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Response of an infinite free plate–liquid system to a moving load: Theoretical stationary response in the subsonic case

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Abstract

An unbounded system consisting of a plate in contact with a liquid is considered. The coupling equations are summarised, accounting for the compressibility of the liquid and the Mindlin–Reissner plate theory. An analytical solution is sought in the one-dimensional case, for a load that travels at a constant velocity, subsonic with respect to the liquid sound velocity. The work completes results already obtained for the supersonic case. The steady-state solution is theoretically established for a moving force and is explicit for displacements, transverse deflexion and cross-sectional rotation, for flexural and shearing stresses in the plate and for pressure and velocities in the liquid. All results are in a non-dimensional form and contain as few parameters as possible. Graphs and curves are added which provide results for any aluminium or steel plate coupled with water, whatever the subsonic velocity of loading.

A numerical part completes the study and reveals that, when a moving force travels across a large coupled plate at a constant velocity, a part of the response evolves progressively towards the steady-state response. Comparison shows that the transient numerical solution to a stationary loading converges to the theoretical stationary response.

1. Introduction

Fluid-structure coupling problems can take a variety of forms. The present one concerns the response of a plate in contact with a liquid when a load moves on the plate. This problem itself

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can be envisaged with a very large scope of hypotheses depending on the loading velocity range. If the point of application of the load moves sufficiently fast, it is necessary to consider acoustic wave propagation. A previous study [1], summarised the interest of the problem and studied the response of such a system submitted to a high-velocity travelling loading, possibly of the same order as the velocities of sound in the plate or in the liquid. The response was found and the analysis revealed that the stationary response was of great interest. It was established that, for constant loading velocity, greater than the velocity of sound of the liquid, the response contained a stationary part which converged with a quasi steady-state response.

The previous work suggested that the same would be true for reduced load velocities, i.e. less than the velocity of sound in the liquid. For such so-called subsonic velocities, the nature of the governing equations changes and those describing the liquid become elliptic. The problem cannot be solved in the same manner.

The purpose of this paper is to investigate stationary responses for coupled systems loaded by subsonic travelling pressure fields.

The existence of a stationary response presupposes that the sizes of the system are large enough to avoid waves reflected at the boundaries coming back to the neighbourhood of the observation point. This is generally possible with a favourable compromise between the sizes of the system and the time of observation.

Although several physical problems need such an analysis, the main purpose of this work is to analyse the response of a plate in contact with a liquid under explosive loading.

The previous cited work established a validated numerical method able to predict the response of a coupled plate for any dynamic loading, especially a detonation. A running detonation is able, in some conditions, to produce a constant load profile, travelling at a constant velocity. This special case reinforces the necessity of studying the possibility of a stationary response following the loading. For real explosions, the conditions are not so simple that they could be summarised by a one-dimensional analysis. Nevertheless, the study of real explosions expanding on plates has shown that a cylindrical expansion gave rise to responses, and especially stresses, very close to those observed in the one-dimensional expansion. So, the one-dimensional analysis seems a good compromise to permit a theoretical solution.

This dynamic problem, takes account of acoustic waves in a compressible liquid. Moreover, the shortness of wavelengths requires that Mindlin plate theory be considered (or, the one-dimensional equivalent, the Timoshenko beam). These two hypotheses are essential and sufficient. For this reason, works found in the literature about coupling are not really useful because most of them concern either uncompressible liquid, either the classical flexion theory and often, both together [2,3]. Moreover, some of them consider a finite constant depth of liquid, like the work of Nugroho [4] who studied the behaviour of an ice sheet loaded by moving trucks or aeroplanes; all these hypotheses do not lend themselves to finding high-frequency stationary responses.

The best analysis is that of the Timoshenko beam on an elastic foundation, a classical problem revisited recently by Felszeghy [5] who summarises the essential elements of the results and which underlines clearly that the classical flexion theory is not suitable.

The problem of a beam (or plate) coupled to a continuous medium is somewhat different from that of a beam on an elastic foundation in the sense that the continuity of the medium ties the two parts of the beam, ahead or rear of the force. In that case, the velocity of the loading must be compared, not only with the characteristic bar (or plate) velocities, but also with the characteristic velocities of waves in the medium (Shear and Rayleigh waves). Dieterman and Metrikine have contributed to many works in this field. The nearest of them concerns the steady-state displacement of a beam on an elastic half-space due to a uniformly moving constant load [6].

The coupling to a compressible liquid, in the case of subsonic velocity of loading, reveals a new difficulty. The solutions of the coupling equations are different from those previously encountered. The present analytical resolution has used Fourier Sine and Cosine Transforms of polynomial ratios.

2. Statement of problem

The problem is envisaged as being one-dimensional. All necessary details useful for the statement of equations can be obtained in the previously cited work. In this context, only the definition of functions, variables and parameters is recalled, as the correspondence between dimensional and non-dimensional values (noted in capital letters).

Fig. 1 presents the geometry and coordinates of the plate–liquid coupled system.

2.1. Plate governing equations

The plate is assimilated to a strip of unit width. It has a mass density ρ, a thickness h, a Young's modulus E, a Poisson's ratio v, a shear modulus G and a shear correction factor κ.

The displacements necessary to describe the motion of the strip are: $w(x, t)$, the flexural displacement of the neutral axis, and $\Psi(x, t)$, the angular rotation of the cross-section.

The stresses induced by the bending are: σ, the flexural stress on the external surface and τ the average shear stress in the cross-section. The general form of pressure loading is a function $p(x, t)$.

The following non-dimensional variables are used where A represents the cross-sectional area and I its moment of inertia:

$$r_0 = \sqrt{\frac{I}{A}} = \frac{h}{\sqrt{12}}, \qquad X = \frac{x}{r_0}, \qquad W = \frac{w}{r_0}. \tag{1}$$

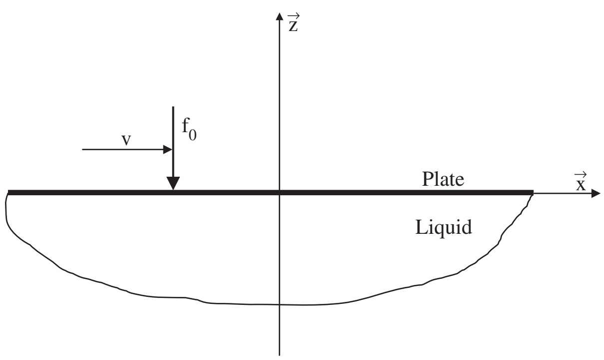

Fig. 1. The coupled system and its loading.

Using the velocity of longitudinal propagation in a plate

$$v_p = \sqrt{\frac{E}{\rho(1-\upsilon^2)}}, \tag{2}$$

and the modified shear wave velocity

$$v_s = \sqrt{\frac{\kappa G}{\rho}} \quad \text{with } \kappa = 0.86 \tag{3}$$

[7] one sets

$$\theta = \frac{v_s}{v_p}, \qquad T = t\frac{v_p}{r_0}, \qquad P = \frac{p}{\rho v_p^2\sqrt{12}}, \qquad \Sigma = \frac{\sigma}{\rho v_p^2\sqrt{12}}. \tag{4}$$

The non-dimensional equations of the movement are recalled from Eq. [8]

$$\frac{\partial^2 W}{\partial T^2} = \theta^2\left(\frac{\partial^2 W}{\partial X^2} - \frac{\partial \Psi}{\partial X}\right) + P(X,T), \tag{5}$$

$$\frac{\partial^2 \Psi}{\partial T^2} = \frac{\partial^2 \Psi}{\partial X^2} + \theta^2\left(\frac{\partial W}{\partial X} - \Psi\right). \tag{6}$$

The non-dimensional forms Σ and Γ correspond to the dimensional forms σ and τ.

$$\Sigma = -\frac{1}{2}\frac{\partial \Psi}{\partial X}, \qquad \Gamma = \frac{\theta^2}{\sqrt{12}}\left(\frac{\partial W}{\partial X} - \Psi\right). \tag{7}$$

2.2. Liquid governing equations

The velocity of acoustic waves in the liquid is noted v_l; its mass density is ρ_l.
Using the potential function φ, one obtains

$$\frac{\partial^2 \varphi}{\partial t^2} = v_l^2 \nabla^2 \varphi. \tag{8}$$

The pressure and speeds are deduced by

$$p = \rho_l \frac{\partial \varphi}{\partial t}, \qquad \frac{\partial u}{\partial t} = -\frac{\partial \varphi}{\partial x}, \qquad \frac{\partial w}{\partial t} = -\frac{\partial \varphi}{\partial z}, \tag{9}$$

u and w are the displacements in the liquid according, respectively, to the x and z coordinates.
The new non-dimensional variables are introduced

$$\Phi = \varphi/v_p r_0, \qquad Z = z/r_0, \qquad \delta = v_l/v_p, \qquad \mu = \rho_l/\rho\sqrt{12}. \tag{10}$$

And the new non-dimensional equations are deduced

$$\frac{\partial^2 \Phi}{\partial T^2} = \delta^2\left(\frac{\partial^2 \Phi}{\partial X^2} + \frac{\partial^2 \Phi}{\partial Z^2}\right), \tag{11}$$

$$\frac{\partial U}{\partial T} = -\frac{\partial \Phi}{\partial X}, \qquad \frac{\partial W}{\partial T} = -\frac{\partial \Phi}{\partial Z}, \qquad P = \mu \frac{\partial \Phi}{\partial T}. \tag{12}$$

2.3. Coupling equations

Introducing the continuity of stresses and normal displacement at the interface, the following two equations take place:

$$P_{\text{int}} = \mu \left(\frac{\partial \Phi}{\partial T} \right)_{(Z=0)} \quad \text{and} \quad \left(\frac{\partial W}{\partial T} \right)_{(\text{Plate})} = -\left(\frac{\partial \Phi}{\partial Z} \right)_{(Z=0)}. \tag{13}$$

P_{int} is the internal pressure, under the plate, and P_{ext} the external loading, on the plate.

