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Short Communication

Investigation of the properties of the period for the nonlinear oscillator $\ddot{x} + (1 + \dot{x}^2)x = 0$

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Abstract

The mathematical properties of a nonlinear oscillator, having the square of the angular frequency depend quadratically on the velocity, is presented. The major conclusion is that the exact period $T(A)$ is well-defined for all values of the amplitude (where the initial conditions are taken to be $x(0) = A$ and $\dot{x}(0) = 0$). An approximate expression is derived for $T(A)$.

A fundamental property of any nonlinear oscillator is the dependence of the period on the initial amplitude, i.e., $T = T(A)$, where $x(0) = A$ and $\dot{x}(0) = 0$. The nonlinear oscillator

$$\ddot{x} + (1 + \dot{x}^2)x = 0 \tag{1}$$

has the interesting feature that its angular frequency, $\omega(A) = 2\pi/T(A)$, is singular or not defined at finite values of A when standard perturbation procedures [1–4] are used to calculate $\omega(A)$. For example, the first-order harmonic balance method [2] gives [5]

$$\omega(A) = \frac{2}{\sqrt{4 - A^2}}. \tag{2}$$

Observe that this expression defines $\omega(A)$ only for $0 \leqslant |A| < 2$; outside of this interval, the angular frequency is complex-valued, indicating that no periodic oscillations occur. One interpretation of this and related results for other methods is that these techniques for calculating $\omega(A)$ are restricted to small amplitudes [1,5]. However, this observation raises issues as to whether it can be demonstrated that the exact value of the angular frequency for Eq. (1) exists for all (real) values of the amplitude A. The main purpose of this Short Communication is to obtain the exact expression for the period $T(A)$ and study its properties. The period is the more fundamental quantity and, as indicated above, is related to the angular frequency by the relation

$$T(A) = \frac{2\pi}{\omega(A)}. \tag{3}$$

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To proceed, note should be made to a result obtained by Beatty and Mickens [5]: All the nontrivial solutions of Eq. (1) are periodic for arbitrary initial conditions. They demonstrate this by applying methods from the qualitative theory of differential equations [2] to the $(x, y = \dot{x})$ phase-space of Eq. (1), i.e.,

$$\frac{dx}{dt} = y, \quad \frac{dy}{dt} = -(1 + y^2)x. \tag{4}$$

Using Eqs. (4), a first-integral can be calculated for Eq. (1). It is given integrating the expression [2]

$$\frac{dy}{dx} = -\frac{(1 + y^2)x}{y}, \tag{5}$$

to obtain

$$\left(\frac{1}{2}\right)\ln(1 + y^2) + \left(\frac{1}{2}\right)x^2 = \frac{A^2}{2} \tag{6}$$

for the initial conditions $x(0) = A$ and $y(0) = 0$. Solving for y and using $y = dx/dt$, it follows that the period $T(A)$ is [2]

$$T(A) = 4\int_0^A \frac{dx}{[e^{(A^2 - x^2)} - 1]^{1/2}}. \tag{7}$$

The linear transformation, $x = Au$, reduces this equation to the form

$$T(A) = 4A\int_0^1 \frac{du}{[e^{A^2(1-u^2)} - 1]^{1/2}}. \tag{8}$$

The number $T(0)$ represents the period for infinitesimal values of A. For such values of A, it follows that

$$e^{A^2(1-u^2)} - 1 = A^2(1 - u^2) + O(A^4) \tag{9}$$

and

$$T(0) = \lim_{A \to 0} 4A\int_0^1 \frac{du}{\sqrt{A^2(1 - u^2)}} = 4\int_0^1 \frac{du}{\sqrt{1 - u^2}} = 2\pi, \tag{10}$$

where the value of the last integral is $\pi/2$ [6].

A lower bound on $T(A)$ can be found by making use of the inequality

$$\frac{1}{[e^{(A^2 - x^2)} - 1]^{1/2}} \geqslant \frac{1}{[e^{A^2} - 1]^{1/2}}. \tag{11}$$

Using this in Eq. (7) gives

$$T(A) \geqslant 4\int_0^A \frac{dx}{\sqrt{e^{A^2} - 1}} = \frac{4A}{\sqrt{e^{A^2} - 1}}. \tag{12}$$

Thus, $T(A)$ is bounded from below.

