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Short Communication

Theoretical natural frequencies and mode shapes for thin and thick curved pipes and toroidal shells

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Abstract

The finite element and differential quadrature methods are used to determine the natural frequencies and mode shapes of thin and thick curved pipes and toroidal shells. The methods of the study are extensively validated against previously published results. New results are given for standard thin and thick curved pipes and toroidal shells, that indicate the influence of bend angle, support conditions, and wall thickness on the free vibration characteristics. The new results serve as convenient benchmarks that should be of interest to researchers and design engineers.

1. Introduction

In recent years, there has been a resurgence of interest in the study of the vibrations of curved pipes and toroidal shells. Contributing to the interest have been continuing and new applications of these structures in the fields of satellite antennas, fusion reactor vessels, nuclear piping systems, etc. Vibration results for these shells are now available from a number of studies for both thin-wall [1–5] and thick-wall geometries [6,7]. The information available is in a form useful mostly for researchers, rather than design engineers.

The present study gives an extensive summary of existing information on natural frequencies of toroidal geometries, and contributes new theoretical results for two standard geometries of interest to engineers. The analysis of the present study is conducted using the finite-element method (FEM), and the differential quadrature method (DQM). Results are given for thin and thick curved pipes extending through a circumferential angle α of 90° or 180°, and for toroidal shells ($\alpha = 360°$). Boundary conditions considered are of the fixed, diaphragm, and completely free types. For the thin shells the analysis is based on the Love–Kirchoff or on the shear-deformation theory, while for the thick shells the analysis is based on the theory of elasticity. The new results are for two standard components, a '5s' thin one and an '80s' thick one. A single one of these standard components forms a 90° curved pipe (elbow), two welded together form a 180° curved pipe, and four welded together form a complete toroidal shell. Mode shapes determined using the FEM

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are presented for the two standard components. The vibration results presented herein for these geometries can serve as benchmarks for researchers, and as reference values for design engineers.

2. Analysis using the DQM

A DQM solution has been given for the vibration analysis of thin curved pipes of circumferential angle less than 360° [5]. It is based on the Sanders–Budiansky thin shell theory which is of the Love–Kirchoff type. This solution has been used in the present study for analysis of thin curved pipes with fixed end supports. It has been adapted as well for the analysis of completely free thin toroidal shells. The adaptation has produced a mathematically one-dimensional semi-analytical solution, permitting the determination of natural frequencies pertaining to specified harmonics in the circumferential direction. By taking results for even circumferential harmonics only, a solution is given also for a 90° curved pipe with diaphragm supports [1,2].

A DQM solution, based on the three-dimensional theory of elasticity, for the vibration analysis of a smooth shell of revolution of constant but arbitrary thickness, has recently been presented [7]. The semi-analytical solution developed is mathematically a two-dimensional one, permitting the determination of natural frequencies pertaining to specified harmonics in the circumferential direction. It has been adapted in the present study for the analysis of completely free thick toroidal shells.

3. Analysis using the FEM

The commercial FEM program ADINA [8] was used in this study to determine the natural frequencies of curved pipes and toroidal shells. For the analysis of thin curved pipes and shells a 4-node 24-degree-of-freedom shell element was used. This element is based on the shear-deformation shell theory. For the analysis of thick curved pipes and shells a 27-node 81-degree-of-freedom brick element was used. This element is based on the theory of elasticity.

In the FEM calculations of this study geometric symmetry was not accounted for, i.e. full models were made in every case. The new results cited are generally for the finest meshes that could be run on the available computer system. The eigenvalue analysis in all cases was carried out using the subspace iteration method. Mode shapes were plotted, using the FEM, for the '5s' and '80s' curved pipes and shells.

4. Validation and results for thin shells

A validation study was conducted to summarize the existing information about the vibration of thin curved pipes and toroidal shells, and also to establish the accuracy of the current methods of analysis. The properties of ten thin pipe/shell validation cases are given in Table 1, covering the work of four previous researchers. In this table h is the wall thickness, r the cross-sectional radius, R the bend radius, and $\Gamma = r^2/(hR)$ the pipe

Table 1
Properties of thin curved pipe and shell cases for validation study

Case	h (m)	r (m)	R (m)	r/h	R/r	Γ	Ref.
1	0.010	1.0	2.5465	100.0	2.5465	39.27	[1]
2	0.003	0.1	1.0	33.3	10.0	3.33	[2]
3	0.001	0.1	1.0	100.0	10.0	10.00	[3]
4	0.002	0.1	1.0	50.0	10.0	5.00	[3]
5	0.004	0.2	1.0	50.0	5.0	10.00	[3]
6	0.001	0.00913	0.0913	9.13	10.0	0.91	[4]
7	0.001	0.0204	0.204	20.4	10.0	2.04	[4]
8	0.001	0.0289	0.289	28.9	10.0	2.89	[4]
9	0.001	0.0645	0.645	64.5	10.0	6.45	[4]
10	0.001	0.0913	0.913	91.3	10.0	9.13	[4]

factor. Results for these cases, as given in the literature, and as established by the current methods are given in Tables 2–4. The results given are natural frequencies in Hz and correspond to a material with the properties

$$E = 0.207 \times 10^{12}\,\text{Pa}, \quad v = 0.3, \quad \rho = 7800\,\text{kg/m}^3, \tag{1}$$

where E is the Young's modulus, v the Poisson's ratio, and ρ the mass density.

