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Out-of-plane free vibration of a circular arch with uniform cross-section: Exact solution

Ekrem Tufekci[a,*], O. Yasar Dogruer[b]

[a]*Faculty of Mechanical Engineering, Istanbul Technical University, Gumussuyu TR-34437, Istanbul, Turkey*
[b]*Anadolu Isuzu Otomotiv San. Yedipinarlar Mevkii, Sekerpinar TR-41400, Gebze-Kocaeli, Turkey*

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Abstract

In this study, free out-of-plane vibrations of a circular arch with uniform cross-section are investigated by taking into account the effects of transverse shear and rotatory inertia due to both flexural and torsional vibrations. The governing differential equations for out-of-plane vibration of uniform circular beams are solved exactly by using the initial value method. The solution does not depend on the boundary conditions. The same solution procedure is also used to obtain the results of other cases in which each effect is considered individually in order to assess its importance. The frequency coefficients are obtained for the first five modes of arches with various slenderness ratios and opening angles. The results show that the flexural and torsional rotatory inertia and shear deformation have very important effects on resonance frequencies, even if slender shallow arches are considered. It is concluded that the torsional rotatory inertia effect is the most significant effect to be included in the analysis. A phenomenon known as transition of modes from torsional into flexural is characterized by the sharp increment in resonance frequencies of modes that occurs at certain combinations of curvature and length of the arch. The mode transition phenomenon is shown in figures. Vibration problems for circular beams that have been analysed in the literature are solved and the results are compared in tables. The comparison shows good agreement between the results.

*Corresponding author. Tel.: +212 293 1300/2650; fax: +212 245 0795.
E-mail addresses: tufekcie@itu.edu.tr (E. Tufekci), yasar.dogruer@isuzu.com.tr (O.Y. Dogruer).

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1. Introduction

Arches have long been widely used as structural elements in many mechanical, aerospace and civil engineering applications such as spring design, brake shoes within drum brakes, tire dynamics, piping systems, turbo-machinery blades, curved wires in missile guidance floated gyroscopes, aerospace structures, stiffeners in aircraft structures, arch bridges, curved girder bridges, long span roof structures and earthquake resistant structures. Hence, the dynamic behaviour of arches has been of interest to many researchers since the nineteenth century. More than 600 articles have been summarized in review articles [1–4]. It is noteworthy that the majority of the papers on this subject are restricted to the planar case where the deformations include the bending and stretching in the plane of curvature of the arch. Much less research has been focused on the out-of-plane behaviour of arches.

In general, the in-plane and out-of-plane vibrations of a planar arch are coupled. However, based on the Bernoulli–Euler hypothesis, if the cross-section of an arch is uniform and doubly symmetric, i.e., the shear centre and centroid coincide, then the in-plane and out-of-plane vibrations are uncoupled. However, the out-of-plane bending and torsional responses will still be coupled.

It is often difficult and sometimes impossible to find a general closed-form solution for the vibration problem of an arch, since the governing differential equations possess variable coefficients. The exact solution of the governing equations exists only for a circular beam of uniform cross-section. The previous studies are based upon the classical theory in which neither rotatory inertia nor shear deformation are taken into account. Timoshenko beam theory considers the effects of shear deformation and rotatory inertia due to both flexural and torsional vibrations and provides a better approximation to the actual arch behaviour. Many techniques have been considered in the papers on out-of-plane vibrations of arches. The Ritz method with different types of trial functions has often been applied in determining the natural frequencies of arches. With the advancement of computer technology and several programs, the finite element method has been used widely to solve for more general geometry and a number of curved elements have been developed. If the behaviour of the arch is non-planar, usual finite element or finite difference model becomes very complicated.

The classical governing equations of in-plane and out-of-plane vibrations of arches are derived and the analytical solutions of the equations for a circular arch are given in the book by Love [5]. Volterra and Morell [6] used the Rayleigh–Ritz method to determine the lowest natural frequency of elastic arches with clamped ends. The arches have the centerlines in the forms of a circle, a cycloid, a catenary and a parabola. The shear deformation effect is neglected. Irie et al. [7] investigated the steady state out-of-plane response of a Timoshenko arch with internal damping in response to a sinusoidal point force or moment by using the transfer matrix method. In another work by Irie et al. [8], the transfer matrix method was used to study the out-of-plane free vibration of Timoshenko arches of constant radius. The results for clamped–clamped arches with circular and square cross-sections are given. Wang et al. [9] obtained the dynamic stiffness matrix for an arch. The effects of shear deformation and rotatory inertia due to bending vibration are included, but the effect of rotatory inertia due to torsional vibration is neglected. Bickford and Maganty [10] investigated the out-of-plane vibrations of thick circular rings by considering the effects of shear deformation and rotatory inertias. In Ref. [11], an analytical model where the dynamic stiffness matrix elements allow for inclusion of the full effects of rotatory inertia and shear deformation is presented and compared with experimental results.

