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Acoustic pulse reflection at a time-reversal mirror

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Abstract

To investigate the physical properties of fields reflected by time-reversal mirrors, one resorts to a gedanken experiment where a mirror realizes exactly the time inversion of the incident field. Working with incident rectangular and windowed harmonic pulses and using an integral equation approach recently developed to deal with scattering from obstacles, the reflection of plane and spherical pulses on time-reversal mirrors is analysed. It is proved that according to the form of the incident field, such mirrors may be transparent or behave as a dual source of pulses propagating in the opposite direction to incident pulses.

1. Introduction

Time-reversal mirrors (TRM) described by Fink [1] and Deroche et al. [2], are used to convert an acoustic wave field from a source into a wave field at the source position. The areas of applications include medical imagery, lithotripsy and nondestructive testing. The practical realization of TRM is discussed at length in Fink [1] and mathematical works on their properties are flourishing, among the most recent ones being Bardos and Fink [3], Bal and Ryzbik [4] and Klibanov and Timonov [5].

This work is concerned with the theoretical properties of TRM waves and, as in the gedanken experiment devised by Stokes many years ago, mentioned in Fink [1] and Hecht [6], an ideal TRM carrying out exactly the inversion of time T is devised to investigate plane wave reflection and transmission at a time-reversal interface. Using an integral equation approach to deal with scattering from obstacles, investigations focus on rectangular-windowed harmonic plane and spherical pulses with as objective the reflection laws of these acoustic pulses on TRMs.

The integral equation approach in this work is not the conventional one. This point is made clear on the scattering of harmonic plane waves, solutions of the Helmholtz equation $\nabla^2\psi + k^2\psi = 0$, by a perfectly reflected surface located in the $z = 0$ plane. The Sommerfeld terminology is used: $\mathbf{x} = (x, y, z), \mathbf{x}' = (x', y', z')$, denote, respectively, the action and source points for the Green's functions $G(\mathbf{x}, \mathbf{x}'; k)$ acting as kernels in integral equations; similarly, Σ, Σ', denote the surface S of action and source points.

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Then, assuming $\partial_{z'}\psi(\mathbf{x}',k)$ known on the plane $\Sigma' = \{z' = 0\}$, the classical formalism, used for instance by Sneddon [7], starts with the integral equation

$$\psi_s(\mathbf{x},k) = \iint_{-\infty}^{\infty} \mathrm{d}x'\,\mathrm{d}y'[\partial_{z'}\psi(\mathbf{x}',k)G_N(\mathbf{x},\mathbf{x}',k)]_{\Sigma'},$$

where the Green's function satisfies the Neumann boundary condition on Σ', $[\partial_{z'}G_N(\mathbf{x},\mathbf{x}',k)]_{\Sigma'} = 0$. If $\psi(\mathbf{x}',k)$ is known on Σ', the integral equation is defined with $\partial_{z'}G_D(\mathbf{x},\mathbf{x}',k)$, $\Sigma' G_D$ satisfying the Dirichlet boundary condition $[G_D(\mathbf{x},\mathbf{x}',k)]_{\Sigma'} = 0$.

In the approach recently developed by Hillion [8], the total field $\psi(\mathbf{x},k) = \psi_i(\mathbf{x},k) + \psi_r(\mathbf{x},k)$ and the Green's function $g_N(\mathbf{x},\mathbf{x}',k)$ satisfy the Neumann boundary condition on the plane $\Sigma = \{z = 0\}$ and the integral equation is

$$\psi(\mathbf{x},k) = -\iint_{-\infty}^{\infty} \mathrm{d}x'\,\mathrm{d}y'[\psi(\mathbf{x}',k)\partial_{z'}g_N(\mathbf{x},\mathbf{x}',k)]_{\Sigma'}.$$

If ψ and g_D satisfy on Σ the Dirichlet boundary condition, the integral equation is defined with $\partial_{z'}\psi$ and g_D.

In short, fields and Green's functions are imposed on Σ' in the conventional approach and on Σ in the second approach.

