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Journal of Sound and Vibration 291 (2006) 1261–1268

JOURNAL OF SOUND AND VIBRATION

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Short Communication

Response statistics of strongly nonlinear system to random narrowband excitation

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Received 12 March 2005; accepted 3 July 2005
Available online 27 October 2005

Abstract

A technique coupling with the parameter transformation method and the multiple scales method is presented for determining the primary resonance response of strongly nonlinear Duffing–Rayleigh oscillator subject to random narrowband excitation. By introducing a new expansion parameter $\alpha = \alpha(\varepsilon, u_0)$, the multiple scales method is adapted to determine the equations describing the modulation of response amplitude and phase. The effect of the random excitation on the stable periodic response is analyzed as a perturbation. By the moment method steady-state mean square response is obtained and its local stability is checked by Routh–Hurwitz criterion. Theoretical analyses and numerical calculations show that when the intensity of random excitation increases, the steady-state solution may change from a limit cycle to a diffused limit cycle. Under some conditions the system may have two steady-state solutions. The results obtained for strongly nonlinear oscillator complement previous results in the literature for weakly nonlinear case.

1. Introduction

The study on the response of stochastic nonlinear systems is of considerable important. Up to now there have been a copious list of texts [1–3] and reference on random vibration of nonlinear system (see Refs. [5–10], to mention a few). Some methods such as the equivalent linearization method [4], the quasic-static method [5], the multiple scales method [6–8], the path integral method [9], and the stochastic averaging method [10] have been used to investigate the weakly stochastic nonlinear system. It is realized that many problems of physical interest are concerned with the motion of strongly nonlinear system (see, Refs. [11–13], to mention a few). However, there is a smaller literature discussing strongly nonlinear system subject to random narrowband excitation. This letter is to explore the strongly nonlinear Duffing–Rayleigh oscillator under random narrowband excitation and the governing differential equation is

$$\ddot{u} + u - \varepsilon\delta_1\dot{u} + \tfrac{1}{3}\varepsilon\delta_2\dot{u}^3 + \varepsilon\beta u^3 = \varepsilon\xi(t), \tag{1}$$

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0022-460X/$ - see front matter
doi:10.1016/j.jsv.2005.07.050

where dots denote derivatives with respect to the time t, the positive parameter ε may be "not small", δ_1, δ_2 are damping coefficients , β denotes the intensity of nonlinear terms, and $\xi(t)$ is an ergodic stochastic process with zero mean governed by the following equation proposed by Wedig [14]:

$$\xi(t) = p\cos(\Omega t + \gamma W(t)), \tag{2}$$

where $W(t)$ is a standard Wiener process, and the power spectrum density of $\xi(t)$ is

$$S_\xi(\omega) = \frac{1}{2}\frac{p^2\gamma^2(\Omega^2 + \omega^2 + \gamma^4/4)}{(\Omega^2 - \omega^2 + \gamma^4/4)^2 + \omega^2\gamma^4}.$$

For the extreme limiting case $\gamma \to 0$, the fluctuation spectrum $S_\xi(\omega)$ is vanishing over the entire frequency range except at the singular frequency $\omega = \pm\Omega$, where $S_\xi(\pm\Omega)$ goes to infinity. This is a typical spectrum of random narrowband noise. The aim of this letter is to investigate the behavior and stability of steady-state response of system (1).

2. Analysis with the modified multiple scales method (MSM)

The modified MSM, suggested by Burton [16] while studying deterministic strongly nonlinear system, is a valid perturbation method when the nonlinearity $\varepsilon\beta$ is not small compared with unity. This present letter is to put forward his idea to investigate system (1). The first step is to introduce a new time variable $\tau = \Omega t$, so that Eq. (1) becomes

$$\Omega^2 u'' + u - \varepsilon\Omega\delta_1 u' + \tfrac{1}{3}\varepsilon\Omega^3\delta_2 u'^3 + \varepsilon\beta u^3 = \varepsilon\xi(\tau), \tag{3}$$