Kawakami et al. [12] presented an approximate method to study both in-plane and out-of-plane free vibrations of horizontal arches with arbitrary shapes and variable cross-sections. Based on the Timoshenko theory, Kang et al. [13] applied the differential quadrature method in the computation of the eigenvalues for the in-plane and out-of-plane vibrations of circular arches. Howson et al. [14] presented a converging method where the exact member theory in conjunction with the dynamic stiffness technique is used. The effects of shear deformation and rotatory inertia due to torsional vibration are included, but the rotatory inertia effect due to bending vibration is neglected. In the paper by Howson and Jemah [15], a method for finding the exact out-of-plane frequencies of curved Timeshenko beams is presented. The effects of shear deformation and rotatory inertia due to both torsional and flexural vibrations are included in the equations. Huang et al. [16] presented a method for out-of-plane dynamic behaviour of non-circular arches. The viscous damping and effects of shear deformation and rotatory inertia are considered by using Laplace transformation. Then, Huang et al. [17] extended the work on uniform arches [16] to investigate the out-of-plane responses arches with arbitrary shapes and cross-sections. The dynamic stiffness matrix and equivalent nodal force vector based on the general series solution of the differential equations are derived. In Ref. [18], the out-of-plane vibrations of non-uniform circular arches were investigated without considering the effects of shear deformation, warping and the rotatory inertia due to bending vibration. By introducing two physical parameters to simplify the analysis, the explicit relations between the flexural and torsional displacements for the out-of-plane vibrations are derived.

Recently, Rubin and Tufekci [19] investigated small deformation three-dimensional free vibrations of a circular arch with uniform rectangular cross-section by using different theoretical approaches. Special emphasis focused on the formulation by using theory of Cosserat point. Finite element results were also presented and some experiments were conducted to verify the theoretical and finite element results.

The main purpose of this paper is to present the exact solution to the governing differential equations of out-of-plane vibrations for a circular arch with uniform cross-section. The effects of shear deformation and rotatory inertia due to the flexural and torsional vibrations are taken into account. But the warping deformation of the cross-section is neglected. The initial value method is used in order to solve the governing differential equations. The same solution procedure, which is given by Tufekci and Arpaci [20], for in-plane free vibrations of a circular arch with uniform cross-section, is applied. The solution does not depend on the boundary conditions. The variations of the frequency coefficients with respect to the opening angle are presented for a certain slenderness ratio and several boundary conditions. The examples given in the literature are solved and the results are compared.

For in-plane vibration of arches, a phenomenon of transition of modes from extensional into inextensional, which occur with increase in beam curvature, has been observed by several authors [21–23]. The similar phenomenon can also be observed for out-of-plane vibrations of arches. The transition phenomenon is characterized by the sharp increase in frequencies of modes that occurs at certain combinations of curvature and length of the arch. This increase in mode frequency is accompanied by a significant change in the mode shapes. There is still no comprehensive analysis of the transition phenomenon and there are no proper explanations and methods for predicting the frequencies of an arch. This is possibly due to the fact that numerical simulations, commonly employed for the analyses, provide little analytical insight into the vibrational problem. In this study, the analysis of the transition phenomenon in vibrational behaviour of a shallow circular arch with uniform cross-section is also presented by using the exact solution of the governing equations.

2. Analysis

The out-of-plane behaviour of elastic arches (Fig. 1) with account taken of rotatory inertias and shear deformation is formulated by several authors as

$$
\begin{aligned}
&\frac{\mathrm{d}v}{\mathrm{d}\phi} + R\Omega_n - \frac{R}{GA/k_b}F_b = 0\\
&\frac{\mathrm{d}\Omega_n}{\mathrm{d}\phi} + \Omega_t - \frac{R}{EI_n}M_n = 0\\
&\frac{\mathrm{d}\Omega_t}{\mathrm{d}\phi} - \Omega_n - \frac{R}{GJ}M_t = 0,\\
&\frac{\mathrm{d}M_n}{\mathrm{d}\phi} + M_t - RF_b + R\mu\frac{I_n}{A}\omega^2\Omega_n = 0,\\
&\frac{\mathrm{d}M_t}{\mathrm{d}\phi} - M_n + R\mu\frac{I_p}{A}\omega^2\Omega_t = 0,\\
&\frac{\mathrm{d}F_b}{\mathrm{d}\phi} + R\mu\omega^2 v = 0,
\end{aligned}
\tag{1}
$$

where v is the out-of-plane displacement; Ω_n and Ω_t are the rotation angles about the normal and tangential axes; ϕ is the angular coordinate; R is the radius of curvature of undeformed beam axis; F_b is the binormal component of internal force; k_b is the factor of shear distribution along the binormal axis; E and G are Young's and shearing moduli; M_n and M_t are the internal moment about the normal and tangential axes; μ is the mass per unit length; A is the cross-sectional area; I_n

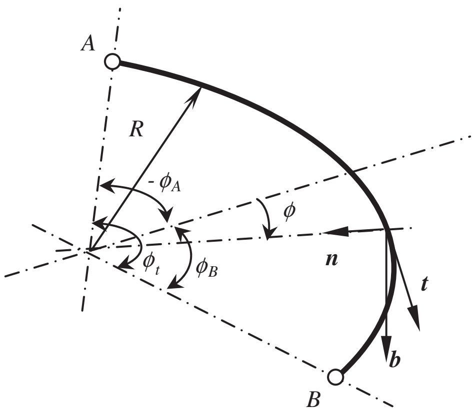

Fig. 1. Geometry and the coordinate system of a circular arch.

is the moment of inertia with respect to the normal axis; I_p is the polar moment of inertia, J is the torsional constant and ω is the angular frequency.