Now for acoustic pulses as considered here, solutions of the wave equation $\nabla^2\psi - c^{-2}\partial_t^2\psi = 0$ it is better to work with the Laplace transform $\psi(\mathbf{x},s)$ of fields and $\nabla^2\psi - c^{-2}s^2\psi = 0$ of the wave equation to avoid mathematical difficulties. Then, changing formally ik into s in the previous integral equations provides an approach to the scattering of pulses by planes on condition to justify the formal exchange $k \Rightarrow \mathrm{i}s$ and to perform the inverse Laplace transform of $\psi(\mathbf{x},s)$.

This paper is organized as follows: Section 2 is devoted to a presentation of the integral equation approach used to analyse reflection from surfaces on which the total field satisfies the Neumann or the Dirichlet boundary condition. Section 3 is concerned with the reflection of 2D-rectangular, unit step and windowed plane harmonic pulses both on conventional and TRMs while the reflection of windowed harmonic spherical pulses is discussed in Section 4. Conclusive comments are given in Section 5 and Appendix A and B complete this paper.

2. Integral equation approach

Since the time inversion $T : t \Rightarrow -t$ is exactly carried out by the ideal TRM considered here and since rectangular and truncated pulses require unit step functions U, it is needed to analyse the behaviour of U not only under T but also under the parity operator $P : \mathbf{x} \Rightarrow -\mathbf{x}$. That the Dirac distribution δ is an even function implies

$$T\{\delta(ct - z)\} = \delta(-ct - z) = \delta(ct + z) = P\{\delta(ct - z)\}, \tag{1}$$

so that $PT = I$ the identity operator. For the unit step function U, the following definition is used:

$$U(x) = \int_{\infty}^{x} \delta(\xi)\,\mathrm{d}\xi = 0 \quad \text{for } x<0,\ 1 \text{ for } x>0, \tag{2}$$

which entails $U(-x) = 1 - U(x)$, so

$$T\{U(ct - z)\} = U(-ct - z) = 1 - U(ct + z) = 1 - P\{U(ct - z)\}. \tag{3}$$

From now on, pulses are supposed launched at $z = 0$ from a source located at $z_0 > 0$ above a mirror in the plane $z = 0$: so, along this work t and z are positive and the Sommerfeld terminology defined in the introduction is used, c is the sound velocity and acoustic pulses are solutions of the wave equation $\nabla^2\psi - c^{-2}\partial_t^2\psi = 0$.

Then, the total field, incident plus reflected is denoted $\psi(\mathbf{x},t)$

$$\psi(\mathbf{x},t) = \psi_i(\mathbf{x},t) + \psi_r(\mathbf{x},t), \tag{4}$$

ψ and the Green's function g are supposed to satisfy on the plane $\Sigma = \{z = 0\}$ the Neumann (hard) boundary condition

$$[\partial_z \psi(\mathbf{x}, t)]_\Sigma = 0, \quad [\partial_z g(\mathbf{x}, t; \mathbf{x}', t')]_\Sigma = 0. \tag{5}$$

The integral equation approach summarized in the introduction (see Ref. [9] for an application to acoustics) is in fact a technique to solve boundary-value problems of the wave equation for which integral equations with Green's functions as kernels are known for a long time: see, for instance, Courant and Hilbert [10]. For a field satisfying the Neumann boundary condition on the plane $z = 0$, the integral equation is given by Hillion [8] as

$$\psi(\mathbf{x}, t) = -\int_{-\infty}^{\infty} \mathrm{d}t' \iint_{-\infty}^{\infty} \mathrm{d}z'\, \mathrm{d}y' [\psi(\mathbf{x}', t')\partial_{z'} g(\mathbf{x}, t; \mathbf{x}', t')]_{z'=0}, \tag{6}$$

where $g(\mathbf{x}, t; \mathbf{x}', t')$ is the inverse Laplace transform of the space Green's function $G(\mathbf{x}, \mathbf{x}'; s)$ given by Schlegrov and Scott Carney [11] and justifying the formal change $\mathrm{i}k \Rightarrow s$

$$g(\mathbf{x}, t; \mathbf{x}', t') = (\mathrm{i}c/16\pi^3) \int_{Br} \mathrm{d}s \exp[s(ct - ct')] G(\mathbf{x}, \mathbf{x}'; s), \tag{7}$$

$$G(\mathbf{x}, \mathbf{x}'; s) = \iint_{-\infty}^{\infty} \mathrm{d}\beta\, \mathrm{d}\gamma\, s_z^{-1} \exp[\mathrm{i}\beta(x - x') + \mathrm{i}\gamma(y - y')]\{\exp(s_z|z - z'|) + \exp(s_z|z + z'|)\} \tag{7a}$$

with $s_z = (s^2 + \beta^2 + \gamma^2)^{1/2}$. The Bromwich contour Br in integral (7) is made of a line L parallel to the imaginary axis of the s-plane with all the singularities of the integrand on its left and of a half circle.