2.4. Final equation system

According to the previous conditions, the coupled equations take the form

$$\frac{\partial^2 W}{\partial T^2} = \theta^2 \left(\frac{\partial^2 W}{\partial X^2} - \frac{\partial \Psi}{\partial X} \right) + \mu \left(\frac{\partial \Phi}{\partial T} \right)_{(Z=0)} + P_{\text{ext}}(X, T), \tag{14}$$

$$\frac{\partial^2 \Psi}{\partial T^2} = \frac{\partial^2 \Psi}{\partial X^2} + \theta^2 \left(\frac{\partial W}{\partial X} - \Psi \right), \tag{15}$$

$$\left(\frac{\partial W}{\partial T} \right) = -\left(\frac{\partial \Phi}{\partial Z} \right)_{(Z=0)}, \tag{16}$$

$$\frac{\partial^2 \Phi}{\partial T^2} = \delta^2 \left(\frac{\partial^2 \Phi}{\partial X^2} + \frac{\partial^2 \Phi}{\partial Z^2} \right). \tag{17}$$

The first three equations are valid for the plate and on the boundary of the liquid while the fourth one is valid in the fluid domain and on its boundary.

3. The steady-state case

3.1. Equation formulation

Let now the loading of the system be the dimensional force f_0 travelling at constant velocity v on an infinitely long strip, from left to right. The force starts from $x = -\infty$ and goes towards $x = +\infty$. If, for an observer who moves with the load, the motion of the coupled system appears frozen, a steady-state solution exists.

The formulation of a steady-state solution needs some abstract analysis. While no boundary conditions are introduced, the system is free. The gravity forces are also omitted. The movement of the system is assumed to have existed for a very long time. In these conditions, the values of diverse functions can be infinite at the ends of the plate. Furthermore, a steady-state movement of the system, loaded by a vertical force, can verify that it has been uniformly falling for a very long time. Under this assumption, a non determined very large constant is added to the vertical displacement.

Let V and F_0 be the non-dimensional form of the loading velocity ($V = v/v_p$) and of the moving force.

For an observer moving with the load, only one variable Y is useful to describe the motion of the system. This relative variable Y is related to the absolute variable X by: $Y = X - V.T$, then

$$W(X,T) = \hat{W}(Y), \qquad \Psi(X,T) = \hat{\Psi}(Y), \qquad \Phi(X,Z,T) = \hat{\Phi}(Y,Z)$$

with

$$P(X,T) = \hat{P}(Y) = F_0 \cdot \delta(Y)$$

Replacing partial derivatives by ordinary derivatives, $\partial/\partial X = \mathrm{d}/\mathrm{d}Y$, $\partial/\partial T = -V(\mathrm{d}/\mathrm{d}Y)$, and omitting the sign ˆ on the functions, the system of coupled equations becomes

$$V^2 \frac{\mathrm{d}^2 W}{\mathrm{d}Y^2} = \theta^2 \left(\frac{\mathrm{d}^2 W}{\mathrm{d}Y^2} - \frac{\mathrm{d}\Psi}{\mathrm{d}Y} \right) - \mu V \left(\frac{\partial \Phi}{\partial Y} \right)_{(Z=0)} + F_0 \delta(Y), \tag{18}$$

$$V^2 \frac{\mathrm{d}^2 \Psi}{\mathrm{d}Y^2} = \frac{\mathrm{d}^2 \Psi}{\mathrm{d}Y^2} + \theta^2 \left(\frac{\mathrm{d}W}{\mathrm{d}Y} - \Psi \right), \tag{19}$$

$$V \frac{\mathrm{d}W}{\mathrm{d}Y} = \left(\frac{\partial \Phi}{\partial Z} \right)_{(Z=0)}, \tag{20}$$

$$V^2 \frac{\partial^2 \Phi}{\partial Y^2} = \delta^2 \left(\frac{\partial^2 \Phi}{\partial Y^2} + \frac{\partial^2 \Phi}{\partial Z^2} \right). \tag{21}$$

3.2. Solution procedure

Eqs. (18)–(21) form a linear system. Its solution can be searched as the sum of a solution of the forced equations and of a solution of the homogeneous associated equations.

The last equation of the system is transformed by setting $\Omega^2 = \left(\delta^2 - V^2\right)/\delta^2$ (22)

$$\text{in: } \Omega^2 \frac{\partial^2 \Phi}{\partial Y^2} + \frac{\partial^2 \Phi}{\partial Z^2} = 0. \tag{23}$$

The hypothesis $V < \delta$ confirms that Eq. (23) is a Laplace equation. That is the very difference with the supersonic case. Nevertheless, the analysis remains highly applicable for a range of very high load velocities, since the velocity of acoustic waves in water is about 1500 m/s.

Eq. (23) is valid in the half-plane $Z \leqslant 0$. Its solution in the whole half-plane can be found by knowing only $\Phi(Y, 0)$, the value of Φ on the boundary $Z = 0$, as recalled by Haberman [9] in its consideration about infinite-domain problems.

Using the Fourier transform with the following definition:

$$\overline{\Phi}(\xi, Z) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \Phi(Y, Z)\mathrm{e}^{\mathrm{i}\xi Y}\,\mathrm{d}Y, \tag{24}$$

$$\Phi(Y, Z) = \int_{-\infty}^{+\infty} \overline{\Phi}(\xi, Z)\mathrm{e}^{-\mathrm{i}\xi Y}\,\mathrm{d}\xi \tag{25}$$

and transforming Eq. (23), one obtains

$$\frac{\partial^2\overline{\Phi}}{\partial Z^2} - \xi^2\Omega^2\overline{\Phi} = 0, \tag{26}$$

which has for general solution

$$\overline{\Phi}(\xi, Z) = a(\xi)\mathrm{e}^{\xi\Omega Z} + b(\xi)\mathrm{e}^{-\xi\Omega Z}. \tag{27}$$

Z being non-positive, the solution $\overline{\Phi}(\xi, Z)$ must be interpreted as

$$\overline{\Phi}(\xi, Z) = \left| \begin{array}{ll} a(\xi)\mathrm{e}^{\xi\Omega Z} & \text{for } \xi \geqslant 0, \\ b(\xi)\mathrm{e}^{-\xi\Omega Z} & \text{for } \xi < 0. \end{array} \right. \tag{28}$$

To sum up

$$\overline{\Phi}(\xi, Z) = C(\xi)\mathrm{e}^{|\xi|\Omega Z}. \tag{29}$$

This general solution is used to solve Eqs. (18), (19) by the way of Fourier transform.

$$-\xi^2 V^2\overline{W} = \theta^2\left(-\xi^2\overline{W} + \mathrm{i}\xi\overline{\psi}\right) - \mu V(-\mathrm{i}\xi)\overline{\Phi}_{Z=0} + \frac{F_0}{2\pi}, \tag{30}$$

$$-\xi^2 V^2\overline{\psi} = -\xi^2\overline{\psi} + \theta^2\left(-\mathrm{i}\xi\overline{W} - \overline{\psi}\right) \tag{31}$$

$$\text{with} \quad \overline{\Phi}_{Z=0} = \overline{\Phi}(\xi, 0) = C(\xi). \tag{32}$$

The condition of coupling, Eq. (20) gives

$$\left(\frac{\partial\overline{\Phi}}{\partial Z}\right)_{Z=0} = -\mathrm{i}\xi V\overline{W}. \tag{33}$$

Using Eq. (29)

$$\left(\frac{\partial\overline{\Phi}}{\partial Z}\right)_{Z=0} = \Omega|\xi|C(\xi) \tag{34}$$

and after comparison

$$C(\xi) = -\mathrm{i}\,\mathrm{sgn}(\xi)\frac{V}{\Omega}\overline{W}. \tag{35}$$

Reintroducing this value of $C(\xi)$, system (30), (31) becomes

$$\overline{W}\left[\xi^2\left(\theta^2 - V^2\right) - |\xi|\frac{\mu V^2}{\Omega}\right] - \left[\mathrm{i}\xi\theta^2\right]\overline{\psi} = \frac{F_0}{2\pi}, \tag{36}$$

$$\overline{W}\left[-\mathrm{i}\xi\theta^2\right] + \left[-\theta^2 + \xi^2\left(V^2 - 1\right)\right]\overline{\psi} = 0. \tag{37}$$

To solve the system, the determinant is necessary and resembles the following polynomial:

$$P(\xi) = \xi\left[\xi^3\left(\theta^2 - V^2\right)\left(V^2 - 1\right) + \xi\theta^2 V^2 - \operatorname{sgn}(\xi)\mu\frac{V^2}{\Omega}\left(\xi^2\left(V^2 - 1\right) - \theta^2\right)\right], \tag{38}$$

where it is convenient to distinguish

$P_+(\xi)$ corresponding to $\xi \geqslant 0$, from $P\_(\xi)$ corresponding to $\xi < 0$.

Thus, it is possible to obtain the original functions

$$W(Y) = \int_{-\infty}^{+\infty} \overline{W}(\xi)\mathrm{e}^{-\mathrm{i}\xi Y}\,\mathrm{d}\xi = \frac{F_0}{2\pi}\int_{-\infty}^{+\infty}\frac{\xi^2\left(V^2 - 1\right) - \theta^2}{P(\xi)}\mathrm{e}^{-\mathrm{i}\xi Y}\,\mathrm{d}\xi, \tag{39}$$

$$\Psi(Y) = \int_{-\infty}^{+\infty} \overline{\Psi}(\xi)\mathrm{e}^{-\mathrm{i}\xi Y}\,\mathrm{d}\xi = \frac{F_0}{2\pi}\int_{-\infty}^{+\infty}\frac{\mathrm{i}\theta^2\xi}{P(\xi)}\mathrm{e}^{-\mathrm{i}\xi Y}\,\mathrm{d}\xi, \tag{40}$$

$$\Phi(Y,0) = \int_{-\infty}^{+\infty} \overline{\Phi}(\xi,0)\mathrm{e}^{-\mathrm{i}\xi Y}\,\mathrm{d}\xi = \frac{F_0}{2\pi}\int_{-\infty}^{+\infty}\frac{V}{\Omega}\frac{-\mathrm{i}\,\operatorname{sgn}(\xi)\left(\xi^2\left(1 - V^2\right) - \theta^2\right)}{P(\xi)}\mathrm{e}^{-\mathrm{i}\xi Y}\,\mathrm{d}\xi. \tag{41}$$

For example, developing Ψ

$$\Psi(Y) = \frac{F_0}{2\pi}\int_{-\infty}^{0}\frac{\mathrm{i}\theta^2\xi}{P_-(\xi)}\mathrm{e}^{-\mathrm{i}\xi Y}\,\mathrm{d}\xi + \frac{F_0}{2\pi}\int_{0}^{\infty}\frac{\mathrm{i}\theta^2\xi}{P_+(\xi)}\mathrm{e}^{-\mathrm{i}\xi Y}\,\mathrm{d}\xi, \tag{42}$$

$$\Psi(Y) = \frac{F_0}{\pi}\int_{0}^{\infty}\frac{\theta^2\xi}{P_+(\xi)}\sin(\xi Y)\,\mathrm{d}\xi, \tag{43}$$

where

$$\int_{0}^{\infty}\frac{\theta^2\xi}{P_+(\xi)}\sin(\xi Y)\,\mathrm{d}\xi$$

represents the inverse Fourier Sine Transform of $\theta^2\xi/P_+(\xi)$.