In Table 2 are given the ten lowest frequencies for the thin curved pipe cases 1–2 for diaphragm and fixed end supports. Three sets of results are given, the previous shell theory (ST) and FEM results, and the current DQM results. The ST solutions employed trigonometric and beam-type trial functions, and the Galerkin method [1,2]. The DQM results agree to within 11% of the previous FEM results of case 1, and to within 0.7% of the previous FEM results of case 2. For this and subsequent tables the percent values that are cited are based on calculations in which the reference value was taken as the mean of the two values being compared.

In Table 3 are given the results for the ten lowest frequencies for the thin shell cases 3–5 for completely free conditions. The Runge–Kutta method (RKM) values stem from the study by Balderes and Armenakas [3]. The current DQM results agree with the previous results within 0.7%. In Table 4, the results for the six lowest frequencies corresponding to the 2nd circumferential harmonic for thin shell cases 6–10 for completely free conditions are given. The power series expansion (PSE) values stem from the study of Kosawada et al. [4]. The current DQM results agree with the previous results within 0.5%.

Table 2
Frequencies (Hz) in validation study for thin 90° curved pipes [1,2]

Mode	Diaphragm ends						Fixed ends		
	Case 1			Case 2			Case 2		
	ST	FEM	DQM	ST	FEM	DQM	ST	FEM	DQM
1	27.20	26.68	26.05	67.96	67.41	66.98	347.76	336.73	334.64
2	29.86	29.29	28.53	71.50	71.43	70.98	530.21	509.54	506.58
3	65.14	66.79	62.90	277.09	276.33	275.04	539.10	510.72	507.77
4	65.17	66.81	62.92	277.15	276.34	275.05	540.08	530.60	527.39
5	110.53	119.80	107.75	470.67	468.51	467.24	548.21	536.71	533.52
6	110.63	119.80	107.75	471.00	468.84	467.56	677.46	673.02	669.26
7	139.69	144.99	137.90	523.30	522.91	519.53	684.12	674.40	671.03
8	140.21	145.64	137.91	592.10	591.79	587.86	801.54	798.59	793.59
9	140.29	150.04	138.42	678.49	673.83	673.58	811.33	808.49	804.59
10	140.50	150.06	138.86	678.56	673.90	673.65	811.42	808.92	805.03

Table 3
Frequencies (Hz) in validation study for completely free thin toroidal shells [3]

Mode	Case 3		Case 4		Case 5	
	RKM	DQM	RKM	DQM	RKM	DQM
1	38.46	38.41	54.51	54.38	77.83	77.58
2	41.48	41.38	58.32	58.12	78.47	78.22
3	95.62	95.61	139.14	138.67	100.42	100.43
4	97.36	97.04	141.06	140.58	157.50	156.77
5	99.74	99.70	145.52	145.61	159.41	158.67
6	143.89	143.56	216.30	215.62	224.09	222.88
7	143.89	143.66	216.46	215.72	224.84	223.53
8	184.43	184.03	285.20	284.71	296.75	294.76
9	184.61	184.15	286.38	284.96	296.75	294.88
10	226.48	225.72	358.32	357.14	375.72	373.95

Table 4
Frequencies (Hz) in validation study for completely free thin toroidal shells ($m = 2$) [4]

Mode	Case 6		Case 7		Case 8		Case 9		Case 10	
	PSE	DQM	PSE	DQM	PSE	DQM	PSE	DQM	PSE	DQM
1	1344.3	1348.0	426.7	426.9	250.2	250.0	74.2	74.2	44.1	44.0
2	1352.7	1352.6	442.5	444.5	262.9	263.9	79.3	79.6	47.3	47.4
3	9247.4	9290.5	2827.3	2834.3	1839.5	1841.6	745.5	745.9	509.0	509.1
4	10042.7	10042.2	3155.5	3156.1	2076.2	2075.9	850.7	850.8	577.7	577.7
5	13428.1	13426.9	5315.9	5329.4	2993.4	2996.6	969.8	970.7	630.6	630.8
6	19720.1	19720.9	5331.2	5333.8	2999.5	2997.7	970.5	970.7	631.2	631.3