As it is well known, the Coulomb theory of torsion gives the exact solution for a circular shaft. It is assumed that the cross-sections of the bar remain plane and rotate without any distortion during twist. In other words, the shearing stress at any point of the cross-section is perpendicular to the radius r and proportional to the length r and to the angle of twist per unit length of the shaft. Then, the torsional constant is obtained as the polar moment of inertia of the cross-section. However, for an arbitrary geometry of cross-section, the torsional moment causes warping of the cross-section. This effect is deeper than the shear deformation effect. The correct solution of the problem of torsion of prismatic bars by couples applied at the ends was given by Saint-Venant. In this theory, the deformation of the bar consists of rotations and warping of the cross-section. The warping is the same for all cross-sections. If there is no warping, J will be equal to the polar moment of inertia of the cross-section.

The torsional moment of inertia for a rectangular cross-section is given in Refs. [24,25] as

$$J = \frac{WH^3}{3}\left[1 - \frac{192H}{\pi^5 W}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^5}\tanh\frac{\pi(2n-1)W}{2H}\right] \quad \text{for } W \geqslant H, \tag{2}$$

where W and H are the dimensions of the cross-section.

Instead, the approximate formula, which gives extremely good results, can also be used:

$$J = \frac{WH^3}{3}\left[1 - 0.63\frac{H}{W}\left(1 - \frac{H^4}{12W^4}\right)\right] \quad \text{for } W \geqslant H. \tag{3}$$

Some authors used the polar moment of inertia instead of the torsional constant [18,26]. Taking the polar moment of inertia as the torsional moment of inertia leads to an erroneous conclusion that the maximum shearing stress occurs at the corner of the rectangular cross-section where the stress is zero. Yang et al. [27] used the torsional constant in the terms for both torsional stiffness and rotatory inertia due to torsional vibrations.

Eq. (1) can be written also in matrix form as

$$\mathrm{d}y(\phi)/\mathrm{d}\phi = \mathbf{A}\mathbf{y}(\phi), \tag{4}$$

where

$$\mathbf{y} = \begin{bmatrix} v \\ \Omega_n \\ \Omega_t \\ M_n \\ M_t \\ F_b \end{bmatrix}, \quad \mathbf{A} = \begin{bmatrix} 0 & -R & 0 & 0 & 0 & \dfrac{k_b R}{GA} \\ 0 & 0 & -1 & \dfrac{R}{EI_n} & 0 & 0 \\ 0 & 1 & 0 & 0 & \dfrac{R}{GJ} & 0 \\ 0 & -R\mu\omega^2\dfrac{I_n}{A} & 0 & 0 & -1 & R \\ 0 & 0 & -R\mu\omega^2\dfrac{I_p}{A} & 1 & 0 & 0 \\ -R\mu\omega^2 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \tag{5}$$

The solution of this set of equations can be found to be

$$\mathbf{y}(\phi) = \mathrm{e}^{\mathbf{A}\phi}\mathbf{y}_0, \tag{6}$$

provided that the initial values vector $\mathbf{y}_0 = \mathbf{y}(\phi_0)$ at the reference coordinates $\phi = \phi_0$ is known. The term $\mathrm{e}^{\mathbf{A}\phi}$ can be expressed exactly by using the procedure given by Tufekci and Arpaci [20].

The initial value vector $\mathbf{y}_0$ must be obtained in order to specify the solution vector $\mathbf{y}(\phi)$. The six elements of the vectors can be found by using six equations obtained from the boundary conditions at the end points A and B.

For the end A as shown in Fig. 1:

$$\begin{array}{llll} \text{Clamped end}: & V(-\phi_A) = 0; & \Omega_n(-\phi_A) = 0; & \Omega_t(-\phi_A) = 0. \\ \text{Free end}: & M_t(-\phi_A) = 0; & M_n(-\phi_A) = 0; & F_b(-\phi_A) = 0. \end{array} \tag{7}$$

This yields six simultaneous linear equations in terms of the initial values at reference coordinate $\phi = \phi_0$. Since these six equations are homogeneous, the determinant of the coefficient matrix of the system must equal to zero in order to get the non-trivial solution. This requirement will give the natural frequencies. It is also possible to apply this solution procedure to the other cases in which some effects are neglected. For example, $k_b R/GA$ term in Eq. (1) must vanish in order to neglect the effect of shear deformation.

3. Numerical results and comparisons

The dimensionless frequency coefficients $c_i = \omega_i R^2 \phi_t^2 (\mu/EI_n)^{1/2}$ are calculated for five different cases:

Case 1: No effect is considered.
Case 2: All effects are considered.
Case 3: Only shear deformation effect is considered.
Case 4: Only rotatory inertia effect due to bending vibration is considered.
Case 5: Only rotatory inertia effect due to torsonal vibration is considered.

