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Short Communication

Small amplitude, transverse vibrations of circular plates elastically restrained against rotation with an eccentric circular perforation with a free edge

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Abstract

Rayleigh–Ritz variational method is applied to the determination of the first four frequency coefficients for the title problem by making use of coordinate functions which identically satisfy the boundary conditions at the outer edge. Good stability and convergence properties are found. The mathematical model seems to be realistic and accurate, within the realm of the classical theory of vibrating plates.

1. Introduction

The title problem deals with a situation which may appear in real-life vibrating systems. A small degree of eccentricity in an internal boundary may be caused by a human error in some instances. In others, it may originate by a practical reason like passage of a cable or a conduit of small diameter.

The present study makes use of the methodology employed by the senior authors (Avalos, Laura) in previous publications where the coordinate function does not satisfy the natural boundary conditions.

This is admissible since Ritz variational method is used.

2. Approximate analytical solution

In the case of normal modes of vibration of the vibrating system shown in Fig. 1, one takes

$$w'(r', \theta, t) = W'(r', \theta)\mathrm{e}^{\mathrm{i}\omega t} \tag{1}$$

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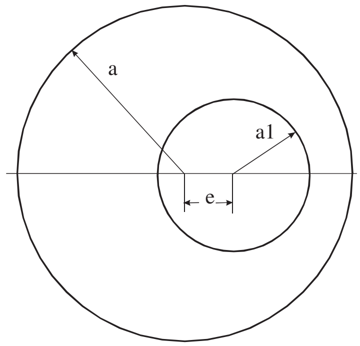

Fig. 1. Vibrating mechanical system. Case of an eccentric cutout with a free edge.

Table 1
Values of the first four frequency coefficients in the case of a circular plate with a circular cutout of radius $a_1/a = 0.1$, for different values of the non-dimensional eccentricity e/a when the cutout is displaced along a radial line

Eccentricity e/a	$a/(\phi D)$	Ω_1	Ω_2	Ω_3	Ω_4
	0	4.0053	13.880	25.416	29.488
0.1	1	6.0053	14.945	26.459	30.531
	10	8.7114	18.513	30.602	34.989
	∞	10.175	21.220	34.577	39.554
	0	4.8755	13.855	25.485	29.383
0.2	1	6.0249	14.929	26.536	30.673
	10	8.7407	18.502	30.706	35.112
	∞	10.206	21.212	34.716	39.626
	0	4.9030	13.853	25.573	29.672
0.3	1	6.0529	14.937	26.631	30.740
	10	8.7778	18.534	30.835	35.165
	∞	10.255	21.268	34.892	39.674
	0	4.9182	13.871	25.653	29.636
0.4	1	6.0668	14.965	26.718	30.686
	10	8.7909	18.597	30.956	35.108
	∞	10.269	21.358	35.052	39.621
	0	4.9249	13.892	25.706	29.606
0.5	1	6.0706	14.993	26.779	30.650
	10	8.7872	18.651	31.043	35.108
	∞	10.258	21.427	35.158	39.669
	0	4.9274	13.904	25.722	29.622
0.6	1	6.0698	15.010	26.801	30.678
	10	8.7744	18.674	31.078	35.191
	∞	10.235	21.436	35.185	39.802
	0	4.9272	13.904	25.700	29.668
0.7	1	6.0659	15.010	26.783	30.734
	10	8.7563	18.653	31.057	35.279
	∞	10.206	21.375	35.125	39.889
	0	4.9268	13.8965	25.653	29.714
0.8	1	6.0620	14.9958	26.738	30.771
	10	8.7354	18.5889	30.980	35.296
	∞	10.173	21.2464	34.967	39.843

for the plate transverse displacement, and then introduces the following approximation, convenient in the case of both axisymmetric and antisymmetric modes of vibration, see for example [1].

$$W'(r', \theta) \cong W'_a(r', \theta) = \sum_{j=0}^{j} A_{j0}(\alpha_{jk} r'^4 + \beta_{jk} r'^2 + 1) r'^{2j}$$

$$+\sum_{k=1}^{K}\cos(k\theta)\sum_{j=0}^{j}A_{jk}(\alpha_{jk}r'^4+\beta_{jk}r'^2+1)r'^{j+k}, \tag{2}$$

where α's and β's of each coordinate function are determined substituting each functional relation in the governing boundary conditions at the external contour, i.e., for the elastically restrained edge against rotation,

$$W'(a,\theta)=0, \quad \frac{\partial W'(r',\theta)}{\partial r}(a,\theta)=\phi M_r(r',\theta), \tag{3}$$

M_r being the radial flexural moment, ϕ is the flexibility coefficient of the rotational boundary spring and a is the radius of the circular plate.