Table 5
Frequencies (Hz) for '5s' curved pipes and toroidal shell

Support	Fixed ends				Completely free			
α	90°		180°		90°	180°	360°	
Mesh	80 × 48	28 × 27	80 × 96	28 × 27	80 × 48	80 × 96	80 × 192	200
Mode	FEM	DQM	FEM	DQM	FEM	FEM	FEM	DQM
1	3152	3147	1279	1278	526	464	659	657
2	3422	3411	2220	2216	533	591	845	844
3	3482	3468	2386	2384	623	612	845	—
4	3614	3599	2437	2429	628	613	935	933
5	3962	3943	2450	2442	1359	724	935	—
6	4103	4080	2466	2458	1363	724	1639	1635
7	4110	4087	2886	2877	1451	1206	1639	—
8	4279	4257	2939	2937	1452	1230	1650	1645
9	4295	4265	2993	2986	1915	1468	1650	—
10	4300	4270	3161	3150	1919	1469	2300	2301
Fig.	1		2		3	4	5	

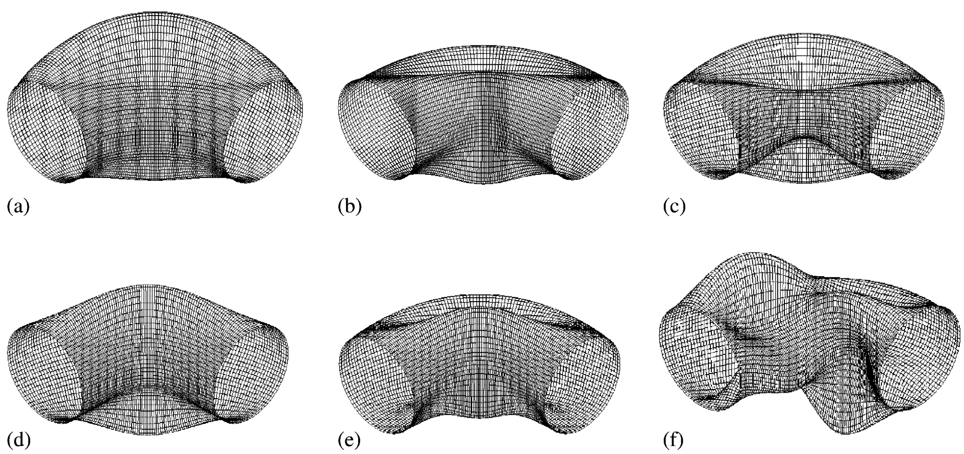

Fig. 1. Mode shapes 1–6 for 90° fixed–fixed '5s' shell (frequencies in Hz): (a) $f_1 = 3152$, (b) $f_2 = 3422$, (c) $f_3 = 3482$, (d) $f_4 = 3614$, (e) $f_5 = 3962$ and (f) $f_6 = 4103$.

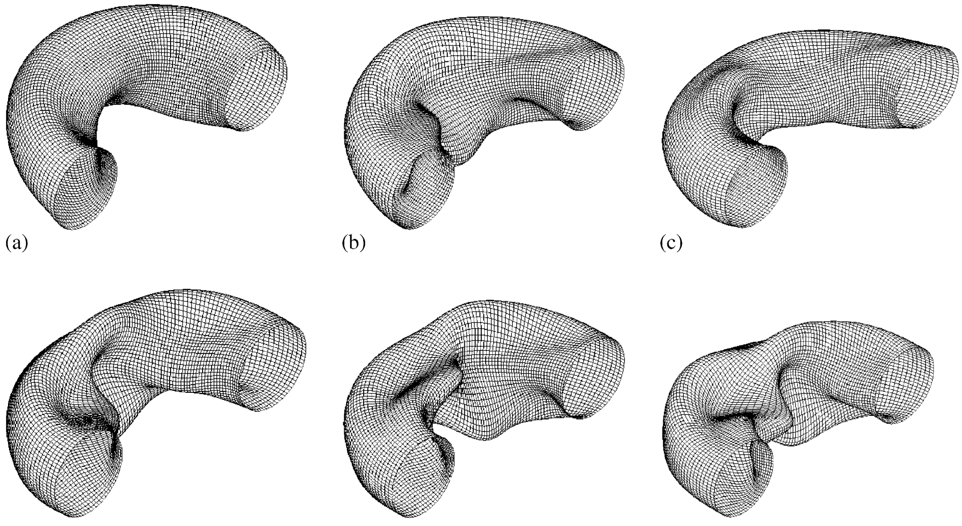

Fig. 2. Mode shapes 1–6 for 180° fixed–fixed '5s' shell (frequencies in Hz): (a) $f_1 = 1279$, (b) $f_2 = 2220$, (c) $f_3 = 2386$, (d) $f_4 = 2437$, (e) $f_5 = 2450$ and (f) $f_6 = 2466$.