where primes represent differentiation with respect to "time" τ. This step accommodates the eventual expansion of Ω^2 in the inertia term. Now a steady-state response with the fundamental harmonic of amplitude u_0 when $\gamma = 0$ may be expected. Here u_0 is used together with ε to define a new expansion parameter $\alpha = \varepsilon u_0^2/(4 + 3\varepsilon u_0^2)$. In terms of α, the original parameter is replaced by

$$\varepsilon = \frac{1}{u_0^2}\left(\frac{4\alpha}{1 - 3\alpha}\right). \tag{4}$$

The detuning parameter σ, as a deviation from the so-called, reasonably accurate approximate backbone curve illustrated in Ref. [16] in detail, is now introduced into Ω^2 as

$$\Omega^2 = \frac{1 + \alpha\sigma}{1 - 3\alpha}. \tag{5}$$

Substituting Eqs. (2), (4) and (5) into Eq. (3) and non-dimensionalizing u by setting $v = u/u_0$, we have

$$(1 + \alpha\sigma)v'' + v - 2\alpha\mu_1 v' + \alpha(2\mu_2 v'^3 + 4\beta v^3 - 3v) = \frac{4p}{u_0^3}\alpha\cos(\tau + \gamma W(\tau)), \tag{6}$$

where the damping terms have been redefined as

$$\mu_1 = \frac{2\delta_1\Omega}{u_0^2}, \quad \mu_2 = \frac{2\delta_2\Omega^3}{3}.$$

Obviously, the eventual steady-state fundamental harmonic of v must be of amplitude unity when $\gamma = 0$. Now the usual steps in the standard multiple scales method may be applied to Eq. (6). Then a uniformly approximate solution of Eq. (6) is sought in the form of power series

$$v = v_0(T_0, T_1) + \alpha v_1(T_0, T_1) + \cdots. \tag{7}$$

Denoting $D_0 = \partial/\partial T_0, D_1 = \partial/\partial T_1$, the ordinary-time derivatives can be transformed into partial derivatives as

$$\frac{d}{d\tau} = D_0 + \alpha D_1 + \cdots, \quad \frac{d^2}{d\tau^2} = D_0^2 + 2\alpha D_0 D_1 + \cdots. \tag{8}$$

Substituting Eqs. (7) and (8) into Eq. (6) and comparing coefficients of α with equal powers leads to the following differential equations:

$$D_0^2 v_0 + v_0 = 0, \tag{9}$$

$$D_0^2 v_1 + v_1 = -\sigma D_0^2 v_0 - 2D_0 D_1 v_0 + 2\mu_1 D_0 v_0 - 2\mu_2 (D_0 v_0)^3 - 4\beta v_0^3 + 3v_0 + \frac{4p}{u_0^3}\cos(\tau + \gamma W(\tau)). \tag{10}$$

The general solution of Eq. (9) can be written in the following form:

$$v_0 = \tfrac{1}{2} a(T_1) \exp\, \mathrm{i}(T_0 + \varphi(T_1)) + \mathrm{cc}, \tag{11}$$

where cc denotes the complex conjugate of its preceding terms.

Applying Eq. (11) into Eq. (10) and eliminating the secular term, it is required that a and φ vary in the slow time scale according to

$$a' = \mu_1 a - \frac{3}{4}\mu_2 a^3 - \frac{2p}{u_0^3}\sin(\varphi - \gamma W(T_1)),$$

$$a\varphi' = -\frac{1}{2}a(3+\sigma) + \frac{3}{2}\beta a^3 - \frac{2p}{u_0^3}\cos(\varphi - \gamma W(T_1)). \tag{12}$$