In a similar way, the other original functions are found, which use Fourier Sine and Cosine transform [10].

$$W(Y) = \frac{F_0}{\pi}\int_{0}^{\infty}\frac{\xi^2\left(V^2 - 1\right) - \theta^2}{P_+(\xi)}\cos(\xi Y)\,\mathrm{d}\xi, \tag{44}$$

$$\Phi(Y,0) = \frac{F_0}{\pi}\int_{0}^{\infty}\frac{V}{\Omega}\frac{\theta^2 - \xi^2\left(V^2 - 1\right)}{P_+(\xi)}\sin(\xi Y)\,\mathrm{d}\xi. \tag{45}$$

And the most useful derivatives to obtain stresses, according to Eq. (7), and pressure, according to Eq. (12)

$$\frac{\mathrm{d}W}{\mathrm{d}Y}(Y) - \Psi(Y) = \frac{F_0}{\pi}\int_{0}^{\infty}\frac{-\xi^3\left(V^2 - 1\right)}{P_+(\xi)}\sin(\xi Y)\,\mathrm{d}\xi, \tag{46}$$

$$\frac{\mathrm{d}\Psi}{\mathrm{d}Y}(Y) = \frac{F_0}{\pi}\int_0^\infty \frac{\xi^2\theta^2}{P_+(\xi)}\cos(\xi Y)\,\mathrm{d}\xi, \tag{47}$$

$$\left(\frac{\mathrm{d}\Phi_{z=0}}{\mathrm{d}Y}\right)(Y) = \frac{F_0}{\pi}\frac{V}{\Omega}\int_0^\infty \frac{\xi\left(\theta^2 - \xi^2\left(V^2-1\right)\right)}{P_+(\xi)}\cos(\xi Y)\,\mathrm{d}\xi, \tag{48}$$

$P_+(\xi)$ can be written

$$P_+(\xi) = \left(\theta^2 - V^2\right)\left(V^2-1\right)\xi Q(\xi) \tag{49}$$

with

$$Q(\xi) = \left((\xi-\alpha)^2 + \beta^2\right)(\xi-\gamma) \text{ case (a)} \quad \text{or}$$
$$Q(\xi) = (\xi-\alpha_1)(\xi-\alpha_2)(\xi-\gamma) \text{ case (b)}. \tag{50}$$

The study of the roots of $Q(\xi)$ for any value of V, $0<V<\theta$, confirms the following properties: $\alpha<0$; $\beta>0$; $\gamma>0$ or $\alpha_1<0$; $\alpha_2<0$; $\gamma>0$. For an aluminium plate coupled with water, the conditions of case (a) are always verified. For high values of μ, the two cases can occur if the velocity of loading is taken on a large interval; which is the case for steel plates.

Case (b) is much less difficult to study than (a). The further developments will correspond to choice (a) so that the conclusion would be generalised to any case.

The research of inverse transforms requires a long and delicate development which is detailed in appendix.

This work defines a series of coefficients and primary integrals which allow all of the fundamental functions of the response to be constructed, which finally take the form

$$\begin{aligned}W(Y) = \frac{F_0}{\pi}\Bigg\{&\frac{\gamma C_2}{\theta^2 - V^2}\left[I_{\gamma c}(Y) - I_{1\alpha\beta c}(Y) + C_6 I_{0\alpha\beta c}(Y)\right] \\ &- \frac{\theta^2}{\gamma\left(\theta^2 - V^2\right)\left(V^2-1\right)}\left[-C_1 I_{0c}(Y) + C_2 I_{\gamma c}(Y)\right. \\ &\left. + C_1C_2C_3I_{1\alpha\beta c}(Y) + C_1C_2C_4I_{0\alpha\beta c}(Y)\right]\Bigg\},\end{aligned} \tag{51}$$

$$\Psi(Y) = \frac{F_0}{\pi}\frac{\theta^2}{\left(\theta^2 - V^2\right)\left(V^2-1\right)}C_2\left[I_{\gamma s}(Y) - I_{1\alpha\beta s}(Y) + C_5 I_{0\alpha\beta s}(Y)\right], \tag{52}$$

$$\begin{aligned}\Phi(Y,0) = \frac{F_0}{\pi}\frac{V}{\Omega}\Bigg\{&\frac{\theta^2}{\gamma\left(\theta^2 - V^2\right)\left(V^2-1\right)} \\ &\left[-C_1 I_{0s}(Y) + C_2 I_{\gamma s}(Y) + C_1C_2C_3I_{1\alpha\beta s}(Y) + C_1C_2C_4I_{0\alpha\beta s}(Y)\right] \\ &- \frac{\gamma C_2}{\left(\theta^2 - V^2\right)}\left[I_{\gamma s}(Y) - I_{1\alpha\beta s}(Y) + C_6 I_{0\alpha\beta c}(Y)\right]\Bigg\}.\end{aligned} \tag{53}$$

The useful functions, $\mathrm{d}W/\mathrm{d}Y - \Psi$ and $\mathrm{d}\Psi/\mathrm{d}Y$, allow the stresses in the plate to be obtained, and $\mathrm{d}\Phi_{(Y,0)}/\mathrm{d}Y$ allow the pressure immediately under the plate to be determined. It can be deduced, either directly or from the derivatives of W, Ψ and Φ. The derivatives of primary integrals are contained in the appendix.

Observing the whole results in detail, it is verified that the continuity of the functions W, Ψ, ϕ and $\mathrm{d}\Psi/\mathrm{d}Y$ can be extended in $Y = 0$, which is physically realistic.

On the contrary, $\mathrm{d}W/\mathrm{d}Y$ is not continuous in $Y = 0$, like the shear force. The derivative $\mathrm{d}\Phi_{(z=0)}/\mathrm{d}Y$ is not finite in $Y = 0$, due to the fact that the slope of $W(Y)$ is not continuous in $Y = 0$. In fact, it takes the form of a very narrow sharp peak much less wide than the plate thickness. This is mathematically correct, but physically unacceptable. The discrepancy comes from the fact that a three-dimensional theory would be indispensable to describe the flexing of a beam in the immediate neighbourhood of the point of application of the force.

For large values of Y, ($Y = \pm\infty$), all the functions have a zero average value, except $\Phi(Y,0)$. In fact, any constant value can be added to Φ without consequences, as can be seen in the initial system, Eqs. (18)–(21).

If it is chosen that the average value of $\Phi(+\infty,0)$ is zero, then the average value of $\Phi(-\infty,0)$ will be $F_0/\mu V$. This value verified by the calculated function could be deduced directly from Eq. (18) of the initial system after integration between $Y = -\infty$ and $+\infty$.

All the functions calculated up to now form a particular solution to the forced system, Eqs. (18)–(21). It is clear that any solution which would satisfy the homogenous associated system could be added.

Since no singularities exist in the absence of loading, a solution of homogeneous equations must be valid over the interval $[-\infty, +\infty]$ and must present a single definition on this whole domain. Among all the mathematical possibilities, those of type $A\cos((\alpha + \mathrm{i}\beta)Y)$ are not convenient because they are not finite, either for $Y = +\infty$ or for $Y = -\infty$. Only one remains acceptable, of type $A\cos(\gamma Y)$. Such a solution is generally called "free wave".

A so called free wave is theoretically acceptable. Furthermore, the forced solution presented above is obviously incomplete because it appears exactly symmetrical with respect to negative or positive Y, without depending on the direction of the moving of the load, which is not physically realistic.

A previous work [8], which is similar in some aspects, but which studies plates alone, without coupled liquid, established that, in this range of velocity of loading, vibrations took place only ahead of the load front.

The research of free waves is based on the characteristic equation of the associated homogeneous system and on the compatibility of functions W, Ψ and Φ between each other. This analysis has been achieved; its development is quite long but leads without ambiguity to an acceptable simple form of free waves.

The free waves obtained are given below, and verify exactly the homogeneous system. Their amplitudes have been adjusted to cancel any harmonic vibrations behind the position of the loading force.

$$W_{\text{free}} = -F_0 \frac{C_2}{\gamma} \frac{\theta^2 + (1 - V^2)\gamma^2}{(\theta^2 - V^2)(1 - V^2)} \sin(\gamma Y), \tag{54}$$

$$\Psi_{\text{free}} = -F_0 C_2 \frac{\theta^2}{\left(\theta^2 - V^2\right)\left(1 - V^2\right)} \cos(\gamma Y), \tag{55}$$

$$\Phi_{\text{free}(Z=0)} = -F_0 \frac{V}{\Omega} \frac{C_2}{\gamma} \frac{\theta^2 + \left(1 - V^2\right)\gamma^2}{\left(\theta^2 - V^2\right)\left(1 - V^2\right)} \cos(\gamma Y). \tag{56}$$

The addition of these free waves to the forced solution completes the response of the system. The knowledge of $\Phi(Y,0)$ makes it possible to obtain $\Phi(Y,Z)$ for any negative value of Z. The method using a Green function is recalled in [9].