The examples are solved for clamped–clamped, free–free and clamped–free end conditions. Effects of the opening angle ϕ_t and the slenderness ratio $\lambda = R/i$ (where $i = (I_n/A)^{1/2}$ is the radius of gyration) on the natural frequencies are studied for several boundary conditions. The frequency coefficients are calculated for the lowest five modes of vibration.

In Fig. 2, the variation of the first frequency coefficient with the opening angle ϕ_t is given for a free-free arch with slenderness ratio $\lambda = 50$. It can be seen that the results obtained for the cases are considerably different from each other for small opening angles. The frequency coefficient for case 2 increases sharply then decreases slightly. The sharp increase in the frequency is due to the mode transition phenomenon from torsional into flexural. It can be observed near the opening angle of 90° for this example. The results of different cases become more consistent for larger opening angles. It can be seen easily in the figure that the rotatory inertia due to torsional vibration is the most important effect. As it is expected, the curves of different cases become closer to each other for larger slenderness ratios, and also the value of the opening angle in which the mode transition phenomenon is observed decreases.

In Fig. 3, the influence of opening angle on the first frequency coefficient is given for a clamped–clamped arch with the slenderness ratio $\lambda = 50$. The curves of different cases are much more consistent than those for a free-free arch. It shows that this vibration mode is flexural dominant for all opening angles.

In Fig. 4, the dimensionless frequency for the first mode is given for a clamped–free arch. It is interesting to note that the dimensionless frequency increases as the opening angle increases, in the contrary to the other boundary conditions. But this cannot be observed in higher modes. The results of different cases are very consistent even for a shallow arch.

Fig. 5 shows the fifth frequency coefficient for a clamped–clamped arch. As it can be seen in the figure, the results of several cases are considerably different even for a deep arch (up to $\phi_t = 150°$). It is also concluded from Fig. 5 that the effects of shear deformation and rotatory inertia due to bending vibration become more important with increasing mode numbers and decreasing opening angles of arch. The figures for more slender arches or for other boundary conditions are very similar to those given in this paper and they are not presented here for the brevity.

In Figs. 6–8, the frequency coefficients of the lowest five modes are given for clamped–clamped, free–free and clamped–free arches with $\lambda = 50$, respectively. The effects of shear deformation, and rotatory inertias due to both flexural and torsional vibrations are considered. A sharp increase in the frequency coefficient which is due to mode transition phenomenon can be seen in these figures. The mode shapes change from torsional into flexural where the frequency curves approach each other.

In Table 1, the frequency parameters $\omega R^2\sqrt{\mu/(EI_n)}$ of a clamped–clamped circular arch are presented with the parameters provided by Huang et al. [17] and Irie et al. [8]. A circular arch with opening angle $\phi_t = 80°$ and the slenderness ratio $\lambda = 20$ is investigated in this example. Huang et al. [17] used the dynamic stiffness matrix method, while Irie et al. [8] solved the governing equations by using transfer matrix method. In Table 1, the lowest four natural frequency parameters of circular uniform arches are compared with those given by Irie et al. [8] and Huang et al. [17]. The comparison shows that the results are very consistent.

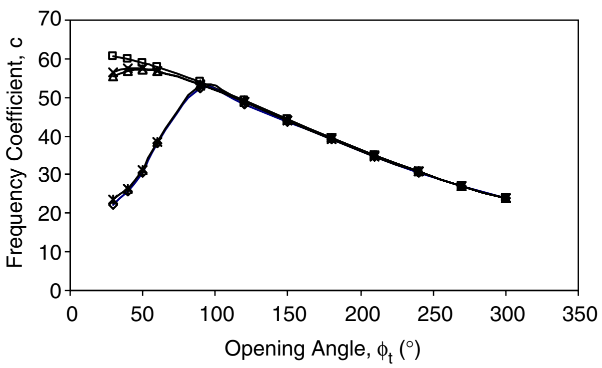

Fig. 2. Variation of the first frequency coefficient $c_1 = \omega_1 R^2 \phi_t^2 (\mu/EI_n)^{1/2}$ of free–free arch with the opening angle (the slenderness ratio $\lambda = 50$), —□— case 1; —◇— case 2; —△— case 3; —×— case 4; —✳— case 5.

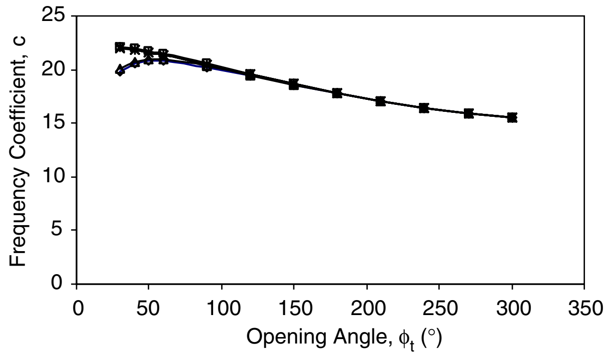

Fig. 3. Variation of the first frequency coefficient $c_1 = \omega_1 R^2 \phi_t^2 (\mu/EI_n)^{1/2}$ of clamped–clamped arch with the opening angle (the slenderness ratio $\lambda = 50$), —□— case 1; —◇— case 2; —△— case 3; —×— case 4; —✳— case 5.