After solving a and φ, the first-order uniform expansion for the solution of Eq. (1) is given by

$$u(t) = u_0 a(\alpha\tau)\cos(\tau + \varphi(\alpha\tau)) + O(\alpha).$$

3. Steady-state response and their stability

The response of Eq. (12) when $\gamma = 0$ is firstly considered. The steady-state solutions require in Eq. (12) that $a' = 0, \varphi' = 0$, $a = a_0 = 1$ and this yields

$$\mu_1 - \frac{3}{4}\mu_2 = \frac{2p}{u_0^3}\sin\varphi,$$

$$-\frac{1}{2}(3+\sigma) + \frac{3}{2}\beta = \frac{2p}{u_0^3}\cos\varphi. \tag{13}$$

Eliminating φ from Eq. (13) leads to the frequency response equation

$$\left(\mu_1 - \frac{3}{4}\mu_2\right)^2 + \left[\frac{3}{2}\beta - \frac{1}{2}(\sigma+3)\right]^2 = \frac{4p^2}{u_0^6}. \tag{14}$$

The frequency response relation provides a one-parameter family of primary response curves. By denoting $\rho = \frac{1}{4}u_0^2$, Figs. 1 and 2 present the amplitudes response curves for different values of the external excitation and the unstable and stable domain in the $\rho - \sigma$ plane for the parameters $\varepsilon = 1.0,\ \delta_1 = 1.0,\ \delta_2 = 0.5$.

In the case of $\beta = 0$, system (1) is reduced to a Rayleigh oscillator. When $p = 0$, the curves of the family degenerate into the point $(-3, 2)$. As p increases, the curves first consist of two branches—a branch running near the σ-axis and a branch consisting of an approximate oval. As p increases, the approximate ovals expand and the branches near the σ-axis move away from this axis. When p reaches a critical value $p_1 = 1.0897$, the two branches coalesce, and the resultant curve has a double point, as shown in Fig. 1. As p increases further, the response curves become open curves, which continue to be multiple valued functions until p exceeds the second critical value $p_2 = 1.19$. Beyond this critical value, the response curves are single valued function for all values of σ. For $\beta = 1.0$, representative response curves with the same variety trend as that of the Rayleigh oscillator are displayed in Fig. 2.

Not all of the solutions given by frequency relation are realizable and have a physical sense because some of them are unstable. Through examining the stability of each fixed point by the usual way, one conclusion can

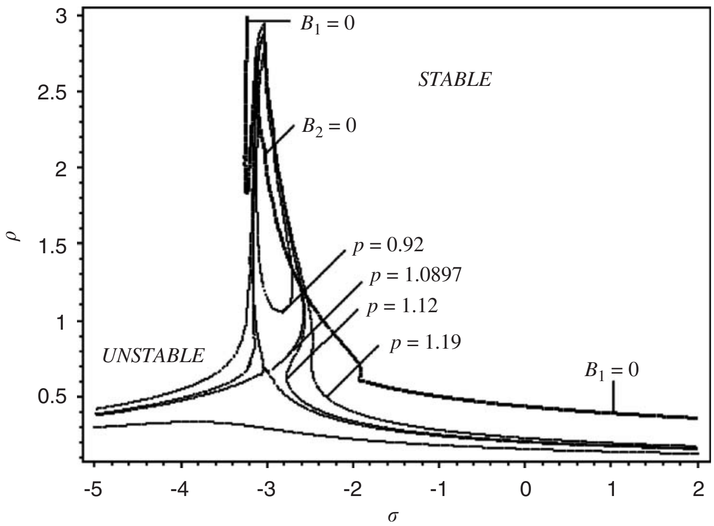

Fig. 1. Frequency response curves and stable domain for the primary resonances of the Rayleigh oscillator.