A simple calculation, see Hillion [8], gives on the plane $\Sigma' = \{z' = 0\}$

$$[\partial_{z'} g(\mathbf{x}, t; \mathbf{x}', t')]_{z'=0} = (\mathrm{i}c/8\pi^3) \int_{Br} \mathrm{d}s \exp[s(ct - ct')] G_0(\mathbf{x}, \mathbf{x}'; s), \tag{8}$$

$$G_0(\mathbf{x}, \mathbf{x}'; s) = \iint_{-\infty}^{\infty} \mathrm{d}\beta\, \mathrm{d}\gamma \exp[\mathrm{i}\beta(x - x') + \mathrm{i}\gamma(y - y')] \cosh(s_z z) \tag{8a}$$

and taking into account Eq. (8) relation (6) becomes for $0 \leqslant t < \infty$

$$\psi(\mathbf{x}, t) = (1/8\mathrm{i}\pi^3) \int_{Br} \mathrm{d}s \exp(sct) \iint_{-\infty}^{\infty} \mathrm{d}\beta\, \mathrm{d}\gamma \exp(\mathrm{i}\beta x + \mathrm{i}\gamma y) \cosh(s_z z) F(\beta, \gamma, s), \tag{9}$$

$$F(\beta, \gamma, s) = \iint_{-\infty}^{\infty} \mathrm{d}x'\, \mathrm{d}y' \exp(-\mathrm{i}\beta x' - \mathrm{i}\gamma y') \int_{-\infty}^{\infty} c\, \mathrm{d}t' \exp(-sct')[\psi(x', t')]_{z'=0}. \tag{9a}$$

Introducing the symbol L^{-1} of the inverse Laplace transform, Eq. (9) may be written

$$\psi(\mathbf{x}, t) = L^{-1}\{\Psi(\mathbf{x}, s)\}, \quad \Psi(\mathbf{x}, s) = \iint_{-\infty}^{\infty} \mathrm{d}\beta\, \mathrm{d}\gamma \exp(\mathrm{i}\beta x + \mathrm{i}\gamma y) \cosh(s_z z) F(\beta, \gamma, s). \tag{10}$$

The integral equation for a scalar field satisfying the Dirichlet (soft) boundary condition $[\psi(\mathbf{x}, t)]_{\Sigma=0}$, $[g(\mathbf{x}, t; \mathbf{x}', t')]_\Sigma = 0$ is obtained by changing in Eq. (9) $\cosh(s_z z)$ into $\sinh(s_z z)$ and in Eq. (9a) $[\psi(\mathbf{x}', t')]_{z'=0}$ into $[s_z^{-1}\partial_{z'}\psi(\mathbf{x}', t')]_{z'=0}$. But from now on, the Dirichlet boundary condition is left aside.

This integral equation approach requires three steps

1. the definition of the total field $[\psi(\mathbf{x}', t')]_{z'=0}$ on the Σ' surface,
2. the use of Eq. (9a) to obtain the form factor $F(\beta, \gamma, s)$,
3. the substitution of $F(\beta, \gamma, s)$ into Eq. (9) to get the total field outside obstacles.

Then, supposing as just said, an acoustic point source located at $(0, 0, z_0)$, $z_0 > 0$ and launching at $t = 0$ either a rectangular or a windowed harmonic pulse, the integral equation (9) is used to get the total field after reflection on a TRM located in the $z = 0$ plane. Reflection of 2D-planar pulses is first analysed before considering spherical pulses.