The Rayleigh–Ritz variational approach requires minimization of the functional

$$J[W']=U[W']-T[W'], \tag{4}$$

where $U[W']$ is the maximum strain energy and $T[W']$ is the maximum kinetic energy for the (true) displacement amplitude W' of the plate.

Table 2
Values of the first four frequency coefficients in the case of a circular plate with a circular cutout of radius $a_1/a=0.2$, for different values of the non-dimensional eccentricity e/a when the cutout is displaced along a radial line

Eccentricity e/a	$a/(\phi D)$	Ω_1	Ω_2	Ω_3	Ω_4
	0	4.7199	13.606	24.881	30.878
0.1	1	5.9528	14.661	25.904	31.973
	10	8.8373	18.152	29.936	35.440
	∞	10.402	20.741	33.753	39.656
	0	4.7272	13.673	25.003	29.899
0.2	1	5.9601	14.760	26.030	30.910
	10	8.8355	18.407	30.112	35.290
	∞	10.384	21.192	33.998	39.770
	0	4.7467	13.756	25.311	29.396
0.3	1	5.9784	14.899	26.365	30.149
	10	8.8355	18.738	30.639	34.365
	∞	10.362	21.699	34.795	38.616
	0	4.7858	13.833	25.664	28.983
0.4	1	6.0132	15.020	26.899	29.989
	10	8.8483	18.975	31.481	34.295
	∞	10.354	21.981	36.001	38.710
	0	4.8273	13.882	26.141	27.591
0.5	1	6.0455	15.099	27.315	30.213
	10	8.8446	19.072	31.958	34.861
	∞	10.321	22.007	36.361	39.695
	0	4.8569	13.895	26.224	28.910
0.6	1	6.0608	15.125	27.396	30.552
	10	8.8068	19.031	31.905	35.397
	∞	10.245	21.829	36.096	40.216
	0	4.8749	13.869	26.125	29.638
	1	6.0608	15.094	27.218	30.754
0.7	10	8.7433	18.858	31.594	35.529
	∞	10.142	21.488	35.582	40.134
	0	4.8865	13.830	25.774	29.629
0.8	1	6.0510	15.027	26.941	30.781
	10	8.6676	18.580	31.151	35.374
	∞	10.040	21.034	34.878	39.782

As has been shown elsewhere, see for example Ref. [2], in the case of a circular plate, each term in Eq. (4) can be written as

$$U[W'] = \frac{D}{2}\int\int\left\{\left[\left(\frac{\partial^2 W'}{\partial r'^2}+\frac{1}{r'}\frac{\partial W'}{\partial r'}\right)+\left(\frac{1}{r'^2}\frac{\partial^2 W'}{\partial \theta^2}\right)\right]^2 - 2(1-\mu)\left[\frac{\partial^2 W'}{\partial r'^2}\left(\frac{1}{r'}\frac{\partial W'}{\partial r'}+\frac{1}{r'^2}\frac{\partial^2 W'}{\partial \theta^2}\right) - \left(\frac{1}{r'}\frac{\partial^2 W'}{\partial\theta\partial r'}-\frac{1}{r'^2}\frac{\partial W}{\partial \theta}\right)^2\right]\right\} r'\,\mathrm{d}r'\,\mathrm{d}\theta, \tag{5}$$

where D is the flexural rigidity of the plate, μ its Poison's ratio and

$$T[W'] = \frac{\rho\omega^2 h}{2}\int\int W'^2 r'\,\mathrm{d}r'\,\mathrm{d}\theta. \tag{6}$$

The integrals in expressions (5) and (6) extend over the actual area of the double connected plate under study. Introducing the non-dimensional variables

$$W = W'/a, \quad r = r'/a. \tag{7}$$

Table 3
Values of the first four frequency coefficients in the case of a circular plate with a circular cutout of radius $a_1/a = 0.3$ for different values of the non-dimensional eccentricity e/a when the cutout is displaced along a radial line