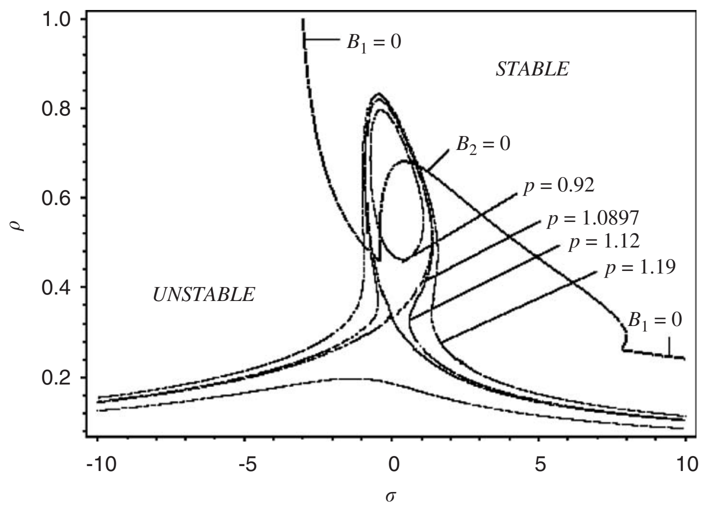

Fig. 2. Frequency response curves and stable domain for the primary resonances of the Duffing–Rayleigh oscillator.

be drawn that when $B_1 > 0$ and $B_2 > 0$ the steady-state solutions are stable where $B_1 = (3/2)\mu_2 - \mu_1$ and

$$B_2 = (\mu_1 - \tfrac{9}{4}\mu_2)(\mu_1 - \tfrac{3}{4}\mu_2) + \tfrac{1}{4}(\sigma + 3)^3 - 3\beta(\sigma + 3) + \tfrac{27}{4}\beta^2.$$

The thick lines (corresponding to the curves $B_1 = 0$ and $B_2 = 0$) in Figs. 1 and 2 separate the stable solutions from the unstable ones: all solutions corresponding to points above the thick lines are stable and those below the thick lines are unstable. The steady-state response and its stability when there is no noise for strongly nonlinear system (1) are similar to the weakly nonlinear cases [9].

Next is to analyze the effect of the noise, i.e., $\gamma \neq 0$ on the deterministic steady-state motion. For this purpose, letting $\eta = \varphi - \gamma W(t)$, Eq. (12) can be rewritten as

$$a' = \mu_1 a - \frac{3}{4}\mu_2 a^3 - \frac{2p}{u_0^3}\sin\eta,$$

$$a\eta' = -\frac{1}{2}a(3+\sigma) + \frac{3}{2}\beta a^3 - \frac{2p}{u_0^3}\cos\eta - a\gamma W'(T_1). \quad (15)$$

Since it is difficult to obtain the exact analytical solution of Eq. (15), the perturbation method may be employed to solve Eq. (15) when γ is small. Let

$$a = a_0 + a_1, \ \ \eta = \varphi_0 + \eta_1, \quad (16)$$

where a_0, φ_0 are given by Eq. (13) and a_1, η_1 are perturbation terms. Inserting Eq. (16) into Eq. (15) and neglecting nonlinear terms, we can obtain the following Itô equation

$$\mathrm{d}a_1 = [(\mu_1 - \tfrac{9}{4}\mu_2)a_1 - (-\tfrac{1}{2}(\sigma+3) + \tfrac{3}{2}\beta)\eta_1]\,\mathrm{d}T_1,$$

$$\mathrm{d}\eta_1 = [(-\tfrac{1}{2}(\sigma+3) + \tfrac{9}{2}\beta)a_1 + (\mu_1 - \tfrac{3}{4}\mu_2)\eta_1]\,\mathrm{d}T_1 - \gamma\,\mathrm{d}W(T_1). \quad (17)$$

Using the moment method [2], the steady-state moments are given by,

$$Ea_1 = E\eta_1 = 0,$$

$$Ea_1^2 = \frac{\left[-\frac{1}{2}(\sigma+3) + \frac{3}{2}\beta\right]^2\gamma^2}{4\left(\frac{3}{2}\mu_2 - \mu_1\right)\left[\left(\mu_1 - \frac{9}{4}\mu_2\right)\left(\mu_1 - \frac{3}{4}\mu_2\right) + \frac{1}{4}(\sigma+3)^2 - 3\beta(\sigma+3) + \frac{27}{4}\beta^2\right]}. \quad (18)$$

