant. The inertial term ρh is likewise assumed to be a constant. The variation of the flexural rigidity is due to the fact that the modulus of elasticity is a function of the radial coordinate, i.e. $E = E(r)$.

The candidate mode shape for the plate is obtained from Eq. (1) by the following substitution:

$$\xi = (r - r_i)/(r_0 - r_i) \tag{6}$$

and reads:

$$w(r) = [(r-r_i)/(r_0-r_i)]^2 - 2[(r-r_i)/(r_0-r_i)]^3 + [(r-r_i)/(r_0-r_i)]^4. \tag{7}$$

The problem is defined as follows: determine the distribution of the flexural rigidity $D(r)$ that is proportional to the variable modulus of elasticity in Eq. (5), so that the governing equation (1) is satisfied. It must be noted immediately that the distribution of $D(r)$ for the annular plate could be anticipated to be different from that of the clamped beam. This is due to the fact that the governing differential equations differ from each other for these two structural configurations. Contrary to the intuitive anticipation, it will be shown that these two structures possess the same mode shape with transformation in Eq. (6) in mind.

It is seen from the analysis of the governing differential equation that the flexural rigidity in the form of fourth-order polynomial is compatible with the fourth-order polynomial representing the mode shape in Eq. (1); in other words, the flexural rigidity is represented as

$$D(r) = b_0 + b_1(r/R_i) + b_2(r/R_i)^2 + b_3(r/R_i)^3 + b_4(r/R_i)^4. \tag{8}$$

Substituting Eqs. (7) and (8) into the governing equation (1) we get the following equation:

$$\sum X_i r^i/(-R_o + R_i)^4 = 0, \tag{9}$$

where $i = 0, 1, 2, \ldots, 7$ and

$$X_0 = R_i^4(-2b_oR_i^5R_o^2 - 2b_oR_i^6R_o), \tag{10}$$

$$X_1 = 0, \tag{11}$$

$$\begin{aligned} X_2 = R_i^4(&-18b_oR_i^4R_o + 2b_2R_i^3R_o^2 + 2b_2R_i^4R_o - 18b_oR_i^5 + 2b_1R_i^3R_o^2 + 8b_1R_i^4R_o \\ &+ 2b_1R_i^5v + 2b_1R_i^5 + 2b_1R_i^3vR_o^2 + 8b_1R_i^4vR_o - 4b_2R_i^3vR_o^2 - 4b_2R_i^4vR_o), \end{aligned} \tag{12}$$

$$\begin{aligned} X_3 = R_i^4(&-60b_1R_i^3R_o + 4b_3R_i^2R_o^2 + 4b_3R_i^3R_o + 64b_oR_i^4 - 60b_1R_i^4 - 12b_1R_i^4v \\ &+ 8b_2R_i^2R_o^2 + 32b_2R_i^3R_o + 8b_2R_i^4v + 8b_2R_i^4 - 12b_1R_i^3R_ov + 8b_2R_i^2R_o^2v \\ &+ 32b_2R_i^3R_ov - 12b_3R_i^2R_o^2v - 12b_3R_i^3R_ov - \rho h\omega^2R_i^6R_o^2), \end{aligned} \tag{13}$$

$$
\begin{aligned}
X_4 = R_i^4(&-126b_2R_i^2R_o + 6b_4R_iR_o^2 + 6b_4R_oR_i^2 + 132b_1R_i^3 - 126b_2R_i^3 + 126_1R_i^3v \\
&- 36b_2R_i^3v + 18b_3R_iR_o^2 + 72b_3R_i^2R_o + 18b_3R_i^3v + 18b_3R_i^3 - 36b_2R_i^2R_ov + 18b_3R_iR_o^2v \\
&+ 72b_3R_i^2R_ov - 24b_4R_iR_o^2v - 24b_4R_oR_i^2v + 2\rho h\omega^2R_i^5R_o^2 + 2\rho h\omega^2R_i^6R_o),
\end{aligned} \tag{14}
$$

$$
\begin{aligned}
X_5 = R_i^4(&-216b_3R_iR_o - 224b_2R_i^2 - 216b_3R_i^2 + 32b_2R_i^2v - 72b_3R_i^2v \\
&+ 128b_4R_oR_i + 32b_4R_o^2v + 32b_4R_i^2v + 32b_4b_o^2 + 32b_4R_i^2 - 72b_3R_iR_ov \\
&+ 128b_4R_oR_iv - \rho h\omega^2R_i^4R_o^2 - 4\rho h\omega^2R_i^5R_o - \rho h\omega^2R_i^6),
\end{aligned} \tag{15}
$$

$$
\begin{aligned}
X_6 = R_i^4(&340b_3R_i - 330b_4R_o - 330b_4R_i + 60b_3R_iv - 120b_4R_ov - 120b_4R_iv \\
&+ 2\rho h\omega^2R_i^4R_o + 2\rho h\omega^2R_i^5),
\end{aligned} \tag{16}
$$

$$X_7 = R_i^4(480b_4 + 96b_4v - \rho h\omega^2R_i^4). \tag{17}$$

From the requirement $X_7 = 0$, where X_7 is defined in Eq. (17) we obtain the natural frequency squared:

$$\omega^2 = 96(5 + v)b_4/\rho hR_i^4. \tag{18}$$

In order for the natural frequency to be a positive quantity, b_4 must be positive. Thus, whereas the clamped–clamped beam coefficient b_4 must be negative, for the plate it must be positive. This fact constitutes a qualitative difference between the beam and plate results.