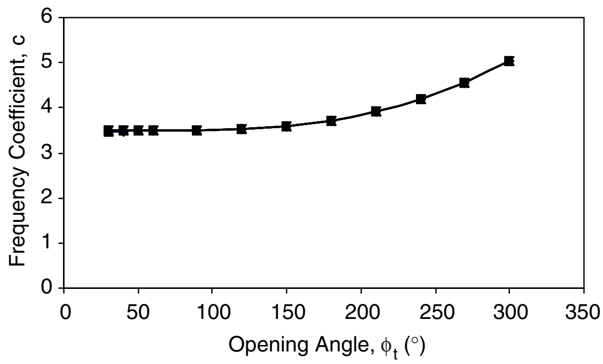

Fig. 4. Variation of the first frequency coefficient $c_1 = \omega_1 R^2 \phi_t^2 (\mu/EI_n)^{1/2}$ of clamped–free arch with the opening angle (the slenderness ratio $\lambda = 50$), —□— case 1; —◇— case 2; —△— case 3; —×— case 4; —✳— case 5.

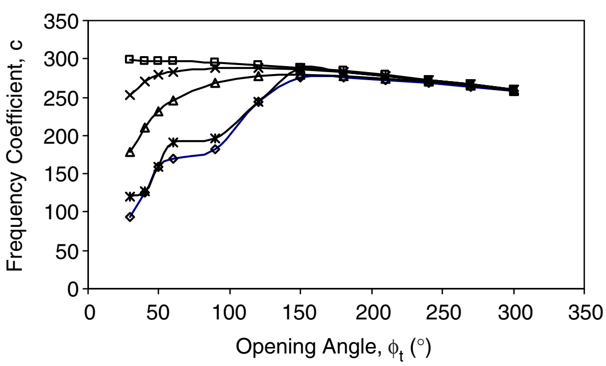

Fig. 5. Variation of the fifth frequency coefficient $c_5 = \omega_5 R^2 \phi_t^2 (\mu/EI_n)^{1/2}$ of clamped–clamped arch with the opening angle (the slenderness ratio $\lambda = 50$), —□— case 1; —◇— case 2; —△— case 3; —×— case 4; —✳— case 5.

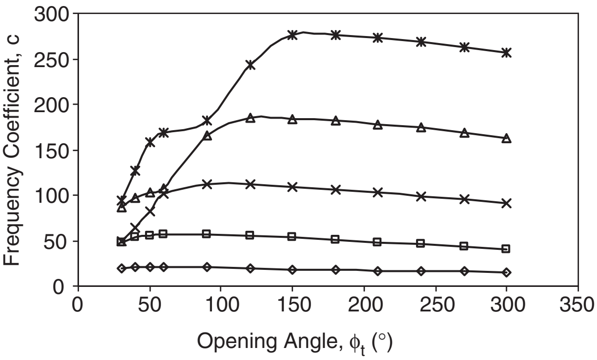

Fig. 6. The first five frequency coefficients of clamped–clamped arch with opening angle (the slenderness ratio $\lambda = 50$). —◇—: first mode, —□—: second mode, —×—: third mode, —△—: fourth mode, —✳—: fifth mode.

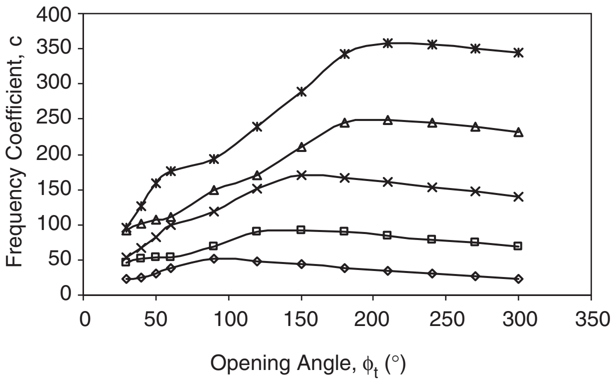

Fig. 7. The first five frequency coefficients of free–free arch with opening angle (the slenderness ratio $\lambda = 50$). —◇—: first mode, —□—: second mode, —×—: third mode, —△—: fourth mode, —✳—: fifth mode.

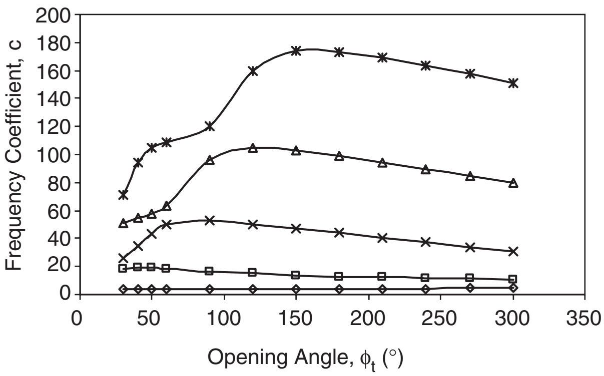

Fig. 8. The first five frequency coefficients of clamped–free arch with opening angle (the slenderness ratio $\lambda = 50$). —◇—: first mode, —□—: second mode, —×—: third mode, —△—: fourth mode, —✳—: fifth mode.