Necessary and sufficient condition for stability of the second-order moments can be derived from Routh–Hurwitz criterion [15]. The characteristic equation of the coefficients matrix of equations for the second order moments Ea_1^2, $Ea_1\eta_1$ and $E\eta_1^2$ is

$$\lambda^3 + A_1\lambda^2 + A_2\lambda + A_3 = 0, \quad (19)$$

where

$$A_1 = 6(\tfrac{3}{2}\mu_2 - \mu_1),$$

$$A_2 = 8(\tfrac{3}{2}\mu_2 - \mu_1)^2 + 4(\mu_1 - \tfrac{9}{4}\mu_2)(\mu_1 - \tfrac{3}{4}\mu_2) + (\sigma+3)^3 - 12\beta(\sigma+3) + 27\beta^2,$$

$$A_3 = 2(\tfrac{3}{2}\mu_2 - \mu_1)[4(\mu_1 - \tfrac{9}{4}\mu_2)(\mu_1 - \tfrac{3}{4}\mu_2) + (\sigma+3)^3 - 12\beta(\sigma+3) + 27\beta^2].$$

The second-order moments are asymptotically stable if all the eigenvalues of Eq. (19) have real parts negative. Hence, if $A_1 > 0, A_3 > 0$ and $A_1A_2 - A_3 > 0$ the second-order moments are stable which is in accord with the condition $B_1 > 0$ and $B_2 > 0$.

Combining Eqs. (16) and (18), the mean square response of system (1) is

$$Ea^2 = u_0^2\left(a_0^2 + \frac{\left[-\frac{1}{2}(\sigma+3) + \frac{3}{2}\beta\right]^2\gamma^2}{4\left(\frac{3}{2}\mu_2 - \mu_1\right)\left[\left(\mu_1 - \frac{9}{4}\mu_2\right)\left(\mu_1 - \frac{3}{4}\mu_2\right) + \frac{1}{4}(\sigma+3)^2 - 3\beta(\sigma+3) + \frac{27}{4}\beta^2\right]}\right). \quad (20)$$

4. Numerical simulation

For the method of numerical calculation, readers can refer to Zhu [2] and Shinozuka [16]. Eq. (1) is integrated numerically by means of fourth Runge–Kutta algorithm. The random process $\xi(t)$ governed by Eq. (2) can be written as

$$\xi(t) = p\cos(\varphi(t)),\ \dot{\varphi}(t) = \Omega + \gamma\zeta(t),\ \zeta(t) = \dot{W}(t),$$

where the formal derivative of unit Wiener process is a Gaussian white noise $\zeta(t)$. It is more convenient to use the pseudorandom signal given by [2] $\xi(t) = \sqrt{4\Omega/N}\sum_{k=1}^{N}\cos[(\Omega/N)(2k-1)t + \varphi_k]$, where φ_k's are independent and uniformly distributed in $(0, 2\pi]$, N is a large positive integer.

The parameters are fixed as $\varepsilon = 1.0$, $\delta_2 = 0.5$, $\beta = 1.0$, $N = 1000$ in this section. When selecting $p = 1.25$, $\Omega = 1.88$, $\gamma = 0.05$ the time history of $\xi(t)$ and its power spectrum density are shown in Fig. 3(a) and (b),