It has been established by Eq. (29) that

$$\overline{\Phi}(\xi, Z) = \overline{\Phi}(\xi, 0) \mathrm{e}^{|\xi|\Omega Z}. \tag{57}$$

The function $\mathrm{e}^{|\xi|\Omega Z}$ can be considered as the Fourier transform of a Green function $g(Y,Z)$, so that

$$\begin{aligned} g(Y,Z) &= \int_{-\infty}^{+\infty} \mathrm{e}^{|\xi|\Omega Z} \mathrm{e}^{-\mathrm{i}\xi Y}\, \mathrm{d}\xi = \int_{-\infty}^{0} \mathrm{e}^{-\xi(\Omega Z + \mathrm{i}Y)}\, \mathrm{d}\xi + \int_{-\infty}^{0} \mathrm{e}^{\xi(\Omega Z - \mathrm{i}Y)}\, \mathrm{d}\xi, (Z<0) \\ &= -\left.\frac{\mathrm{e}^{-\xi(\Omega Z + \mathrm{i}Y)}}{\Omega Z + \mathrm{i}Y}\right|_{-\infty}^{0} + \left.\frac{\mathrm{e}^{\xi(\Omega Z - \mathrm{i}Y)}}{\Omega Z - \mathrm{i}Y}\right|_{0}^{\infty}, \end{aligned} \tag{58}$$

finally,

$$g(Y,Z) = -\frac{2\Omega Z}{Y^2 + \Omega^2 Z^2}, \tag{59}$$

$\overline{\Phi}(\xi, 0)$ being the Fourier transform of $\Phi(Y,0)$, the convolution of functions gives the value of Φ in the whole half space $Z<0$.

$$\Phi(Y,Z) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \Phi\left(\tilde{Y}, 0\right) \frac{-2\Omega Z}{\left(Y - \tilde{Y}\right)^2 + \Omega^2 Z^2}\, \mathrm{d}\tilde{Y}. \tag{60}$$

The same method can be followed to obtain $\partial\Phi/\partial Y(Y,Z)$ from $\partial\Phi/\partial Y(Y,0)$.

Nevertheless, another choice avoids the calculation of any new integral, using the same form of development as the one presented above. The demonstration will be presented on the useful function $\partial\Phi/\partial Y$ only. By derivation of Eq. (57), one obtains

$$\frac{\partial\overline{\Phi}}{\partial Y}(\xi, Z) = -\mathrm{i}\xi\overline{\Phi}(\xi, 0)\mathrm{e}^{|\xi|\Omega Z}, \tag{61}$$

which gives by inverse transform

$$\frac{\partial\Phi}{\partial Y}(Y,Z) = \int_{-\infty}^{\infty} -\mathrm{i}\xi\overline{\Phi}(\xi, 0)\mathrm{e}^{|\xi|\Omega Z}\mathrm{e}^{-\mathrm{i}\xi Y}\, \mathrm{d}\xi. \tag{62}$$

Using previous developments

$$\frac{\partial\Phi}{\partial Y}(Y,Z) = \frac{F_0}{2\pi} \int_{-\infty}^{+\infty} \frac{V}{\Omega} \frac{\mathrm{i}\xi(-\mathrm{i}\,\mathrm{sgn}(\xi))\left(\xi^2\left(V^2 - 1\right) - \theta^2\right)}{P(\xi)} \mathrm{e}^{|\xi|\Omega Z}\mathrm{e}^{-\mathrm{i}\xi Y}\, \mathrm{d}\xi \tag{63}$$

$$= \frac{F_0}{2\pi}\frac{V}{\Omega}\int_{-\infty}^{0}\frac{-\xi\left(\xi^2\left(V^2-1\right)-\theta^2\right)}{P_-(\xi)}\mathrm{e}^{-\xi\Omega Z}\mathrm{e}^{-\mathrm{i}\xi Y}\,\mathrm{d}\xi + \frac{F_0}{2\pi}\frac{V}{\Omega}\int_{0}^{\infty}\frac{\xi\left(\xi^2\left(V^2-1\right)-\theta^2\right)}{P_+(\xi)}\mathrm{e}^{\xi\Omega Z}\mathrm{e}^{-\mathrm{i}\xi Y}\,\mathrm{d}\xi, \tag{64}$$

finally

$$\frac{\partial\Phi}{\partial Y}(Y,Z) = \frac{F_0}{\pi}\frac{V}{\Omega}\int_{0}^{\infty}\frac{\xi\left(\xi^2\left(V^2-1\right)-\theta^2\right)}{P_+(\xi)}\mathrm{e}^{\xi\Omega Z}\cos(\xi Y)\,\mathrm{d}\xi. \tag{65}$$

Using the transformation of $\mathrm{e}^{\xi\Omega Z}$ in complex expression, one obtains

$$\mathrm{e}^{\xi\Omega Z}\cos(\xi Y) = -\frac{\mathrm{i}}{2}[\sin((Y+\mathrm{i}\Omega Z)\xi)-\sin((Y-\mathrm{i}\Omega Z)\xi)] + \frac{1}{2}[\cos((Y+\mathrm{i}\Omega Z)\xi)+\cos((Y-\mathrm{i}\Omega Z)\xi)]. \tag{66}$$

Reintroducing this value in Eq. (65), $\partial\Phi/\partial Y(Y,Z)$ can be obtained by the way of Fourier Sine and Cosine transform, as previously. All the necessary integrals have been already presented explicitly; they must be rearranged only for the use of complex variables.

Finally, the entire theoretical results can be summarised, including the free wave in every function.

The results are valid for any Y, $Y\in[-\infty,+\infty]$

$$W(Y) = \frac{F_0}{\pi}\frac{\gamma C_2}{\theta^2-V^2}\left[I_{\gamma c}(|Y|)-I_{1\alpha\beta c}(|Y|)+C_6 I_{0\alpha\beta c}(|Y|)\right] + \frac{F_0}{\pi}\frac{\theta^2}{\gamma\left(\theta^2-V^2\right)\left(1-V^2\right)}[-C_1 I_{0c}(|Y|)+C_2 I_{\gamma c}(|Y|)+C_1C_2C_3I_{1\alpha\beta c}(|Y|) + C_1C_2C_4I_{0\alpha\beta c}(|Y|) - F_0\frac{\theta^2+\left(1-V^2\right)\gamma^2}{\gamma\left(\theta^2-V^2\right)\left(1-V^2\right)}C_2\sin(\gamma Y), \tag{67}$$

$$\Psi(Y) = -\frac{F_0}{\pi}\frac{\theta^2\,\mathrm{sgn}(Y)}{\left(\theta^2-V^2\right)\left(1-V^2\right)}C_2\left[I_{\gamma s}(|Y|)-I_{1\alpha\beta s}(|Y|)+C_5I_{0\alpha\beta s}(|Y|)\right] - F_0\frac{\theta^2}{\left(\theta^2-V^2\right)\left(1-V^2\right)}C_2\cos(\gamma Y), \tag{68}$$

$$\Phi(Y,0) = -\frac{F_0}{\pi}\frac{V}{\Omega}\frac{\theta^2\,\mathrm{sgn}(Y)}{\gamma\left(\theta^2-V^2\right)\left(1-V^2\right)}\left[-C_1I_{0s}(|Y|)+C_2I_{\gamma s}(|Y|)+C_1C_2C_3I_{1\alpha\beta s}(|Y|) + C_1C_2C_4I_{0\alpha\beta s}(|Y|) - \frac{F_0}{\pi}\frac{V}{\Omega}\frac{\gamma\,\mathrm{sgn}(Y)}{\left(\theta^2-V^2\right)}C_2\left[I_{\gamma s}(|Y|)-I_{1\alpha\beta s}(|Y|)+C_6I_{0\alpha\beta s}(|Y|)\right] - F_0\frac{V}{\Omega}\frac{\theta^2+\left(1-V^2\right)\gamma^2}{\gamma\left(\theta^2-V^2\right)\left(1-V^2\right)}C_2\cos(\gamma Y) - \frac{F_0}{2}\frac{V}{\Omega}\frac{\theta^2}{\gamma\left(\theta^2-V^2\right)\left(1-V^2\right)}C_1. \tag{69}$$

The last constant value added is adjusted to make the average value of $\Phi(+\infty,0)$ equate to zero.

All these functions are displayed below, Fig. 2. All the graphs are drawn with the following choice of parameters :

$\delta = 0.28$; $\theta = 0.55$; $\mu = 0.1$; this first set of parameters corresponds to any aluminium plate coupled with water.

$V = 0.25$; this choice corresponds to a load velocity of approximately 1360 m/s as a dimensional value (for an aluminium plate).

$F_0 = -1$; this choice corresponds to an external unit non-dimensional force pushing the plate against the liquid.

Which results in the following:

$\alpha = -0.167458466453$; $\beta = 0.139521933258$; $\gamma = 0.39274361103$; $\Omega = 0.450340007604$.

Using relationships between the coefficients and roots of polynomials, some simplification can take place and some remarkable values can be highlighted.

Particular values of the functions in some points are noted.

$$\Psi(0) = \frac{F_0\theta^2}{\left(\theta^2 - V^2\right)\left(V^2 - 1\right)} C_2, \tag{70}$$

$$\Phi(0,0) = -F_0 \frac{V}{\Omega} \frac{2\left(\theta^2 + \left(1 - V^2\right)\gamma^2\right) C_2 + \theta^2 C_1}{2\gamma\left(\theta^2 - V^2\right)\left(1 - V^2\right)}, \tag{71}$$

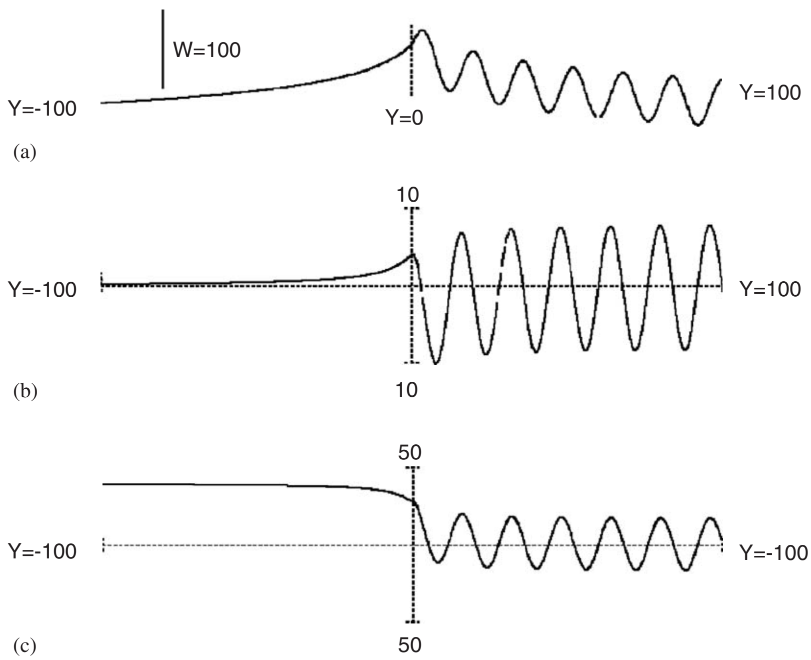

Fig. 2. Theoretical stationary response of the coupled system: (a) transverse displacement $W(Y)$, (b) cross-section angular rotation $\Psi(Y)$ and (c) pressure potential $\Phi(Y,0)$ at the interface.