Tables 2 and 3 show the frequency parameters $\omega R^2\sqrt{\mu/(EI_n)}$ obtained for clamped–clamped circular arches with circular and square cross-sections, respectively. The arches with opening angles $\phi_t = 60°$, $120°$ and $180°$ and the slenderness ratios $\lambda = 20$ and 100 are studied. Poisson's ratio is 0.3 and the shear correction factors are $k_b = 1/0.89$ for the circular cross-section and $k_b = 1/0.85$ for the square cross-section. In these tables, the results obtained in the present study are compared with those given by Irie et al. [8], Kang et al. [13] and Howson and Jemah [15]. It can be seen that there is an excellent agreement between all the results for the arches having both circular and square cross-sections.

E Silva et al. [11] gave an analytical model for the out-of-plane vibration of a circular beam. They also performed some experiments to verify their results. The results are presented for several cases in which some effects are included in the analysis. The model of Wang et al. [9] was also used to solve the problem. In Table 4, the results obtained in the present study by considering the same cases are compared with those presented by E Silva [11]. The assumptions of the cases are as follows:

Case A: The effects of shear deformation and rotatory inertia due to torsional vibrations are considered but the effect of rotatory inertia due to bending vibrations is neglected. This case corresponds to the model of Wang et al. [9].

Case B: Classical beam theory in which all three effects are neglected.

Case C: Only shear deformation effect is neglected while the rotatory inertia effects due to both flexural and torsional vibrations are included in the analysis.

Case D: All effects are taken into account.

It is seen that the results are generally in good agreement. But for all the cases, the differences between the fourth frequencies of both studies are higher than the differences of the first three modes. It is interesting that the fourth natural frequencies of the model in Wang et al. [9] (case A) are higher than those obtained in this study, while all frequencies of the first three modes are lower than the results of this study. For the cases C and D, there are some discrepancies between some results of both studies, while others are consistent. It is thought that these discrepancies exist in the modes in which the torsional vibration is dominant. The results obtained in the present study are better than those presented in Ref. [11], when compared with experimental data. But there still exist some discrepancies between the theoretical and experimental results for the fourth mode. The experimental frequency of the arch with $\phi_t = 180°$ is about 10% higher than the theoretical results, while experimental results present constantly lower values than theoretical ones.

Table 1
The frequency parameters $\omega R^2\sqrt{\mu/(EI_n)}$ for a circular arch with opening angle $\phi_t = 80°$ and slenderness ratio $\lambda = 20$

Mode	Present study	Huang et al. [17]	Irie et al. [8]
1	3.13528	3.13412	3.134
2	5.02582	5.02223	5.022
3	5.58453	5.58418	5.584
4	6.74026	6.73358	6.734

Table 2
The frequency parameters $\omega R^2\sqrt{\mu/(EI_n)}$ for clamped–clamped circular arches with circular cross-sections (shear correction factor $k_b = 1/0.89$ and Poisson's ratio $v = 0.3$)

λ	ϕ (°)	Mode	Present study	Kang et al. [13]	Irie et al. [8]	Howson et al. [15]
20	60	1	16.88495	16.885	16.88	16.885
		2	39.70036	—	39.70	39.700
		3	40.93407	—	40.90	40.934
		4	70.58051	—	70.51	70.581
	120	1	4.309414	4.3094	4.309	4.3094
		2	11.79597	—	11.79	11.796
		3	22.51022	—	22.50	22.510
		4	23.30273	—	23.30	23.303
	180	1	1.790849	1.7908	1.791	1.7908
		2	5.032438	—	5.032	5.032
		3	10.23228	—	10.23	10.232
		4	16.91733	—	16.91	16.917
100	60	1	19.45376	19.454	19.45	19.454
		2	54.14767	—	54.14	54.148
		3	105.86088	—	105.9	105.86
		4	173.15822	—	173.1	173.16
	120	1	4.473080	4.4731	4.473	4.4731
		2	12.89161	—	12.89	12.892
		3	26.08059	—	26.08	26.081
		4	43.68398	—	43.68	43.684
	180	1	1.818173	1.8182	1.818	1.8182
		2	5.241521	—	5.242	5.2415
		3	10.988907	—	10.99	10.989
		4	18.813360	—	18.81	18.813

4. Conclusions

In this study, the equations of free out-of-plane vibrations of a circular arch with uniform cross-section are solved exactly by using the initial value method. The effects of shear deformation and rotatory inertias due to both flexural and torsional vibrations are included in the equations. The same solution procedure is employed for the cases in which some of the effects are considered as well as none. It is obvious that the classical beam theory neglecting all aforementioned effects does not represent the actual arch behaviour if a shallow arch is considered. The effect of rotatory inertia due to torsional vibrations is the major effect for a shallow arch, even if it is slender and lower modes are considered. If the arch is also thick, as it is well known, the effects of shear deformation and rotatory inertia due to bending vibrations should be taken into account even for lower modes.