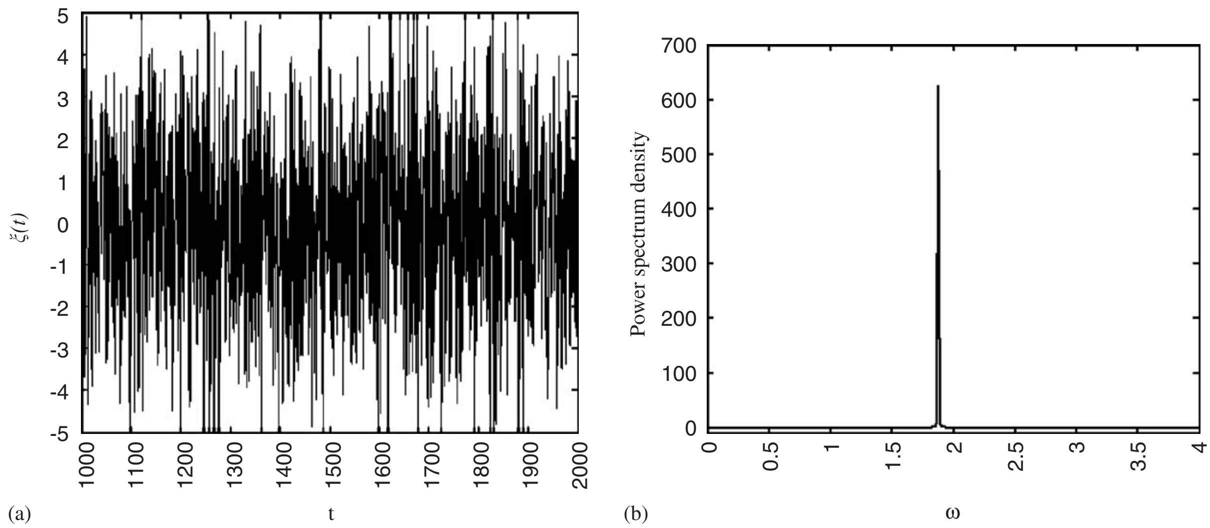

Fig. 3. Time history and Power spectrum density of $\xi(t)$: (a) Time history of $\xi(t)$; (b) power spectrum density of $\xi(t)$.

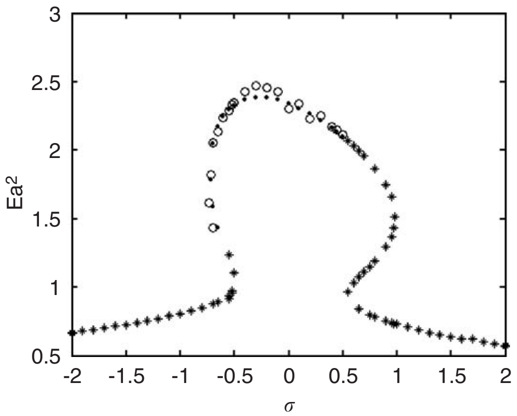

Fig. 4. Steady-state response of system (1) for $p = 0.52$, $\delta_1 = 0.6$, $\gamma = 0.005$. ● ● ● stable theoretically solution, ***unstable theoretically solution, ○ ○ ○ numerical solution.

respectively. Fig. 4 depicts the variation of Eu^2 versus σ. Comparison of the theoretical result obtained by solving Eq. (20) using Newton–Raphson method with the numerical one obtained by numerically integrating Eq. (1) is shown in Fig. 4. One can observe that there are two or three steady-state responses theoretically for some parameter region of σ. However, only one or two can be realized in the numerical simulation, which is identical with the condition $B_1 > 0$ and $B_2 > 0$ for the steady-state moments.

When the initial values are $u(0) = 0.4, \dot{u}(0) = -0.15$, for $\delta_1 = 1.0$, $p = 1.25, \Omega = 1.88$ and $\gamma = 0$, 0.05, 0.1, 0.8, the numerical results of Eq. (1) are shown in Fig. 5. Fig. 5 shows that the random noise $\gamma W(t)$ will change the steady-state response from a limit cycle to a diffused limit cycle and the width of the diffused limit cycle will increase as the intensity of the random excitation increases.