3. Reflection of 2D-planar pulses

To investigate the TRM reflection of 2D-planar pulses, propagation is supposed to be in the (x, z)-plane so that the coordinates y and γ do not intervene. Then, with $\mathbf{u} = (x, z)$ and $s_z = (s^2 + \beta^2)^{1/2}$, integrals (9) and (9a) reduce for $0 \leqslant t < \infty$ to

$$\psi(\mathbf{u}, t) = (1/4\mathrm{i}\pi_2) \int_{Br} \mathrm{d}s \exp(sct) \int_{-\infty}^{\infty} \mathrm{d}\beta \exp(\mathrm{i}\beta x) \cosh(s_z z) F(\beta, s), \tag{11}$$

$$F(\beta, s) = \int_{-\infty}^{\infty} \mathrm{d}x' \exp(-\mathrm{i}\beta x') \int_{-\infty}^{\infty} c\, \mathrm{d}t' \exp(-sct')[\psi(\mathbf{u}', t')]_{z'=0}. \tag{11a}$$

In addition to make calculations easier, pulses are assumed to impinge normally on the mirror.

3.1. Rectangular pulses

A rectangular pulse of duration t_0, launched at $t = 0$ by a source located at $x = 0$, $z = z_0$, which impinges normally on a mirror in the $z = 0$ plane has the form in which U is the unit step function

$$\psi_i(z, t) = U(ct - z_0 + z) - U(ct - ct_0 - z_0 + z). \tag{12}$$

Then, on the plane $\Sigma' = \{z' = 0\}$ the incident field reduces to

$$\psi_i(0, t') = U(ct' - z_0) - U(ct' - ct_0 - z_0) \tag{13}$$

and to complete the first step of the integral equation approach, the expression of the reflected field on this plane is required.

On a conventional mirror $\psi_r(0, t') = \psi_i(0, t')$ and the total field is

$$\psi(0, t') = \psi_r(0, t') + \psi_i(0, t') = 2\psi_i(0, t'). \tag{13a}$$

So using Eq. (13), the second integral in Eq. (11a) with $a = z_0$, $b = ct_0 + z_0$ bounds of the interval inside which $\psi(0, t') \neq 0$, takes the form

$$\begin{aligned} \int_{-\infty}^{\infty} c\, \mathrm{d}t' \exp(-sct')\psi(0, t') &= 2 \int_a^b \exp(-sct') c\, \mathrm{d}t' \\ &= 2 s^{-1} \exp(-sz_0)[1 - \exp(-sct_0)] \end{aligned} \tag{14}$$

and integral (11a) becomes

$$F(\beta, s) = 4\pi\delta(\beta) s^{-1} \exp(-sz_0)[1 - \exp(-sct_0)]. \tag{15}$$

Taking into account the definition of s_z, the β-integral in Eq. (11) is

$$\int_{-\infty}^{\infty} \mathrm{d}\beta \exp(\mathrm{i}\beta x) \cosh(s_z z) F(\beta, s) = 2\pi s^{-1} \exp(-sz_0)[\exp(-sz) + \exp(sz)][1 - \exp(-sct_0)] \tag{16}$$

and using Eq. (16) expression (11) may be written

$$\psi(z, t) = L^{-1}\{\Psi_-(z, s)\} + L^{-1}\{\Psi_+(z, s)\}, \tag{17}$$

$$\Psi_\pm(z, s) = s^{-1} \exp(-sz_0 \pm sz)[1 - \exp(-sct_0)]. \tag{17a}$$

But from the well-known property of the Laplace transform, see, for instance, Doetsch [12] and Erdelyi [13] for $a > 0$, $t \geqslant 0$

$$L^{-1}\{F(s)\} = f(ct) \Rightarrow L^{-1}\{\exp(-as)F(s)\} = f(ct - a)U(ct - a) \tag{18}$$

and since $L^{-1}\{1/s\} = U(ct)$ a simple calculation gives

$$L^{-1}\{\Psi_\pm(z, s)\} = U(ct - z_0 \pm z) - U(ct - ct_0 - z_0 \pm z). \tag{19}$$

Finally, substituting Eq. (19) into Eq. (17) supplies the total field with ψ_r obtained from ψ_i by changing z into $-z$

$$\psi(z,t) = \psi_i(z,t) + \psi_r(z,t). \tag{20}$$

This result, in agreement with the Descartes–Snell law, proves the consistency of the integral equation approach of Section 2.