Table 3
The frequency parameters $\omega R^2\sqrt{\mu/(EI_n)}$ for clamped–clamped circular arches with square cross-sections (shear correction factor $k_b = 1/0.85$ and Poisson's ratio $\nu = 0.3$)

λ	ϕ (°)	Mode	Present study	Kang et al. [13]	Irie et al. [8]	Howson et al. [15]
20	60	1	16.74397	16.744	16.74	16.743
		2	36.94631	—	36.92	36.921
		3	40.45051	—	40.45	40.449
		4	69.61974	—	69.62	69.618
	120	1	4.282547	4.2827	4.282	4.2823
		2	11.69060	—	11.69	11.690
		3	22.05355	—	22.05	22.045
		4	22.38193	—	22.38	22.379
	180	1	1.776521	1.7766	1.776	1.7764
		2	4.981896	—	4.982	4.9814
		3	10.134003	—	10.13	10.133
		4	16.76278	—	16.76	16.762
100	60	1	19.40190	19.402	19.40	19.401
		2	54.02958	—	54.03	54.029
		3	105.64828	—	105.6	105.65
		4	172.77355	—	172.8	172.77
	120	1	4.451450	4.4516	4.451	4.4512
		2	12.82629	—	12.83	12.826
		3	25.98937	—	25.99	26.988
		4	43.57053	—	43.57	43.570
	180	1	1.804340	1.8045	1.804	1.8042
		2	5.197995	—	5.198	5.1975
		3	10.91819	—	10.92	10.917
		4	18.72548	—	18.72	18.725

The natural frequencies for all modes exhibit sharp increases for shallow arches and then decrease gradually with an increase in the opening angle. This is due to mode transition phenomenon which cannot be observed in the classical beam theory. The transition phenomenon is characterized by the sharp increase in frequencies of modes that occurs at certain combinations of curvature and length of the arch. This increase in mode frequency is accompanied by a significant change in the mode shapes.

The behaviours of arches with clamped–clamped and free–free boundary conditions are very similar. For a clamped–free arch, it is seen that there exists a distinct difference in behaviour of the arch when compared with other boundary conditions; the fundamental frequency increases gradually as the opening angle increases for all slenderness ratios. In the other modes, a similar behaviour with other boundary conditions is observed.

The examples given in the literature are solved and the results are compared with the published results. Excellent agreement is found between the results. All examples show that the effects of

Table 4
Comparison of frequencies (Hz) obtained in Ref. [11] using different models

Mode no.	ϕ (°)	Results of this study				Results of E Silva et al. [11]				
		Case A	Case B	Case C	Case D	Case A	Case B	Case C	Case D	Exp. results
1	30	786.6738	800.6474	514.7469	513.2655	782.87	796.75	634.21	632.36	—
	60	719.0624	730.8058	707.8963	703.0553	715.46	727.2	712.36	707.29	—
	90	633.2932	642.6385	617.9104	614.8309	630.2	639.18	623.42	620.25	—
	180	402.5242	407.2038	394.434	393.6718	400.2	400.3	395.81	391.05	387.5
	270	256.5756	259.112	254.0342	253.7477	255.0	254.2	253.7	250.8	250.5
2	30	1425.554	1474.058	786.9517	779.9234	1418.6	1466.1	786.33	779.6	—
	60	1382.723	1428.2	848.4754	845.296	1376.0	1421.2	1053.5	1046.4	—
	90	1317.301	1358.46	1072.914	1065.671	1310.7	1351.8	1203.4	1190.2	—
	180	1053.206	1080.203	998.8414	991.5031	1047.8	1067.2	1023.4	1007.2	967.6
	270	788.136	805.0474	762.2617	759.1717	784.1	792.9	771.19	760.4	747.1
3	30	2508.818	2651.783	1597.856	1565.754	2494.2	2642.5	1605.1	1570.7	—
	60	2435.994	2571.177	1612.849	1584.146	2424.8	2562.5	1700.8	1671.2	—
	90	2339.53	2464.672	1734.47	1711.729	2328.8	2456.0	2045.6	2022.2	—
	180	2002.425	2094.989	1831.355	1804.384	1992.5	2074.1	1927.9	1882.2	1717.5
	270	1646.99	1711.597	1551.65	1534.549	1638.2	1690.5	1596.9	1564.2	1487.5
4	30	3572.732	3868.856	1639.586	1637.249	3675.0	3458.5	2162.4	2153.4	—
	60	3529.953	3818.441	1753.192	1744.951	3459.3	3550.6	2197.8	2167.4	—
	90	3462.539	3739.28	1861.641	1844.976	3451.3	3614.2	2180.2	2136.2	—
	180	3155.901	3383.386	2524.19	2489.824	3143.4	3370.4	2864.3	2818.0	2763.0
	270	2766.249	2940.697	2510.843	2465.595	2752.0	2910.3	2657.8	2580.0	2460.2

shear deformation and rotatory inertia due to both flexural and torsional vibrations on the natural frequencies of stubby beams are substantial. It is also known that these effects become more important for higher natural frequencies.

References

[1] S. Markus, T. Nanasi, Vibration of curved beams, *The Shock and Vibration Digest* 13 (1981) 3–14.