5. Conclusion and discussion

This letter investigated the primary resonance response of strongly nonlinear Duffing–Rayleigh oscillator. The approximate periodic solution is derived by the multiple scales method coupling with the idea of

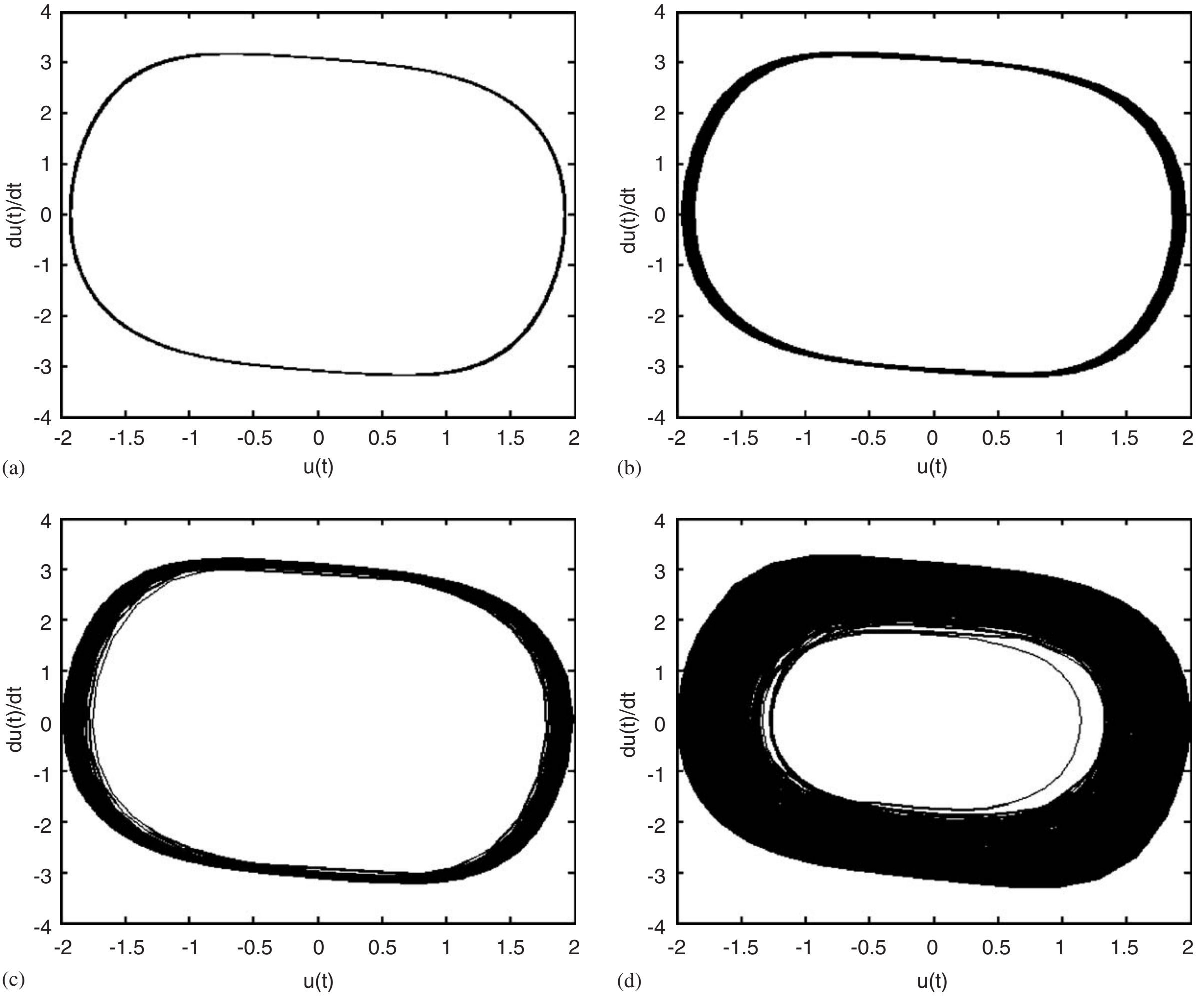

Fig. 5. Phase plot of system (1) for (a) $\gamma = 0$, (b) $\gamma = 0.05$, (c) $\gamma = 0.1$, (d) $\gamma = 0.8$.

parameter transformation. The mean square response is obtained by the moment method and the stability for the steady-state solution is checked by Routh–Hurwitz criterion.