From the equation $X_6 = 0$ we get

$$b_3 = 3b_4(215 + 52v)(R_i + R_o)/10(3v + 17)R_i. \tag{19}$$

Equation $X_5 = 0$ yields

$$
\begin{aligned}
b_2 = \{[&5377v(R_i^2 + R_o^2) + (1474 + 888v)vR_iR_o + 924v^2(R_i^2 + R_o^2) \\
&- 8690R_iR_o + 6535(R_i^2 + R_o^2)]/40(17 + 3v)(7 + v)\}b_4.
\end{aligned} \tag{20}
$$

The coefficient b_1 is obtained from solving equation $X_4 = 0$

$$
\begin{aligned}
b_1 = 3\{[&1944(R_i + R_o)v^3R_iR_o + (146895 + 13126v^2)R_oR_i^2 \\
&+ (9650 + 1224v^3)(R_i^3 + R_o^3) + (27685 + 25701v)(R_i^3 + R_o^3) + 27685R_i^3 \\
&+ (28021 + 113574v)R_iR_o^2/80(17 + 3v)(11 + v)R_i^3\}b_4.
\end{aligned} \tag{21}
$$

Finally, b_0 is derived from the requirement $X_3 = 0$:

$$b_0 = 0. \tag{22}$$

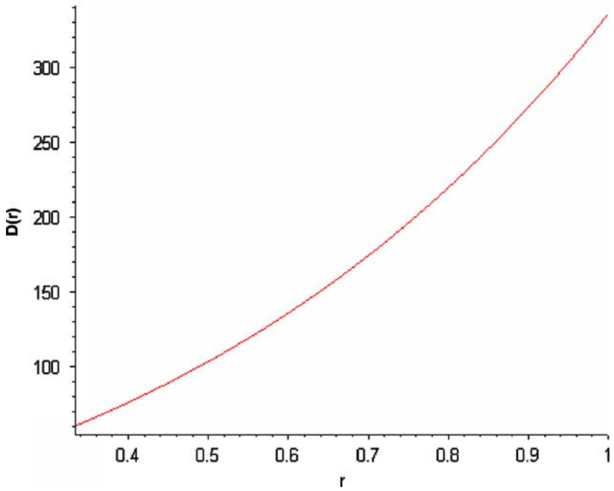

Fig. 1. Vibration in the flexural rigidity vs. the radial coordinate.

Substituting the expressions for b_j into Eq. (8), we get the final expression for the flexural rigidity:

$$\begin{aligned} D(r)/b_4 = {} & 3\{[1944(R_i + R_o)v^3 R_i R_o + (146895 + 13126v^2)R_o R_i^2 + (9650 + 1224v^3)(R_i^3 + R_o^3) \\ & + (27685 + 25701v)(R_i^3 + R_o^3) + 27685R_i^3 \\ & + (28021 + 113574v)R_i R_o^2/80(17 + 3v)(11 + v)R_i^3\}(r/Ri) \\ & + \{[5377v(R_i^2 + R_o^2) + (1474 + 888v)vR_i R_o + 924v^2(R_i^2 + R_o^2) - 8690R_i R_o \\ & + 6535(R_i^2 + R_o^2)]/40(17 + 3v)(7 + v)\}(r/R_i)^2 \\ & + 3(215 + 52v)(R_i + R_o)/10(3v + 17)R_i(r/R_i)^3 + (r/Ri)^4. \end{aligned} \tag{23}$$

For each value of $b_4 > 0$, we get a separate distribution for the flexural rigidity $D(r)$ that corresponds to the heterogeneous plate with mode shape given in Eq. (7). Note that since $b_0 = 0$, $D(r)$ vanishes at $r = 0$; however, the point $r = 0$ is outside the area of the annular plate. In other words, the positivity of $D(r)$ is maintained for $R_i \leqslant r \leqslant R_o$.

For values $\alpha = R_i/R_o = 1/3$, $v = 0.3$, and $b_4 = 1$ we get

$$b_1 = 112,368,408/1,476,571 \approx 76.101,$$
$$b_2 = 722,612/65,335 \approx 11.060,$$
$$b_3 = 13,836/895 \approx 15.459.$$

Fig. 1 depicts the variation of the flexural rigidity versus the radial coordinate r for $R_i = 1$.

3. Conclusion

This paper presents a remarkable case of heterogeneous annular plate that shares the fundamental mode shape with the heterogeneous beam; the annular plate is clamped at inner and outer circumferences, whereas the beam is clamped at both ends.

The finding is of obvious theoretical interest as an amusing fact that takes place for a heterogeneous plate; practical significance of this work lies in the possibility of choosing the free parameter b_4 so that the annular plate possesses the pre-set natural frequency Ω. Indeed, from Eq. (8), we deduce that if

$$b_4 = \rho h R_i^4 \Omega^2/(5+\nu) \tag{24}$$

then the plate possesses the desired natural frequency Ω. Thus, a simple method of vibration tailoring is being presented by this paper, as a by-product of the closed-form analysis.

It is also remarkable that Eq. (18) for the natural frequency squared coincides formally with its counterpart circular plate in Ref. [9]. This sharing of the same natural frequency is usually referred to as isospectrality. Here we established coincidence of analytic expressions for the fundamental frequency, while the corresponding flexural rigidities differ. For other exciting isospectral studies, one can consult the study by Gottlieb [13].

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