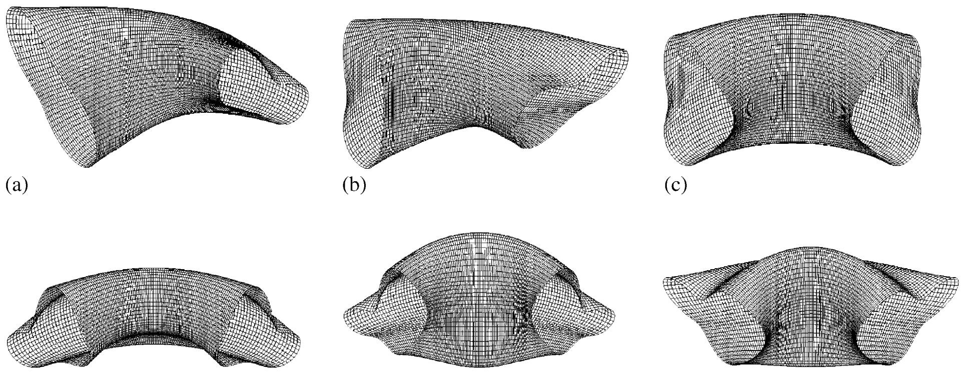

Fig. 3. Mode shapes 1–6 for 90° completely free '5s' shell (frequencies in Hz): (a) $f_1 = 526$, (b) $f_2 = 533$, (c) $f_3 = 623$, (d) $f_4 = 628$, (e) $f_5 = 1359$ and (f) $f_6 = 1363$.

New results are given in Hz for the frequencies of thin curved pipes and a toroidal shell in Table 5. The analysis was made for a stainless-steel material with the following properties:

$$E = 0.193 \times 10^{12}\,\mathrm{Pa}, \quad v = 0.291, \quad \rho = 7850\,\mathrm{kg/m^3}. \tag{2}$$

The geometric parameters of the thin curved pipes and toroidal shell are

$$\begin{aligned} &h = 0.00211\,\mathrm{m}, \quad r = 0.056095\,\mathrm{m}, \quad R = 0.1524\,\mathrm{m}, \\ &r/h = 26.59, \quad R/r = 2.717, \quad \Gamma = 9.785. \end{aligned} \tag{3}$$

These parameters correspond to a commercial product with the '5s' designation. The results are for curved 90° and 180° pipes with fixed and free end supports, and for a completely free toroidal shell. In Table 5, the meshes cited for the FEM give the number of elements in the meridional and circumferential directions, respectively. The meshes cited for the DQM for the curved pipes give the number of sampling points in the meridional and circumferential directions, respectively. The DQM mesh for the toroidal shell gives the number of sampling points in the meridional direction, for a given circumferential harmonic m. Comparing the current DQM and FEM results, the agreement is within 0.8%. The first six mode shapes for these five pipe and shell cases, as determined using the FEM, are presented in Figs. 1–5.

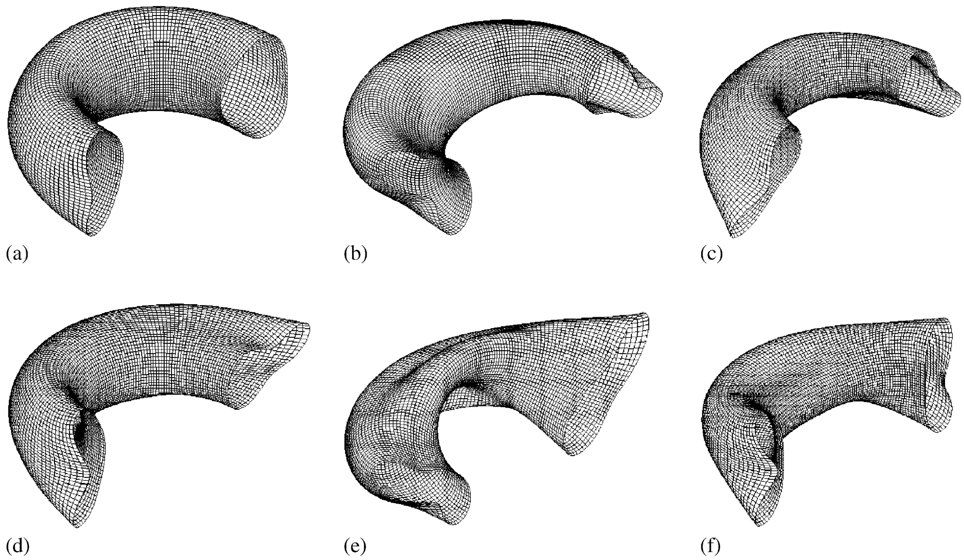

Fig. 4. Mode shapes 1–6 for 180° completely free '5s' shell (frequencies in Hz): (a) $f_1 = 464$, (b) $f_2 = 591$, (c) $f_3 = 612$, (d) $f_4 = 613$, (e) $f_5 = 724$ and (f) $f_6 = 724$.