[2] P.A.A. Laura, M.J. Maurizi, Recent research on vibrations of arch-type structures, *The Shock and Vibration Digest* 19 (1987) 6–9.

[3] P. Chidamparam, A.W. Leissa, Vibrations of a planar curved beams, rings and arches, *Applied Mechanics Reviews* 46 (1993) 467–483.

[4] N.M. Auciello, M.A. De Rosa, Free vibrations of circular arches: a review, *Journal of Sound and Vibration* 174 (1994) 433–458.

[5] A.E.H. Love, *A Treatise on the Mathematical Theory of Elasticity*, 4th ed., Dover Publications, New York, 1944.

[6] E. Volterra, J.D. Morell, Lowest natural frequency of elastic arc for vibrations outside the plane of initial curvature, *Journal of Applied Mechanics* 28 (1961) 624–627.

[7] T. Irie, G. Yamada, I. Takahashi, The steady state out-of-plane response of a Timoshenko curved beam with internal damping, *Journal of Sound and Vibration* 71 (1980) 145–156.

[8] T. Irie, G. Yamada, I. Takahashi, Natural frequencies of out-of-plane vibration of arcs, *Journal of Applied Mechanics* 49 (1982) 910–913.
[9] T.M. Wang, A. Laskey, M. Ahmad, Natural frequencies for out-of-plane vibrations of continuous curved beams considering shear and rotary inertia, *International Journal of Solids and Structures* 20 (1984) 257–265.
[10] W.B. Bickford, S.P. Maganty, On the out-of-plane vibrations of thick rings, *Journal of Sound and Vibration* 108 (1986) 503–507.
[11] J.M.M. E Silva, A.P.V. Urgueira, Out-of-plane dynamic response of curved beams—An analytical model, *International Journal of Solids and Structures* 24 (1988) 271–284.
[12] M. Kawakami, T. Sakiyama, H. Matsuda, C. Morita, In-plane and out-of-plane free vibrations of curved beams with variable sections, *Journal of Sound and Vibration* 187 (1995) 381–401.
[13] K. Kang, C.W. Bert, A.G. Stritz, Vibration analysis of shear deformable circular arches by the differential quadrature method, *Journal of Sound and Vibration* 181 (1995) 353–360.
[14] W.P. Howson, A.K. Jemah, J.Q. Zhou, Exact natural frequencies for out-of-plane motion of plane structures composed of curved beam members, *Computers and Structures* 55 (1995) 989–995.
[15] W.P. Howson, A.K. Jemah, Exact out-of-plane natural frequencies of curved Timoshenko beams, *Journal of Engineering Mechanics* 125 (1999) 19–25.
[16] C.S. Huang, Y.P. Tseng, S.H. Chang, Out-of-plane dynamic responses of non-circular curved beams by numerical Laplace transform, *Journal of Sound and Vibration* 215 (3) (1998) 407–424.
[17] C.S. Huang, Y.P. Tseng, S.H. Chang, C.L. Hung, Out-of-plane dynamic analysis of beams with arbitrarily varying curvature and cross-section by dynamic stiffness matrix method, *International Journal of Solids and Structures* 37 (2000) 495–513.
[18] S.Y. Lee, J.C. Chao, Out-of-plane vibrations of curved non-uniform beams of constant radius, *Journal of Sound and Vibration* 238 (2000) 443–458.
[19] M.B. Rubin, E. Tufekci, Three-dimensional free vibrations of a circular arch using the theory of a Cosserat point, *Journal of Sound and Vibration* 286 (2005) 799–816.
[20] E. Tufekci, A. Arpaci, Exact solution of in-plane vibrations of circular arches with account taken of axial extension, transverse shear and rotatory inertia effects, *Journal of Sound and Vibration* 209 (1998) 845–856.
[21] T. Tarnopolskaya, F.R. De Hoog, N.H. Fletcher, S. Thwaites, Asymptotic analysis of the free vibrations of beams with arbitrarily varying curvature and cross-section, *Journal of Sound and Vibration* 196 (1996) 659–680.
[22] T. Tarnopolskaya, F.R. De Hoog, N.H. Fletcher, Low-frequency mode transition in the free in-plane vibration of curved beams, *Journal of Sound and Vibration* 228 (1999) 69–90.
[23] E. Tufekci, Exact solution of free in-plane vibration of shallow circular arches, *International Journal of Structural Stability and Dynamics* 1 (2001) 409–428.
[24] S. Timoshenko, J.N. Goodier, *Theory of Elasticity*, 2nd ed., McGraw-Hill Book Co., Tokyo, 1951.
[25] M.B. Rubin, *Cosserat Theories: Shells, Rods and Points*, Kluwer Academic Publishers, The Netherlands, 2000.
[26] E. Tufekci, On "Finite-element formulation of geometrically exact three-dimensional beam theories based on interpolation of strain measures [1]", *Computer Methods in Applied Mechanics and Engineering* 193 (2004) 4067–4068.
[27] Y.-B. Yang, C.-M. Wu, J.-D. Yau, Dynamic response of a horizontally curved bean subjected to vertical and horizontal moving loads, *Journal of Sound and Vibration* 242 (2001) 519–537.