Eccentricity e/a	$a/(\Phi D)$	Ω_1	Ω_2	Ω_3	Ω_4
	0	4.6631	12.943	24.116	34.887
0.1	1	6.1017	14.011	25.117	35.703
	10	9.4525	17.521	29.030	40.993
	∞	11.312	20.141	32.675	46.905
	0	4.6570	13.293	24.171	31.376
0.2	1	6.0791	14.486	25.199	32.421
	10	9.3097	18.534	29.260	37.026
	∞	11.036	21.726	33.139	41.906
	0	4.6503	13.804	24.502	29.323
0.3	1	6.0486	15.166	25.629	30.325
	10	9.1315	19.795	30.252	34.611
	∞	10.720	22.895	34.934	39.005
	0	4.6599	14.236	26.132	28.533
0.4	1	6.0181	15.700	26.996	29.254
	10	8.9600	20.377	33.413	33.735
	∞	10.440	23.862	37.534	39.096
	0	4.6601	14.146	27.519	28.675
0.5	1	6.0022	15.779	28.712	29.374
	10	8.8257	20.066	32.976	35.728
	∞	10.231	23.001	37.147	41.304
	0	4.6906	14.143	27.267	29.322
0.6	1	6.0016	15.539	28.377	30.874
	10	8.7232	19.452	32.618	36.364
	∞	10.081	22.072	36.717	41.159
	0	4.7296	13.8727	26.758	29.690
0.7	1	6.0022	15.2338	27.901	31.071
	10	8.6237	18.8477	32.060	35.908
	∞	9.9585	21.2702	35.942	40.352

Eqs. (5) and (6) above can be recast in a non-dimensional form. The functional for the whole system in Fig. 1 is

$$\begin{aligned}\frac{2J[W]}{D} = \int\int \Bigg\{ & \left[\left(\frac{\partial^2 W}{\partial r^2} + \frac{1}{r}\frac{\partial W}{\partial r}\right) + \left(\frac{1}{r^2}\frac{\partial^2 W}{\partial \theta^2}\right)\right]^2 \\ & - 2(1-\mu)\left[\frac{\partial^2 W}{\partial r^2}\left(\frac{1}{r}\frac{\partial W}{\partial r} + \frac{1}{r^2}\frac{\partial^2 W}{\partial \theta^2}\right)\right. \\ & \left.\left. - \left(\frac{1}{r}\frac{\partial^2 W}{\partial \theta \partial r} - \frac{1}{r^2}\frac{\partial W}{\partial \theta}\right)^2\right]\right\} r\,\mathrm{d}r\,\mathrm{d}\theta \\ & - \Omega^2 \int\int W^2 r\,\mathrm{d}r\,\mathrm{d}\theta, \end{aligned} \tag{8}$$

where as usual,

$\Omega_i = \sqrt{\rho h/D}\omega_i a^2$ is the non-dimensional frequency coefficient.

Minimizing the governing functional with respect to the A'_{jk}s expression (8) yields a $(J \times K)$ homogeneous linear system of equations in the A'_{jk}s. A secular determinant in the natural frequency coefficients of the system results from the non-triviality condition. The present study is concerned with the determination of the first four frequency coefficients, Ω_1 to Ω_4 in the case of circular plates with an eccentric circular cutout.

Table 4
Values of the first four frequency coefficients in the case of a circular plate with a circular cutout of radius $a_1/a = 0.4$ for different values of the non-dimensional eccentricity e/a when the cutout is displaced along a radial line

Eccentricity e/a	$a/(\Phi D)$	Ω_1	Ω_2	Ω_3	Ω_4
	0	4.7486	12.253	23.218	37.293
0.1	1	6.5515	13.497	24.234	38.255
	10	10.689	17.699	28.203	42.481
	∞	13.048	21.026	31.953	47.231
	0	4.7058	12.989	23.507	35.506
0.2	1	6.4313	14.530	24.625	36.224
	10	10.130	19.849	29.237	41.105
	∞	12.085	24.148	34.248	42.594
	0	4.64545	14.116	24.284	31.137
0.3	1	6.27936	16.008	25.778	32.196
	10	9.57953	21.908	32.373	36.827
	∞	11.2409	26.074	39.173	41.686
	0	4.5813	15.121	26.056	29.212
0.4	1	6.1292	17.057	28.123	30.433
	10	9.1260	22.259	33.807	37.166
	∞	10.598	25.816	38.105	43.816
	0	4.5258	15.310	27.324	29.620
0.5	1	6.0016	17.006	28.554	31.781
	10	8.7818	21.372	32.890	38.230
	∞	10.142	24.431	37.065	44.215
	0	4.4953	14.821	27.367	30.365
0.6	1	5.9101	16.328	28.400	32.316
	10	8.5462	20.144	32.615	37.799
	∞	9.8560	22.835	36.768	42.757
	0	4.5044	14.194	27.257	30.354
0.7	1	5.8716	15.605	28.266	31.979
	10	8.4198	19.068	32.406	36.830
	∞	9.7107	21.455	36.461	40.276

3. Numerical results

All calculations were performed for simply supported circular plates of uniform thickness, elastically restrained against rotation at the outer edge $r = a$ while the inner border of the eccentric perforation was taken to be a free edge. In all cases, the Poisson coefficient has been taken to be $\mu = 0.3$.