To see whether $T(A)$ is bounded from above, the properties of its derivative, dT/dA, must be determined. A straightforward, but long, calculation of dT/dA, using Eq. (8), leads to the result

$$\frac{dT}{dA} = 4\int_0^1 \frac{[(1 - w)e^w - 1]}{(e^w - 1)^{3/2}} \cdot du, \tag{13}$$

where

$$w = w(A, u) = A^2(1 - u^2). \tag{14}$$

If the integration variable in Eq. (13) is replaced by w, then

$$du = -\left(\frac{1}{2A}\right)\frac{dw}{\sqrt{A^2 - w}} \tag{15}$$

and the w limits of integration are A^2 and 0, i.e.,

$$u = 0 \Longrightarrow w = A^2, \quad u = 1 \Longrightarrow w = 0. \tag{16}$$

Putting all of this into Eq. (13) gives

$$\frac{\mathrm{d}T}{\mathrm{d}A} = \left(\frac{2}{A}\right) \int_0^{A_2} \frac{[(1-w)\mathrm{e}^w - 1]\,\mathrm{d}w}{\sqrt{A^2 - w}(\mathrm{e}^w - 1)^{3/2}}. \tag{17}$$

Now applying the Taylor series expansion [7], it follows that

$$\begin{aligned}(1-w)\mathrm{e}^w - 1 = (\mathrm{e}^w - 1) - w\mathrm{e}^w &= \left[\sum_{n=0}^{\infty} \frac{w^n}{n!} - 1\right] - \sum_{n=0}^{\infty} \frac{w^{n+1}}{n!} \\ &= \sum_{n=1}^{\infty} \frac{w^n}{n!} - \sum_{n=1}^{\infty} \frac{w^n}{(n-1)!} = -\sum_{n=2}^{\infty} \left(\frac{n-1}{n!}\right) w^n. \end{aligned} \tag{18}$$

Since w satisfies the conditions

$$0 \leqslant w \leqslant A^2, \tag{19}$$

the right-side of the last line in Eq. (18) is non-positive, i.e.,

$$(1-w)\mathrm{e}^w - 1 \leqslant 0. \tag{20}$$

This result implies that the integrand in Eq. (17) is non-positive and, as a consequence,

$$\frac{\mathrm{d}T(A)}{\mathrm{d}A} < 0. \tag{21}$$

This means that $T(A)$ is a monotonic decreasing function of A with

$$\operatorname*{Lim}_{A \to \infty} T(A) = 0. \tag{22}$$

Returning back to Eq. (13), where the integration variable is u, and $w = w(A, u)$ is given by Eq. (14), it follows that

$$\begin{aligned}\frac{\mathrm{d}T(0)}{\mathrm{d}A} &= 4 \operatorname*{Lim}_{A \to 0} \int_0^1 \frac{[(1-w)\mathrm{e}^w - 1]}{(\mathrm{e}^w - 1)^{3/2}} \cdot \mathrm{d}u = 4 \operatorname*{Lim}_{A \to 0} \int_0^1 \frac{(-w^2/2)}{w^{3/2}} \cdot \mathrm{d}u \\ &= -2 \operatorname*{Lim}_{A \to 0} A \int_0^1 (1-u^2)^{1/2}\,\mathrm{d}u = 0. \end{aligned} \tag{23}$$

The conclusion is that the slope of $T(A)$ is zero at $A = 0$. Note that combining the results of Eqs. (10), (21), and (22), the period has the bounds

$$\frac{4A}{\sqrt{\mathrm{e}^{A^2} - 1}} \leqslant T(A) \leqslant 2\pi. \tag{24}$$