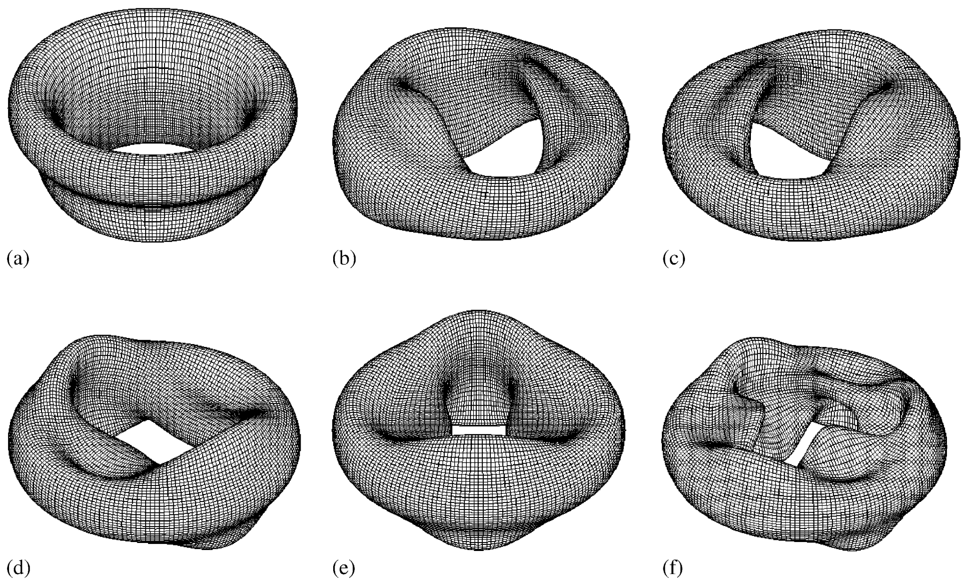

Fig. 5. Mode shapes 1–6 for 360° completely free '5s' shell (frequencies in Hz): (a) $f_1 = 659$, (b) $f_2 = 845$, (c) $f_3 = 845$, (d) $f_4 = 935$, (e) $f_5 = 935$ and (f) $f_6 = 1639$.

5. Validation and results for thick shells

The literature on the vibration of thick toroidal geometries is relatively sparse. There is no known work on thick curved pipes, but a comprehensive study on thick toroidal shells, using the FEM, has recently been published by Buchanan and Liu [6]. Six shell cases from that study, labelled A–F, and with geometric details as given in Table 6, are used for validation of the current analysis. In the table r_i, r_o, and r_m are, respectively, the relative internal, external and mean cross-sectional radii. For the presentation of the results a natural frequency parameter $\Omega = \omega r_o \sqrt{\rho/G}$ is defined, where ω is the circular frequency in rad/s, and $G = E/[2(1+v)]$.

Values for the parameter Ω for the cases A–F are given in Table 7. The results correspond to a material with the properties defined in Eq. (2). The ten lowest frequencies are given for completely free conditions. Two sets

Table 6
Properties of thick shell cases for validation study

Case	r_i	r_o	R	r_m	h	r_m/h	R/r_m	Γ
A	0.25	1	1.5	0.625	0.75	0.833	2.400	0.347
B	0.50	1	1.5	0.750	0.50	1.500	2.000	0.750
C	0.75	1	1.5	0.875	0.25	3.500	1.714	2.042
D	0.25	1	2.5	0.625	0.75	0.833	4.000	0.208
E	0.50	1	2.5	0.750	0.50	1.500	3.333	0.450
F	0.75	1	2.5	0.875	0.25	3.500	2.857	1.225

Table 7
Frequency parameter Ω in validation study for completely free thick toroidal shells [6]