$$\Phi(-\infty, Z) = -\frac{F_0}{\gamma V}. \tag{72}$$

The necessary functions to deduce stresses and pressure take the form

$$\begin{aligned}\frac{\mathrm{d}W}{\mathrm{d}Y} - \Psi = & -\frac{F_0}{\pi}\frac{\gamma^2 \operatorname{sgn}(Y)}{\theta^2 - V^2} C_2\left[I_{\gamma s}(|Y|) + C_7 I_{1\alpha\beta s}(|Y|) + C_8 I_{0\alpha\beta s}(|Y|)\right] \\ & - F_0 \frac{\gamma^2}{\theta^2 - V^2} C_2 \cos(\gamma Y),\end{aligned} \tag{73}$$

$$\begin{aligned}\frac{\mathrm{d}\Psi}{\mathrm{d}Y} = & -\frac{F_0}{\pi}\frac{\theta^2\gamma}{(\theta^2 - V^2)(1 - V^2)} C_2\left[I_{\gamma c}(|Y|) - I_{1\alpha\beta c}(|Y|) + C_6 I_{0\alpha\beta c}(|Y|)\right] \\ & + F_0 \frac{\theta^2\gamma}{(\theta^2 - V^2)(1 - V^2)} C_2 \sin(\gamma Y),\end{aligned} \tag{74}$$

$$\begin{aligned}\left(\frac{\partial\Phi}{\partial Y}\right)_{z=0} = & -\frac{F_0}{\pi}\frac{V}{\Omega}\frac{\theta^2}{(\theta^2 - V^2)(1 - V^2)} C_2\left[I_{\gamma c}(|Y|) - I_{1\alpha\beta c}(|Y|) + C_5 I_{0\alpha\beta c}(|Y|)\right] \\ & -\frac{F_0}{\pi}\frac{V}{\Omega}\frac{\gamma^2}{(\theta^2 - V^2)} C_2\left[I_{\gamma c}(|Y|) + C_7 I_{1\alpha\beta c}(|Y|) + C_8 I_{0\alpha\beta c}(|Y|)\right] \\ & + F_0 \frac{V}{\Omega}\frac{\theta^2 + (1 - V^2)\gamma^2}{(\theta^2 - V^2)(1 - V^2)} C_2 \sin(\gamma Y)\end{aligned} \tag{75}$$

with the particular values

$$\left[\frac{\mathrm{d}W}{\mathrm{d}Y} - \Psi\right]_{0_-} = -\frac{F_0}{2(\theta^2 - V^2)}\left(2\gamma^2 C_2 - 1\right), \tag{76}$$

$$\left[\frac{\mathrm{d}W}{\mathrm{d}Y} - \Psi\right]_{0_+} = -\frac{-F_0}{2(\theta^2 - V^2)}\left(2\gamma^2 C_2 + 1\right), \tag{77}$$

which gives the discontinuity $-F_0/(\theta^2 - V^2)$ in $Y = 0$

$$\left(\frac{\mathrm{d}\Psi}{\mathrm{d}Y}\right)_0 = -\frac{F_0}{\pi}\frac{\theta^2\gamma C_2}{(\theta^2 - V^2)(1 - V^2)}\left(\ln\frac{\sqrt{\alpha^2 + \beta^2}}{\gamma} - C_6 \arctan\frac{\beta}{\alpha}\right). \tag{78}$$

Eqs. (73), (74) and (7) enable stresses to be obtained. They are displayed in Fig. 3.

The pressure in the liquid is deduced from $(\partial\Phi/\partial Y)_{z=0}$. It is displayed in Fig. 4.

Harmonic parts of the responses are of interest because their amplitudes are significant with regard to the whole response, especially so for stresses and pressure.

$$W_\sim = -F_0 \frac{\theta^2 + (1 - V^2)\gamma^2}{\gamma(\theta^2 - V^2)(1 - V^2)} 2C_2 \sin(\gamma Y) = -F_0 A_W \sin(\gamma Y), \tag{79}$$

$$\Psi_{\sim} = -F_0 \frac{\theta^2}{\left(\theta^2 - V^2\right)\left(1 - V^2\right)} 2C_2 \cos(\gamma Y) = -F_0 A_{\Psi} \cos(\gamma Y), \tag{80}$$

$$\Phi_{\sim} = -F_0 \frac{V}{\Omega} \frac{\theta^2 + \left(1 - V^2\right)\gamma^2}{\gamma\left(\theta^2 - V^2\right)\left(1 - V^2\right)} 2C_2 \cos(\gamma Y) = -F_0 A_{\Phi} \cos(\gamma Y), \tag{81}$$

$$\begin{aligned} \Gamma_{\sim} &= \left(\frac{\theta^2}{\sqrt{12}}\right)\left(\frac{\mathrm{d}W}{\mathrm{d}Y} - \Psi\right)_{\sim} = -F_0\left(\frac{\theta^2}{\sqrt{12}}\right)\frac{\gamma^2}{\theta^2 - V^2} 2C_2 \cos(\gamma Y) \\ &= -F_0 A_{\Gamma} \cos(\gamma Y), \end{aligned} \tag{82}$$

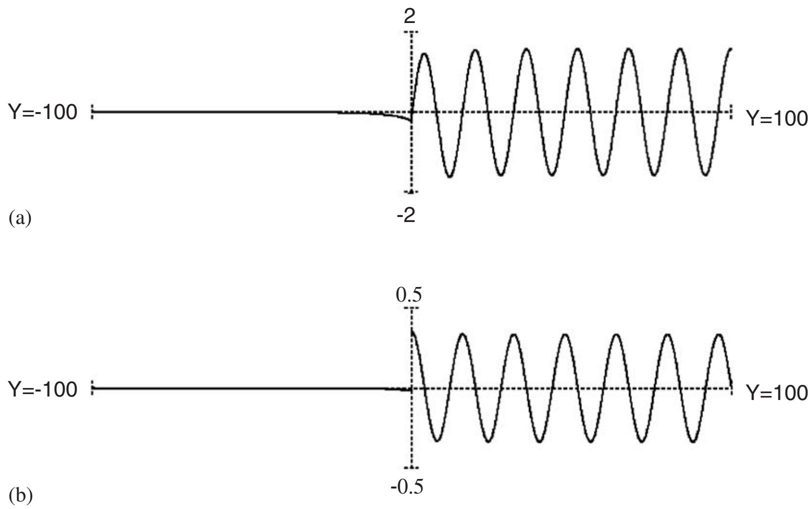

Fig. 3. Theoretical stationary stresses in the plate. (a) flexural stress $\Sigma(Y)$ on the upper face and (b) average shear stress $\Gamma(Y)$.

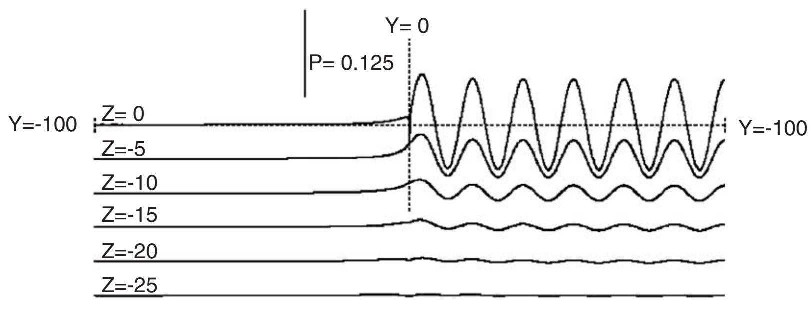

Fig. 4. Pressure in the liquid and its attenuation with depth.

$$\begin{aligned}\Sigma_\sim &= -\tfrac{1}{2}\left(\frac{\mathrm{d}\Psi}{\mathrm{d}Y}\right)_\sim = -F_0\frac{1}{2}\frac{\gamma\theta^2}{\left(\theta^2-V^2\right)\left(1-V^2\right)}2C_2\sin(\gamma Y)\\ &= -F_0A_\Sigma\sin(\gamma Y),\end{aligned} \tag{83}$$

$$\begin{aligned}P_\sim &= -\mu V\left(\frac{\partial\Phi}{\partial Y}\right)_\sim(Y,Z) = -F_0\mu\frac{V^2}{\Omega}\frac{\theta^2+\left(1-V^2\right)\gamma^2}{\left(\theta^2-V^2\right)\left(1-V^2\right)}2C_2\sin(\gamma Y)\mathrm{e}^{\gamma\Omega Z}\\ &= -F_0A_P\sin(\gamma Y)\mathrm{e}^{\gamma\Omega Z}.\end{aligned} \tag{84}$$

All these harmonic functions are characterised by their amplification factors. They all depend on γ and C_2. The latter, according to Eqs. (A.6) and (49), (50) can be expressed in an alternative form

$$C_2 = \left(\theta^2-V^2\right)\left(V^2-1\right)/P'_+(\gamma), \tag{85}$$

which uses root γ only and which is convenient also in case (b) of Eq. (50).

The last function, Eq. (84) gives the oscillation of pressure and also characterises the attenuation of the pressure with the depth.

The whole of the previous analysis corresponds to a unit force, but any constant profile of loading can be taken into account by convolution. The particular case of a constant pressure step is among the simplest, and the corresponding response is exactly the integral of that of a Dirac loading used to represent a single force. In that case, it is particularly easy to obtain the new harmonic parts by integration and to deduce the new amplification factors.

In conclusion to this theoretical analysis, useful curves are presented, which permit the responses of a whole series of plates coupled with liquid to be obtained, for any subsonic loading velocity.