An analytical approximation for the period can be determined by applying the integral mean-value theorem [7] to Eq. (8); doing this gives

$$T_a(A) = \frac{4A}{[\mathrm{e}^{A^2(1-\bar{u}^2)} - 1]^{1/2}}, \tag{25}$$

where $\bar{u}$ is some value for u in the integral $0 \leqslant u \leqslant 1$. While the mean-value theorem tells us that such a value must exist, an important issue generally not discussed is that this value depends on the parameters appearing in the integrand. For this case, $\bar{u}$ is a function of the amplitude A and its value can only be expected to become known when the integral is calculated. But, this integral, see Eq. (8), is not expected to have an exact solution expressed in terms of a finite number of the elementary functions; consequently, $\bar{u} = \bar{u}(A)$ exists, but is unknown. What can be done is to attempt to obtain a value for $\bar{u}$ such that $T_a(A)$ gives an accurate representation for small values of A. This can be done by requiring $T_a(0)$ to be equal to the exact value

Table 1

A	$T_a(A)$[a]	$T_e(A)$[b]	% Error[c]
0.01	6.2831	6.2831	0.00
0.10	6.2768	6.2753	0.03
1.00	5.6584	5.5272	2.4
5.0	0.1261	1.2966	2.8
10.00	6.3301×10^{-8}	0.6328	100

[a] $T_a(A)$ calculated using Eq. (28).
[b] $T_e(A)$ calculated using Eq. (8) by numerical integration.
[c] Percentage error defined as

$$\left|\frac{T_e(A) - T_a(A)}{T_e(A)}\right| \cdot 100.$$

$T(0) = 2\pi$; see, Eq. (10). Therefore,

$$T_a(0) = \underset{A\to 0}{\mathrm{Lim}}\ T_a(A) = \frac{4}{\sqrt{1-\bar{u}^2}} = 2\pi. \tag{26}$$

Since only $(1-\bar{u}^2)$ appears in Eq. (25), it can be calculated from Eq. (26) to be

$$1 - \bar{u}^2 = \frac{4}{\pi^2}. \tag{27}$$

Therefore, our small amplitude approximation is

$$T_a(A) = \frac{4A}{[\mathrm{e}^{4A^2/\pi^2} - 1]^{1/2}}. \tag{28}$$

Table 1 presents the comparison of this approximation to an accurate numerical integration of the period relation given by the integral of Eq. (8). As expected, the error is quite acceptable for amplitude values in the range $0 \leqslant A \lesssim 5$.

In summary, an exact integral formula has been derived for the period of the nonlinear oscillator given by Eq. (1). Examination of this integral and related properties of the period, $T(A)$, shows that it is continuous for all values of $A \geqslant 0$; has a negative derivative, i.e., $\mathrm{d}T(A)/\mathrm{d}A < 0$; and decreases monotonic to zero, with $T(A) > 0$ for $0 \leqslant A < \infty$. Since the angular frequency is

$$\omega(A) = \frac{2\pi}{T(A)}, \tag{29}$$

these results collectively demonstrate that $\omega(A)$ has singularity for $0 \leqslant A < \infty$. Hence, the restrictions on the range of applicable A values for $\omega(A)$, obtained by use of various perturbation procedures [1–5], are only indications of their limitations for calculating the angular frequency for the nonlinear oscillator given by Eq. (1). A future problem is to obtain for Eq. (8) an asymptotic representation of the period for large A.

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References

[1] A. Chatterjee, Harmonic balance based averaging: approximate realizations of an asymptotic technique, *Nonlinear Dynamics* 32 (2003) 323–343.

[2] R.E. Mickens, *Oscillations in Planar Dynamic Systems*, World Scientific, Singapore, 1996.
[3] R.E. Mickens, Iteration procedure for determining approximate solutions to non-linear oscillator equations, *Journal of Sound and Vibration* 116 (1987) 185–187.
[4] R.E. Mickens, A generalized iteration procedure for calculating approximations to periodic solutions of "truly non-linear oscillators", *Journal of Sound and Vibration* 287 (2005) 1045–1051.
[5] J. Beatty, R.E. Mickens, A qualitative study of the solutions to the differential equation $\ddot{x} + (1 + \dot{x}^2) = 0$, *Journal of Sound and Vibration* 283 (2005) 475–477.
[6] H.B. Dwight, *Tables of Integrals and Other Mathematical Data*, MacMillan, New York, 1961.
[7] A.E. Taylor, W.R. Mann, *Advanced Calculus*, third ed., Wiley, New York, 1983.