Mode	Case A		Case B		Case C	
	FEM	DQM	FEM	DQM	FEM	DQM
1	0.5236	0.5235	0.5192	0.5191	0.3741	0.3720
2	0.6975	0.6972	0.6479	0.6471	0.5039	0.5038
3	0.7762	0.7752	0.6611	0.6609	0.5697	0.5691
4	1.0924	1.0922	0.9719	0.9707	0.6320	0.6298
5	1.1308	1.1307	1.0405	1.0403	0.6657	0.6636
6	1.1436	1.1427	1.0529	1.0524	0.6965	0.6944
7	1.1577	1.1577	1.0782	1.0776	0.7130	0.7112
8	1.3468	1.3459	1.1335	1.1316	0.8191	0.8173
9	1.6205	1.6194	1.1505	1.1487	0.9251	0.9242
10	1.6219	1.6201	1.1627	1.1608	0.9270	0.9265
	Case D		Case E		Case F	
	FEM	DQM	FEM	DQM	FEM	DQM
1	0.2537	0.2534	0.2568	0.2567	0.2482	0.2479
2	0.2975	0.2973	0.2965	0.2963	0.2857	0.2854
3	0.4611	0.4609	0.4326	0.4324	0.2942	0.2926
4	0.5929	0.5928	0.5873	0.5872	0.4882	0.4862
5	0.6325	0.6324	0.6121	0.6120	0.5288	0.5263
6	0.6711	0.6709	0.6598	0.6595	0.5357	0.5341
7	0.6917	0.6916	0.6698	0.6698	0.5396	0.5371
8	0.8271	0.8271	0.8039	0.8039	0.5673	0.5647
9	0.9511	0.9510	0.9185	0.9184	0.5771	0.5770
10	1.0081	1.0080	0.9762	0.9759	0.6232	0.6221

of results are given, the previous FEM results [6] and the current DQM results. The agreement in the two sets of values is within 0.6%.

New results are given in Hz for the frequencies of thick curved pipes and a toroidal shell in Table 8. The analysis was made for a stainless-steel material, with the properties of Eq. (2). The geometric parameters of the thick curved pipes and toroidal shell are

$$r_i = 0.04859\,\mathrm{m},\quad r_o = 0.05715\,\mathrm{m},\quad R = 0.1524\,\mathrm{m},$$
$$r_m/h = 6.176,\quad R/r_m = 2.883,\quad \Gamma = 2.143. \tag{4}$$

These values correspond to a commercial product with the '80s' designation. The results are for curved 90° and 180° pipes with fixed and free end supports, and for a completely free toroidal shell. FEM results were obtained for all geometries and support conditions, while DQM results were available only for the completely free shell. In Table 8, the meshes cited for the FEM give the number of elements in the radial, meridional and

Table 8
Frequencies (Hz) for '80s' curved pipes and toroidal shell

Support	Fixed ends		Completely free			
α	90°	180°	90°	180°	360°	
Mesh	2 × 24 × 18	2 × 24 × 36	2 × 24 × 18	2 × 24 × 36	2 × 24 × 72	8 × 40
Mode	FEM	FEM	FEM	FEM	FEM	DQM
1	3547	1376	2038	1098	1620	1611
2	4385	2418	2051	1619	1693	1689
3	4947	2973	2171	1824	1693	—
4	5928	3179	2209	1995	1929	1924
5	6164	3553	3794	2272	1929	—
6	6486	3754	3804	2339	3048	3040
7	7051	3773	5650	3073	3048	—
8	7210	4046	5655	3176	3239	3233
9	7278	4457	6059	3350	3239	—
10	7426	4607	6082	3917	3315	3307
Fig.	6	7	8	9	10	

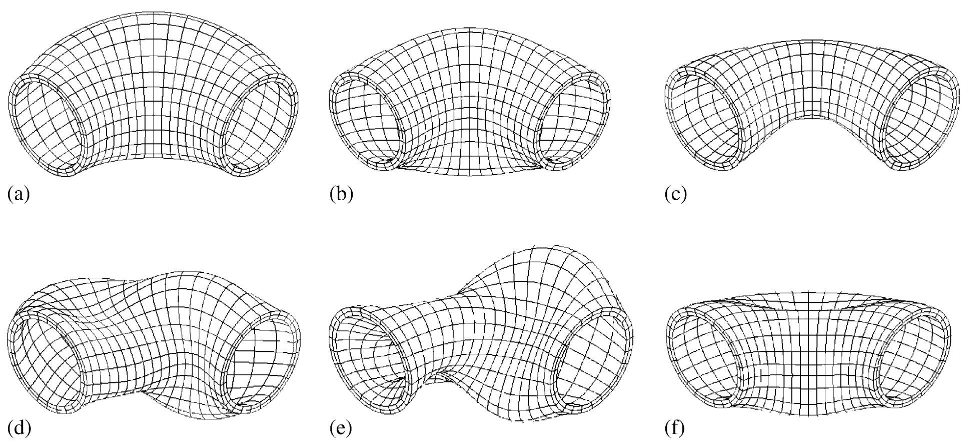

Fig. 6. Mode shapes 1–6 for 90° fixed–fixed '80s' shell (frequencies in Hz): (a) $f_1 = 3547$, (b) $f_2 = 4385$, (c) $f_3 = 4947$, (d) $f_4 = 5928$, (e) $f_5 = 6164$ and (f) $f_6 = 6486$.