Six tables are presented, each with a value of the ratio a_1/a of the radius of the eccentric perforation with respect to the circular plate radius a. Table 1: $a_1/a = 0.1$; Table 2: $a_1/a = 0.2$; Table 3: $a_1/a = 0.3$; Table 4: $a_1/a = 0.4$; Table 5: $a_1/a = 0.5$ and Table 6: $a_1/a = 0.6$. In each table, in turn, four values of the non-dimensional rotational spring coefficient are taken as the center of the eccentric hole is displaced along a radial line.

For the double series, Eq. (2), J and K up to 16×6 terms have been used, that is to say, a secular determinant of order 96 was generated for all situations. Although satisfactory convergence is already achieved for $J = 8$ and $K = 4$, such high values of J and K have been used taking advantage of the speed of modern desktop computers. As usual, special care has been taken to manipulate the numerical solving of the involved determinants, and 80 bits floating point variables (IEEE-standard temporary reals) have been used to satisfy accuracy requirements.

It is worth noting that computations are very stable and all frequency coefficients uniformly converge as the number of terms in the double series is increased.

As a general conclusion one may say that the mathematical model seems to be quite realistic and accurate, within the realm of the classical theory of vibrating plates.

Table 5
Values of the first four frequency coefficients in the case of a circular plate with a circular cutout of radius $a_1/a = 0.5$ for different values of the non-dimensional eccentricity e/a when the cutout is displaced along a radial line

Eccentricity e/a	$a/(\Phi D)$	Ω_1	Ω_2	Ω_3	Ω_4
	0	5.0257	12.028	22.566	35.900
0.1	1	7.4182	13.802	23.751	36.921
	10	12.685	19.979	28.556	41.463
	∞	15.812	25.154	33.520	46.773
	0	4.8956	13.161	23.388	36.826
0.2	1	7.0447	15.669	24.957	38.029
	10	11.329	22.971	32.201	43.713
	∞	13.683	28.003	40.367	51.604
	0	4.7315	15.067	25.576	34.958
0.3	1	6.6535	17.773	28.369	36.150
	10	10.265	24.168	38.129	41.560
	∞	12.158	28.495	45.338	47.903
	0	4.5685	16.370	28.769	31.599
0.4	1	6.3172	18.691	31.065	33.518
	10	9.4824	24.044	36.162	40.835
	∞	11.091	27.967	41.042	47.305
	0	4.4281	16.491	27.891	32.311
0.5	1	6.0455	18.399	29.158	34.605
	10	8.9203	23.055	33.606	40.913
	∞	10.352	26.544	37.932	46.577
	0	4.3183	15.872	27.113	32.549
0.6	1	5.8380	17.459	28.253	34.354
	10	8.5309	21.556	32.575	40.053
	∞	9.8633	24.591	36.749	45.532
	0	4.2053	15.139	27.019	32.180
0.7	1	5.7245	16.395	28.072	33.554
	10	8.2898	20.024	32.340	38.679
	∞	9.5581	22.634	36.4606	43.430

Table 6
Values of the first four frequency coefficients in the case of a circular plate with a circular cutout of radius $a_1/a = 0.6$ for different values of the non-dimensional eccentricity e/a when the cutout is displaced along a radial line

Eccentricity e/a	$a/(\Phi D)$	Ω_1	Ω_2	Ω_3	Ω_4
	0	5.5689	12.660	22.868	35.667
0.1	1	8.8831	15.709	24.644	37.019
	10	15.837	25.503	32.711	43.372
	∞	20.421	32.905	42.603	51.992
	0	5.2509	14.693	24.787	37.480
0.2	1	7.9706	18.505	28.127	39.952
	10	13.222	26.747	40.296	52.803
	∞	16.386	32.547	49.371	72.388
	0	4.9072	16.773	29.648	41.894
0.3	1	7.2083	19.949	34.096	43.303
	10	11.441	26.562	43.330	49.650
	∞	13.809	31.520	51.269	57.573
	0	4.6021	17.644	32.721	35.528
0.4	1	6.6217	20.192	35.127	37.317
	10	10.211	26.005	41.028	44.702
	∞	12.102	30.432	47.097	52.234
	0	4.2920	17.713	30.032	35.185
0.5	1	6.1658	19.760	31.274	37.328
	10	9.3518	25.039	36.065	44.342
	∞	10.946	29.018	40.919	50.946
	0	4.1278	17.645	28.822	35.502
0.6	1	5.7953	18.760	28.910	36.683
	10	8.7531	23.523	33.415	43.159
	∞	10.160	27.002	37.735	49.123

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