Only three non-dimensional parameters accounting for the mechanical properties of the plate and of the liquid are useful: θ, δ and μ. From these, the roots α, β and γ of the characteristic polynomial are deduced and presented Fig 5. They enable the frequency and attenuation of the responses to be found.

The response always contains a harmonic vibration which is present ahead of the force. Its amplitude is characterised by amplification factors. Fig. 6 gives amplification factors of displacement and rotation while Fig. 7 gives those of stresses and pressure.

4. Comparison of stationary and transient responses

The previously cited study [1] established that, in the supersonic range, i.e. for any loading velocity greater than the velocity of acoustic waves in the liquid, the solution in the neighbourhood of the moving front converged to the stationary solution. The comparison was achieved using the theoretical solution for the stationary analysis and the result of a computational method for the transient case.

The same numerical method has been used to calculate the response of a coupled system in the subsonic range. The method uses an explicit scheme of integration in time associated with a finite difference method. It has already been used and improved for shell, plate and beam theories [12] and revealed itself to be very efficient, precise and fast for application to moderately thick

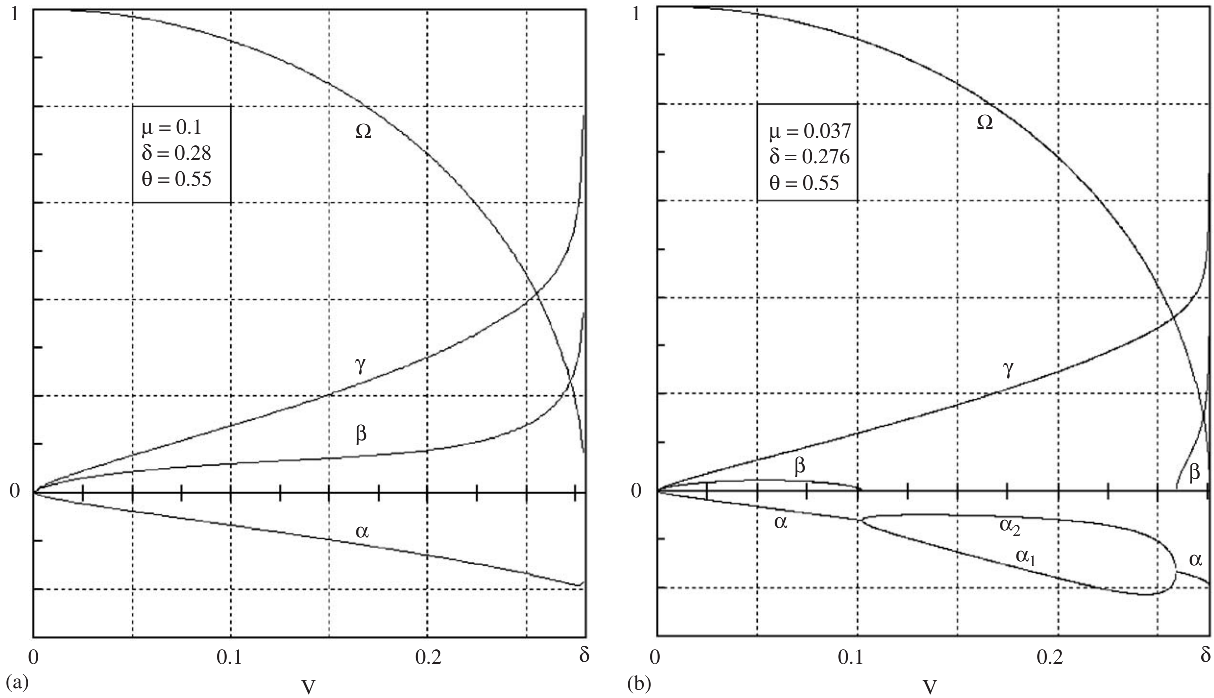

Fig. 5. Evolution of the principal parameters with the load velocity: (a) convenient for aluminium–water coupling and (b) convenient for steel–water coupling.

structures. It is appropriate to find high frequencies, possibly in the range 10^4, $10^5\,s^{-1}$, as confirmed by experiments.

If a theoretical stationary response can be set for an infinite domain, a numerical solution generally needs a finite domain of calculation, which requires boundary conditions. For this purpose, the plate–liquid domain has been chosen to be very large (600 long, 300 deep). A unit force has been supposed, travelling from left to right at a constant velocity ($V = 0.25$). The response has been observed when the force has reached the centre of the plate. At this moment, it has been possible to separate the part of the response depending on the boundary conditions from another part which follows the loading and remains unchanged during translation. Thus, the numerical solution applied over a large domain is able to contain the stationary response to a moving load.

To compare the two stationary solutions, the theoretical curves have been superimposed on the calculated values, in a neighbourhood of force, ranging from $Y = -100$ to $+100$. On the left of this domain, the calculated response contains the effect of boundaries; on the right of the domain, it reveals the progressive construction of the stationary response. Fig. 8 presents the superposition of theoretical and computed curves for the principal functions, W, Ψ, $\Phi_{Z=0}$. The proximity of the transient and stationary responses is evident. For Ψ and Φ, the superposition of curves takes place very quickly. For W, the effect of boundary conditions in the transient response delays the coincidence. Nevertheless, for longer travel times, W itself becomes closer and closer to the stationary solution.

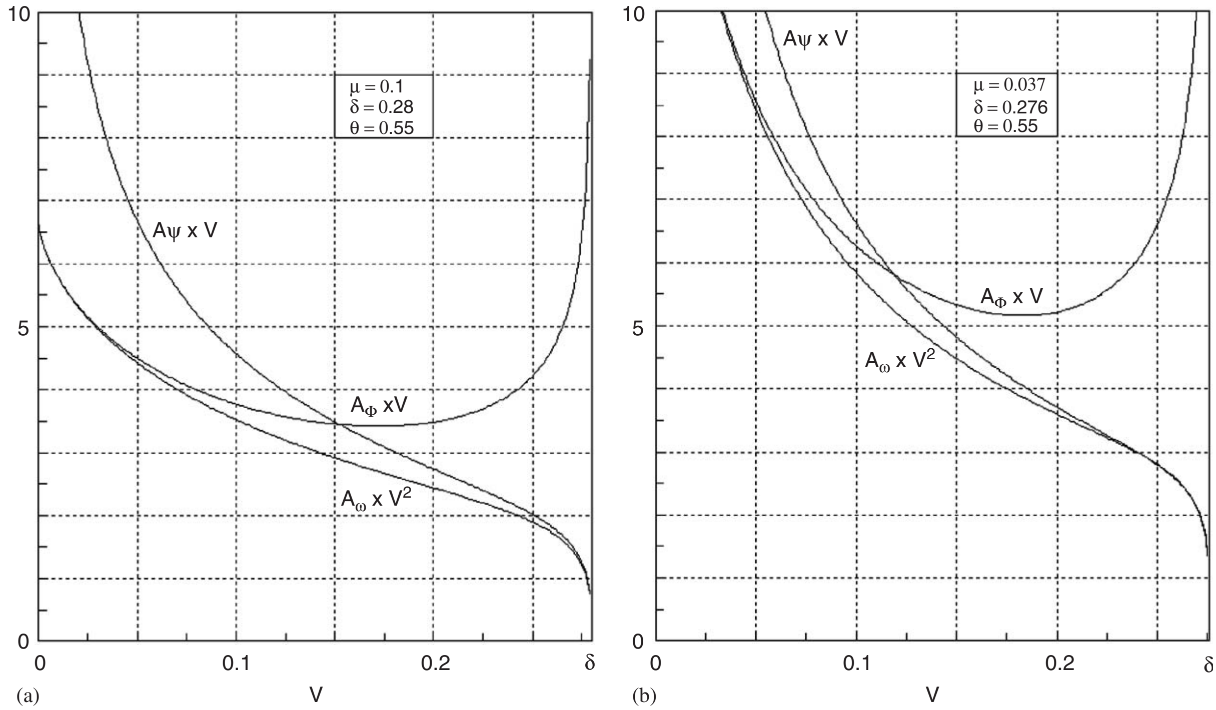

Fig. 6. Evolution of the amplitude of harmonic parts of the response with the load velocity: (a) convenient for aluminium–water coupling and (b) convenient for steel–water coupling.

The comparison has also been observed on stresses in the plate. The Fig. 9 presents the proximity of curves obtained for Σ (flexural stress on the upper face of the plate) and Γ. (average shear stress). The superposition of the curves is almost perfect, down to the smallest detail.

For displacements W and Ψ, the stationary values can be insignificant with respect to values attained in the transient part of the response. On the contrary, for stresses, the values obtained in the stationary responses can be very significant with respect to those reached in the transient part. This result highlights the importance of the stationary solution.

The pressure in the liquid, resulting directly from $\Phi_{Z=0}$, is also the same, in the neighbourhood of the force, for the calculated or theoretical solutions.