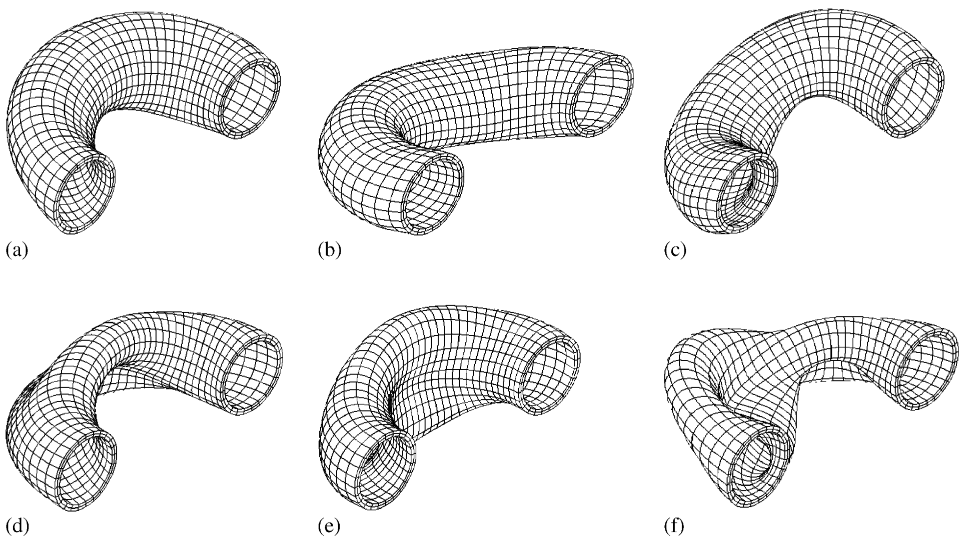

Fig. 7. Mode shapes 1–6 for 180° fixed–fixed '80s' shell (frequencies in Hz): (a) $f_1 = 1376$, (b) $f_2 = 2418$, (c) $f_3 = 2973$, (d) $f_4 = 3179$, (e) $f_5 = 3553$ and (f) $f_6 = 3754$.

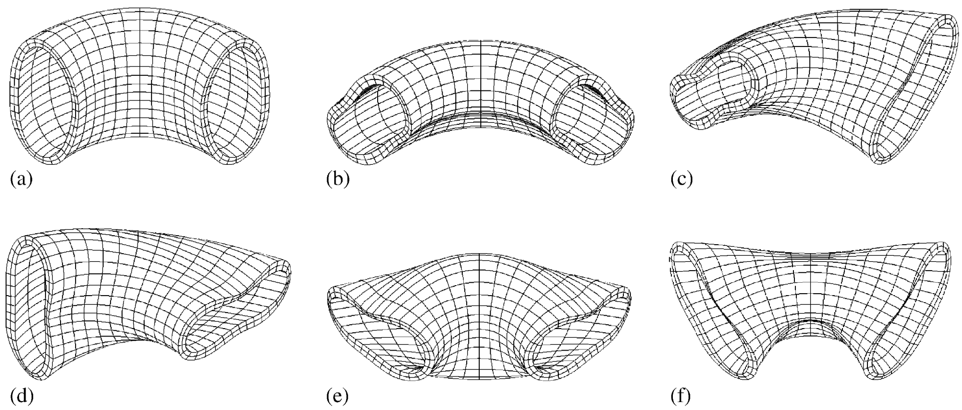

Fig. 8. Mode shapes 1–6 for 90° completely free '80s' shell (frequencies in Hz): (a) $f_1 = 2038$, (b) $f_2 = 2051$, (c) $f_3 = 2171$, (d) $f_4 = 2209$, (e) $f_5 = 3794$ and (f) $f_6 = 3804$.

circumferential directions, respectively. The DQM mesh for the toroidal shell gives the number of sampling points in the radial and meridional directions, respectively, for a given circumferential harmonic m. In the single instance where a comparison was possible, i.e. for the toroidal shell, there is agreement within 0.6%. The first six mode shapes for these five pipe and shell cases, as determined using the FEM, are presented in Figs. 6–10.