The results indicate that combination of the transformation parameter technique and the multiple scales method is an effective approach to investigate the primary resonance response of single-degree-of-freedom strongly nonlinear system under random narrowband excitation of the form $\ddot{u} + u + \varepsilon\alpha u^3 + f(\dot{u}) = \varepsilon\xi(t)$. Further investigation has been devoted to extend the proposed method to multi-degree-of-freedom stochastic strongly nonlinear system and the results will be presented soon.

Acknowledgments

This work was supported by the National Natural Science Foundation of China (Grant No. 10472091) and (Grant No. 10332030), the NSF of Guangdong province (Grant No. 04011640) and Shaanxi Province (Grant No. 2003A03) and the Doctorate Creation Foundation of Northwestern Polytechnical University (Grant No. CX 200516).

References

[1] Y.K. Lin, G.Q. Cai, *Probabilistic Structural Dynamics, Advanced Theory and Applications*, McGraw-Hill, New York, 1995.

[2] W.Q. Zhu, *Random Vibration*, Beijing Science Press, China, 1992.

[3] R.Z. Khasminskii, *Stochastic Stability of Differential Equations*, Sijthoff, Noordhoff, Alphen, den Rijn, 1980.

[4] R.N. lyengar, Response of nonlinear systems to narrow-band excitation, *Structural Safety* 6 (1989) 177–185.

[5] W.C. Lennox, Y.C. Kuak, Narrowband random excitation of a nonlinear oscillator, *Journal of Applied Mechanics* 43 (1976) 340–344.

[6] A.H. Nayfeh, S.J. Serhan, Response statistics of nonlinear systems to combined deterministic and random excitations, *International Journal of Nonlinear Mechanics* 25 (5) (1990) 493–509.

[7] H.W. Rong, W. Xu, X.D. Wang, G. Meng, T. Fang, Principle response of Van der Pol–Duffing oscillator under combined deterministic and random parametric excitation, *Applied Mathematics and Mechanics* 23 (3) (2002) 299–310.

[8] H.W. Rong, W. Xu, T. Fang, Principle response of Duffing oscillator to combined deterministic and narrowband random parametric excitation, *Journal of Sound and Vibration* 210 (4) (1998) 483–515.

[9] J.S. Yu, Y.K. Lin, Numerical path integration of a non-homogeneous Markov Process, *International Journal of Nonlinear Mechanics* 39 (2004) 1493–1500.

[10] Y. Xu, W. Xu, G.M. Mahmoud, On a complex beam–beam interaction model with random forcing, *Physica A* 336 (3–4) (2004) 347–360.

[11] T.D. Burton, Z. Raham, On the mutil-scale analysis of strongly non-linear forced oscillators, *International Journal of Nonlinear Mechanics* 2 (21) (1986) 135–146.

[12] H.W. Rong, W. Xu, X.D. Wang, G. Meng, T. Fang, Response statistic of strongly non-linear oscillator to combined deterministic and random excitation, *International Journal of Nonlinear Mechanics* 39 (2004) 871–878.

[13] Y.K. Cheung, S.H. Chen, S.L. Lau, A modified Lindstedt–Poincare method for certain strongly nonlinear oscillators, *International Journal of Nonlinear Mechanics* 26 (3,4) (1991) 367–378.

[14] W.V. Weding, Invariant measures and Lyapunov exponents for generalized parameter fluctuations, *Structural Safety* 8 (1990) 13–25.

[15] G. Schmidt. A. Tondl, *Nonlinear Vibrations*, Cambridge University Press, Cambridge, 1986.

[16] M. Shinozuka, Simulation of multivariate and multidimensional random processes, *Journal of Sound and Vibration* 19 (1971) 357–367.