5. Conclusion

The purpose of this work was to research the analytical response of a plate coupled with a liquid, for loading travelling at high and constant velocity, but ranging in the subsonic case. It completes the study in the more simple supersonic case, already obtained. The analytical solution is presented for a unit force but the case of any constant profile of loading can be deduced by

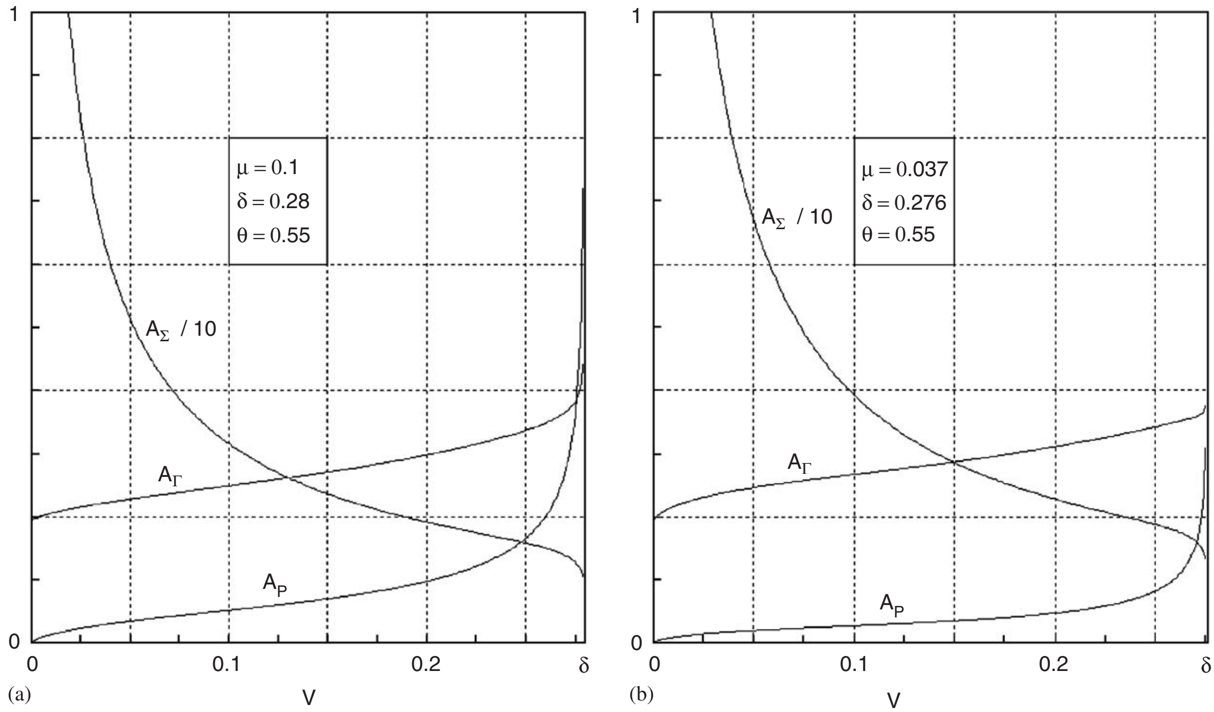

Fig. 7. Evolution of the amplitude of harmonic parts of stresses and pressure with the load velocity: (a) convenient for aluminium–water coupling and (b) convenient for steel–water coupling.

convolution. The analysis of the problem was carried out with non-dimensional functions, so that a minimum number of parameters was necessary. The comparison with the transient solution obtained computationally, for the same load case, shows that the transient response converges towards the stationary response in the vicinity of the load. The two responses quickly become closer and closer on an interval which grows around the position of the load as the latter travels along the plate.

The stresses in the plate and the pressure in the liquid are described very precisely by the stationary theory. Only their harmonic parts are significant and they only occur ahead of the load. For this reason, the theoretical analytical response gives a good estimation of the stresses in the plate, even in real cases of transient loading.

The study of the case of subsonic loading reveals that the pressure vanishes with the depth. This is a significant difference with respect to the supersonic case because the perturbation of pressure in the liquid is localised in the neighbourhood of the plate.

To end with a practical aspect, the paper includes the presentation of the variation of useful parameters and factors with the velocity of loading, which makes it possible to deduce the frequency and amplitude of harmonic waves present in coupled systems. The results are directly applicable to any aluminium or steel plate, whatever their thickness, for subsonic loading velocity (dimensional velocity lower than 1500 m/s).

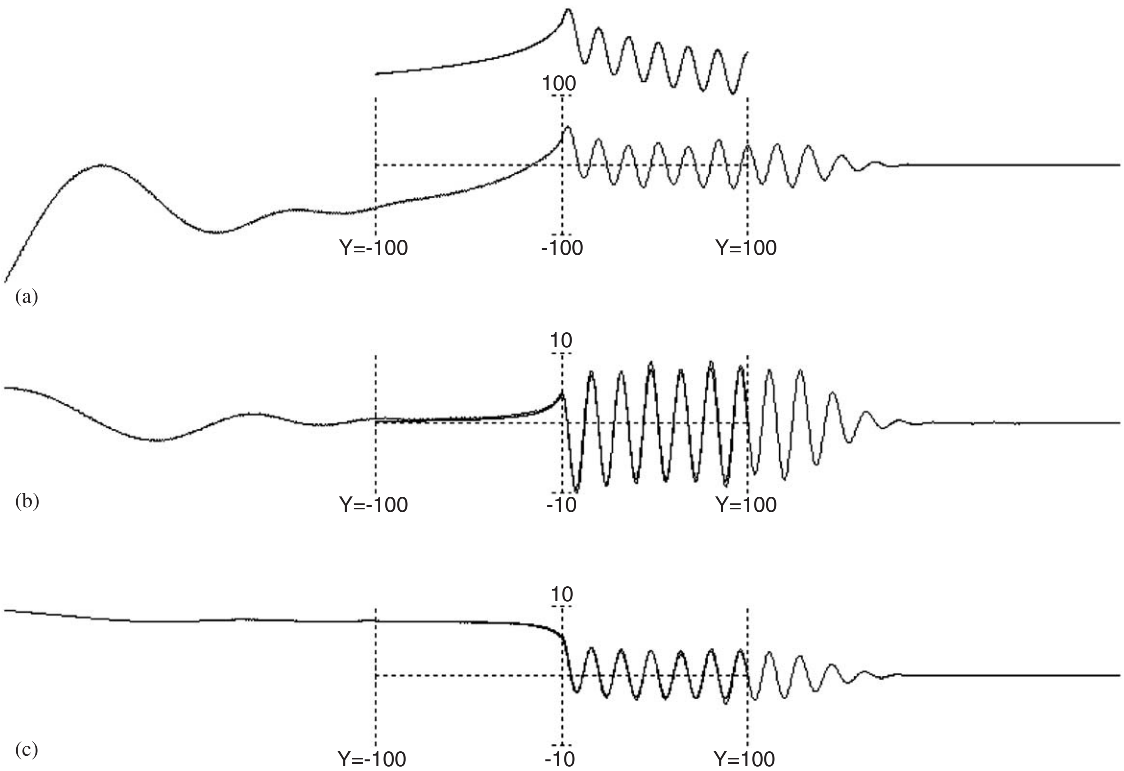

Fig. 8. Comparison between the transient and stationary responses: (a) transverse displacement W, (b) cross-sectional angular rotation Ψ and (c) potential $\Phi_{z=0}$ in the liquid at the interface.

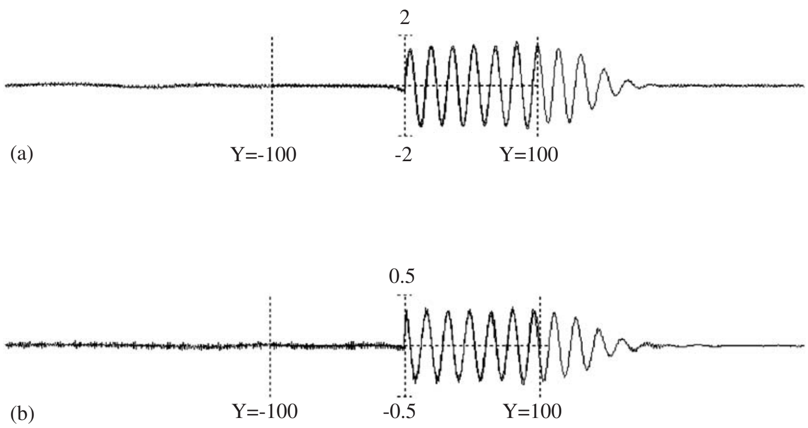

Fig. 9. Comparison between the transient and stationary stresses in the plate: (a) flexural stress on the upper face and (b) average shear stress.

Appendix

To achieve the inverse transforms, the previous polynomial ratios resulting from Eq. (50) have to be converted into partial-fraction expansion

$$\frac{1}{\xi(\xi-\gamma)\left((\xi-\alpha)^2+\beta^2\right)}=\frac{1}{\gamma}\left[-\frac{C_1}{\xi}+\frac{C_2}{\xi-\gamma}+C_1C_2C_3\frac{\xi-\gamma}{(\xi-\alpha)^2+\beta^2}+C_1C_2C_4\frac{\beta}{(\xi-\alpha)^2+\beta^2}\right], \tag{A.1}$$

$$\frac{1}{(\xi-\gamma)\left((\xi-\alpha)^2+\beta^2\right)}=C_2\left[\frac{1}{\xi-\gamma}-\frac{\xi-\alpha}{(\xi-\alpha)^2+\beta^2}+C_5\frac{\beta}{(\xi-\alpha)^2+\beta^2}\right], \tag{A.2}$$

$$\frac{\xi}{(\xi-\gamma)\left((\xi-\alpha)^2+\beta^2\right)}=C_2\gamma\left[\frac{1}{\xi-\gamma}-\frac{\xi-\alpha}{(\xi-\alpha)^2+\beta^2}+C_6\frac{\beta}{(\xi-\alpha)^2+\beta^2}\right], \tag{A.3}$$

$$\frac{\xi^2}{(\xi-\gamma)\left((\xi-\alpha)^2+\beta^2\right)}=C_2\gamma^2\left[\frac{1}{\xi-\gamma}+C_7\frac{\xi-\alpha}{(\xi-\alpha)^2+\beta^2}+C_8\frac{\beta}{(\xi-\alpha)^2+\beta^2}\right] \tag{A.4}$$

with the C_i coefficients

$$C_1=\frac{1}{\alpha^2+\beta^2},\quad C_2=\frac{1}{(\gamma-\alpha)^2+\beta^2},\quad C_3=\gamma(\gamma-2\alpha)$$

$$C_4=\frac{\gamma}{\beta}\left(\alpha^2-\beta^2-\alpha\gamma\right),\quad C_5=\frac{\alpha-\gamma}{\beta},\quad C_6=\frac{\alpha^2+\beta^2-\gamma\alpha}{\beta\gamma},$$

$$C_7=\frac{\alpha^2+\beta^2-2\alpha\gamma}{\gamma^2},\quad C_8=\frac{\gamma\left(\beta^2-\alpha^2\right)+\alpha\left(\alpha^2+\beta^2\right)}{\gamma^2\beta}. \tag{A.5)−(A.12}$$