6. Conclusions

Theoretical results obtained using the methods of this study show excellent agreement with results given in the literature. New results presented for standard steel curved pipes and toroidal shells indicate the influence of bend angle, support conditions, and wall thickness on the free vibration characteristics. In comparing the results for the thin curved pipes (Table 5, Figs. 1–4) it is clear that natural frequencies decrease significantly

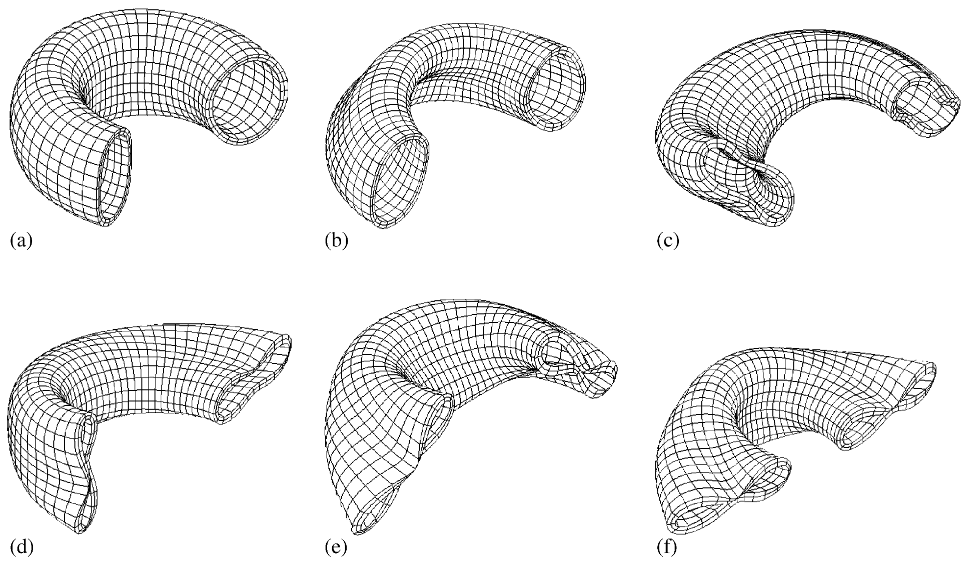

Fig. 9. Mode shapes 1–6 for 180° completely free '80s' shell (frequencies in Hz): (a) $f_1 = 1098$, (b) $f_2 = 1619$, (c) $f_3 = 1824$, (d) $f_4 = 1995$, (e) $f_5 = 2272$ and (f) $f_6 = 2339$.

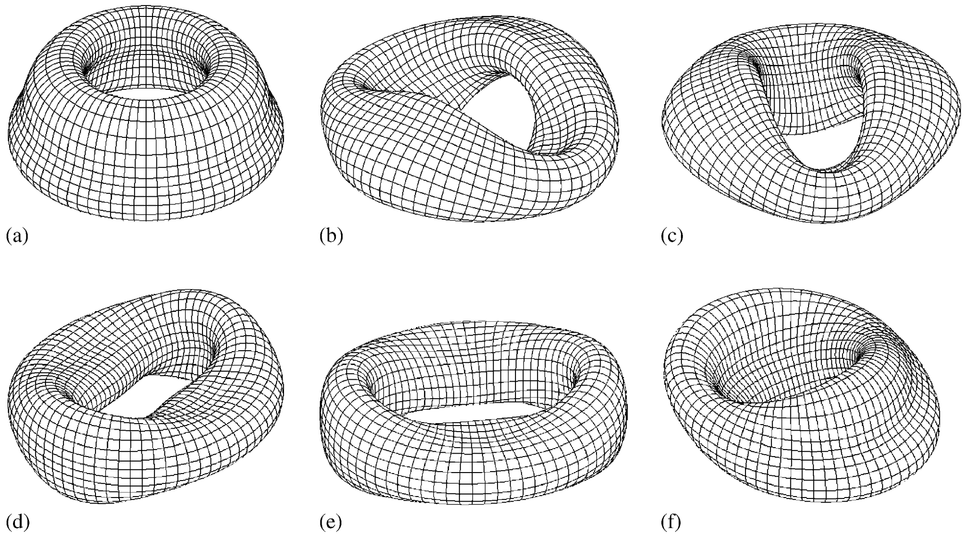

Fig. 10. Mode shapes 1–6 for 360° completely free '80s' shell (frequencies in Hz): (a) $f_1 = 1620$, (b) $f_2 = 1693$, (c) $f_3 = 1693$, (d) $f_4 = 1929$, (e) $f_5 = 1929$ and (f) $f_6 = 3048$.

with an increase in the bend angle, and in passing from fixed end conditions to free end conditions. The mode shapes for these geometries are relatively complex, and cannot readily be identified as radial, meridional, or longitudinal as in the case of straight pipes. In comparing the results for the thick curved pipes (Table 8, Figs. 6–9) with those of the thin ones, it is noted that changes in the bend angle and boundary conditions are more significant in changing the frequencies than changes in the wall thickness. The mode shapes of the thick pipes resemble those of the thin pipes, but are generally simpler. For the toroidal shells (Figs. 5 and 10) it is also seen that the mode shapes for the thick geometry resemble those of the thin geometry. Comparing the free curved pipes and shells it is seen that the 180° pipes are the most flexible, the shells acquiring their stiffness due to the circumferential connectivity. The sample results provided in the study should be of interest to researchers and design engineers.

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