Thus, seeking inverse transforms defined by Eqs (44)–(48) involves 8 integrals only

$$I_{0c}(Y)=\int_o^\infty\frac{\cos(\xi Y)}{\xi}\,\mathrm{d}\xi, \tag{A.13}$$

$$I_{0s}(Y)=\int_o^\infty\frac{\sin(\xi Y)}{\xi}\,\mathrm{d}\xi, \tag{A.14}$$

$$I_{\gamma c}(Y)=\int_o^\infty\frac{\cos(\xi Y)}{\xi-\gamma}\,\mathrm{d}\xi,\qquad \gamma\geqslant 0, \tag{A.15}$$

$$I_{\gamma s}(Y)=\int_o^\infty\frac{\sin(\xi Y)}{\xi-\gamma}\,\mathrm{d}\xi, \tag{A.16}$$

$$I_{0\alpha\beta c}(Y)=\int_0^\infty\frac{\beta\cos(\xi Y)}{(\xi-\alpha)^2+\beta^2}\,\mathrm{d}\xi,\quad \alpha<0;\ \ \beta>0, \tag{A.17}$$

$$I_{0\alpha\beta s}(Y) = \int_o^\infty \frac{\beta \sin(\xi Y)}{(\xi - \alpha)^2 + \beta^2}\,\mathrm{d}\xi, \tag{A.18}$$

$$I_{1\alpha\beta c}(Y) = \int_0^\infty \frac{(\xi - \alpha)\cos(\xi Y)}{(\xi - \alpha)^2 + \beta^2}\,\mathrm{d}\xi, \tag{A.19}$$

$$I_{1\alpha\beta s}(Y) = \int_0^\infty \frac{(\xi - \alpha)\sin(\xi Y)}{(\xi - \alpha)^2 + \beta^2}\,\mathrm{d}\xi. \tag{A.20}$$

These integrals could be named, in this context, primary functions. They are calculated for $Y > 0$ only. The parity consideration enables the values of functions to be extended to negative Y. The values in $Y = 0$ are not always defined. The load itself is also singular in $Y = 0$.

The expression of previous integrals requires using secondary functions named Sine Integral and Cosine Integral, usually found according to the form:

$$\mathrm{Si}(r) = \int_0^r \frac{\sin\, t}{t}\,\mathrm{d}t, \qquad \mathrm{Ci}(r) = \Gamma_E + \ln(r) + \int_0^r \frac{\cos t - 1}{t}\,\mathrm{d}t, \tag{A.21,A.22}$$

where r can be complex and where Γ_E is the Euler Constant ($\Gamma_E = 0.5772156649$).

Precision about Sine and Cosine Integrals could be found in Ref. [11].

The following expressions are finally deduced (for $Y > 0$):

$$I_{0s}(Y) = \frac{\pi}{2}, \tag{A.23}$$

$$I_{\gamma c}(Y) = -\sin(\gamma Y)\mathrm{Si}(\gamma Y) - \cos(\gamma Y)\mathrm{Ci}(\gamma Y) - \frac{\pi}{2}\sin(\gamma Y), \tag{A.24}$$

$$I_{\gamma s}(Y) = \cos(\gamma Y)\mathrm{Si}(\gamma Y) - \sin(\gamma Y)\mathrm{Ci}(\gamma Y) + \frac{\pi}{2}\cos(\gamma Y), \tag{A.25}$$

$$\begin{aligned} I_{0\alpha\beta c}(Y) = {} & \mathrm{i}\frac{\pi}{4}(\sin((\alpha + \mathrm{i}\beta)Y) - \sin((\alpha - \mathrm{i}\beta)Y)) \\ & + \frac{\mathrm{i}}{2}[\mathrm{Ci}((-\alpha - \mathrm{i}\beta)Y)\cos((\alpha + \mathrm{i}\beta)Y) - \mathrm{Si}((-\alpha - \mathrm{i}\beta)Y)\sin((\alpha + \mathrm{i}\beta)Y)] \\ & + \frac{\mathrm{i}}{2}[\mathrm{Si}((-\alpha - \mathrm{i}\beta)Y)\sin((\alpha - \mathrm{i}\beta)Y) - \mathrm{Ci}((-\alpha + \mathrm{i}\beta)Y)\cos((\alpha - \mathrm{i}\beta)Y)], \end{aligned} \tag{A.26}$$

$$\begin{aligned} I_{0\alpha\beta s}(Y) = {} & -\frac{\mathrm{i}\pi}{4}[\cos((\alpha + \mathrm{i}\beta)Y) - \cos((\alpha - \mathrm{i}\beta)Y)] \\ & + \frac{\mathrm{i}}{2}[\mathrm{Si}((-\alpha - \mathrm{i}\beta)Y)\cos((\alpha + \mathrm{i}\beta)Y) - \mathrm{Si}((-\alpha - \mathrm{i}\beta)Y)\cos((\alpha - \mathrm{i}\beta)Y)] \\ & + \frac{i}{2}[\mathrm{Ci}((-\alpha - \mathrm{i}\beta)Y)\sin((\alpha + \mathrm{i}\beta)Y) - \mathrm{Ci}((-\alpha + \mathrm{i}\beta)Y)\sin((\alpha - \mathrm{i}\beta)Y)], \end{aligned} \tag{A.27}$$

$$I_{1\alpha\beta c}(Y) = -\frac{\pi}{4}[\sin((\alpha+i\beta)Y)+\sin((\alpha-i\beta)Y)] + \frac{1}{2}[\mathrm{Si}((-\alpha-i\beta)Y)\sin((\alpha+i\beta)Y)+\mathrm{Si}((-\alpha+i\beta)Y)\sin((\alpha-i\beta)Y)] - \frac{1}{2}[\mathrm{Ci}((-\alpha-i\beta)Y)\cos((\alpha+i\beta)Y)+\mathrm{Ci}((-\alpha+i\beta)Y)\cos((\alpha-i\beta)Y)], \quad \text{(A.28)}$$

$$\mathrm{I}_{1\alpha\beta s}(Y) = \frac{\pi}{4}[\cos((\alpha+i\beta)Y)+\cos((\alpha-i\beta)Y)] - \frac{1}{2}[\mathrm{Si}((-\alpha-i\beta)Y)\cos((\alpha+i\beta)Y)+\mathrm{Si}((-\alpha+i\beta)Y)\cos((\alpha-i\beta)Y)] - \frac{1}{2}[\mathrm{Ci}((-\alpha-i\beta)Y)\sin((\alpha+i\beta)Y)+\mathrm{Ci}((-\alpha+i\beta)Y)\sin((\alpha-i\beta)Y)]. \quad \text{(A.29)}$$

Accounting for the properties of Ci(x) and Si(x)

$$\mathrm{Ci}(x)\sim\Gamma_E+\ln(x) \text{ for } x\sim 0, \quad \lim_{x\to+\infty}\mathrm{Ci}(x)=0, \quad \text{(A.30, A.31)}$$

$$\mathrm{Si}(0)=0, \quad \lim_{x\to+\infty}\mathrm{Si}(x)=\frac{\pi}{2}. \quad \text{(A.32, A.33)}$$

One deduces

$$\lim_{Y\to 0} I_{0s}(Y)=\frac{\pi}{2}, \quad \lim_{Y\to+\infty} I_{0s}(Y)=\frac{\pi}{2}, \quad \text{(A.34, A.35)}$$

$$\lim_{Y\to 0} I_{\gamma s}(Y)=\frac{\pi}{2}, \quad I_{\gamma s}(Y)\sim\pi\cos(\gamma Y) \text{ for } Y\to+\infty, \quad \text{(A.36, A.37)}$$

$$I_{\gamma c}(Y)\sim-\Gamma_E-\ln(\gamma)-\ln(Y) \text{ for } Y\sim 0, \quad \text{(A.38)}$$

$$I_{\gamma c}(Y)\sim-\pi\sin(\gamma Y) \text{ for } Y\to+\infty, \quad \text{(A.39)}$$

$$\lim_{Y\to 0} I_{0\alpha\beta c}(Y)=-\arctan\frac{\beta}{\alpha}, \quad \lim_{Y\to+\infty} I_{0\alpha\beta c}(Y)=0, \quad \text{(A.40, A.41)}$$

$$\lim_{Y\to 0} I_{0\alpha\beta s}(Y)=0, \quad \lim_{Y\to+\infty} I_{0\alpha\beta s}(Y)=0, \quad \text{(A.42, A.43)}$$

$$I_{1\alpha\beta c}(Y)\sim-\Gamma_E-\frac{1}{2}\ln\left(\alpha^2+\beta^2\right)-\ln(Y) \text{ for } Y\sim 0, \quad \lim_{Y\to+\infty} I_{1\alpha\beta c}(Y)=0, \quad \text{(A.44, A.45)}$$

$$\lim_{Y\to 0} I_{1\alpha\beta s}(Y)=\frac{\pi}{2}, \quad \lim_{y\to+\infty} I_{1\alpha\beta s}(Y)=0. \quad \text{(A46, A47)}$$

Useful relations between the functions and their derivatives:

$$\begin{aligned} &I'_{0s} = 0, && I'_{0c} = -\tfrac{1}{Y}, \\ &I'_{\gamma s} = \gamma I_{\gamma c}, && I'_{\gamma c} = -\gamma I_{\gamma s} - \tfrac{1}{Y}, \\ &I'_{1\alpha\beta s} = \alpha I_{1\alpha\beta c} - \beta I_{0\alpha\beta c}, && I'_{1\alpha\beta c} = -\alpha I_{1\alpha\beta s} + \beta I_{0\alpha\beta s} - \tfrac{1}{Y}, \\ &I'_{0\alpha\beta s} = \beta I_{1\alpha\beta c} + \alpha I_{0\alpha\beta c}, && I'_{0\alpha\beta c} = -\beta I_{1\alpha\beta s} - \alpha I_{0\alpha\beta s}. \end{aligned} \qquad (A.48)-(A.55)$$

Comments on some singularities:

The function $I_{0c}(Y)$ occurs only in the development of $W(Y)$. It must be considered attentively. In fact, it is undefined; this property does not contradict the function $W(Y)$ itself which appears only by its derivative in the initial system, (Eqs. (18)–(20)).

Conversely, $\mathrm{d}W/\mathrm{d}Y$ is well defined and $W(Y)$ appears as its undefined integral.

Literally, studying the undefined integral as the limit of a defined integral, it is possible to write $I_{0c}(Y)$ as the sum of a principal value and an undetermined constant.

$$I_{0c}(Y) = -\Gamma_E - \ln(Y) \; + \; \text{undetermined constant}. \qquad (A.56)$$

By convenience, the value $-\Gamma_E - \ln(Y)$ will be used in this context for $I_{0c}(Y)$, ignoring the non-determination.

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