Now what happens on a TRM? According to Eqs. (3) and (13) the reflected field on the $z'=0$ plane for incident pulse (12) is

$$\begin{aligned}\psi_r^*(0,t') = T\psi_i(0,t') &= U(-ct'-z_0) - U(-ct'+ct_0-z_0)\\ &= -[U(ct'+z_0) - U(ct'-ct_0+z_0)]\end{aligned} \tag{21}$$

and from now on, all the quantities pertaining to TRM reflections are starred. Note that $U(ct'+z_0)=1$ since ct' and z_0 are positive and similarly $U(ct'-ct_0+z_0)=1$ if $z_0>ct_0$. In this case $\psi_r^*(0,t')=0$: there is no reflection and the TRM appears as transparent.

So, assuming $ct_0>z_0$, and taking into account Eqs. (13) and (21), the total field on the Σ'-plane which completes the first step of the integral equation approach, has the form

$$\begin{aligned}\psi^*(0,t') &= \psi_i(0,t') - [1 - U(ct'-ct_0+z_0)]\\ &= (1/2)\psi(0,t') + (1/2)\phi(0,t')\end{aligned} \tag{22}$$

in which $\psi(0,t')$ is given by Eq. (13a) and

$$\phi(0,t') = -2[1 - U(ct'-ct_0+z_0)]. \tag{22a}$$

Relation (22) implies that the total field outside the mirror is

$$\psi^*(z,t) = (\tfrac{1}{2})\psi(z,t) + (\tfrac{1}{2})\phi(z,t) \tag{23}$$

with $\psi(z,t)$ supplied by Eq. (20) so that it is sufficient to look for the contribution of $\phi(0,t')$ to Eq. (11).

Now according to Eq. (22a), the second integral in Eq. (11a) where $Z = ct_0 - z_0$ is since $\phi = 0$ for $t'>Z$

$$\begin{aligned}\int_0^\infty c\,\mathrm{d}t' \exp(-sct')\phi(0,t') &= -2\int_0^Z \exp(-sct')c\,\mathrm{d}t\\ &= -2s^{-1}\{1-\exp[-s(ct_0-z_0)]\}\end{aligned} \tag{24}$$

and with Eq. (24), integral (11a) becomes

$$F(\beta,s;\phi) = -4\pi\delta(\beta)s^{-1}\{1-\exp[-s(ct_0-z_0)]\}. \tag{25}$$

Substituting Eq. (25) into the β-integral of Eq. (11) gives the contribution $\phi(z,t)$ to the total pulse in the form

$$\phi(z,t) = L^{-1}\{\Phi_-(z,s)\} + L^{-1}\{\Phi_+(z,s)\}, \tag{26}$$

$$\Phi_\pm(z,s) = -s^{-1}\exp(\pm sz)\{1-\exp[-s(ct_0-z_0)]\}. \tag{26a}$$

With Eq. (18) and $L^{-1}\{1/s\} = U(ct)$, the inverse Laplace transform of $\Phi_-(z,s)$ is easy to perform

$$\phi_-(z,t) = L^{-1}\{\Phi_-(z,s)\} = -[U(ct-z) - U(ct-ct_0+z_0-z)], \tag{27a}$$

but the inverse Laplace transform $L^{-1}\{\Phi_+(z,s)\}$ is more difficult to obtain because

$$\phi_+(z,t) = L^{-1}\{\Phi_+(z,s)\} = L^{-1}\{s^{-1}\exp(sz)\} + U(ct-ct_0+z_0+z) \tag{27b}$$

and because the only relation obtained from tables of inverse Laplace transforms for $a\geqslant 0$ is given by Doetsch [12]

$$L^{-1}\left\{\exp(as)\left[F(s) - \int_0^a \exp(-sc\tau)f(\tau)c\,\mathrm{d}t\right]\right\} = f(ct+a), \quad a\geqslant 0, \tag{28}$$

which is of no help to get $L^{-1}\{s^{-1}\exp(sz)\}$. Then, using elaborate properties of the Laplace transform given in Ref. [13] and the analytical representation of Dirac distributions due to Bremermann [14], the following simple result is obtained in Appendix A (where to simplify calculations c is made unity)

$$L^{-1}\{\exp(as)\} = 2\delta(ct+a), \quad a \geqslant 0 \tag{28a}$$

satisfying Eq. (28) since $F(s)=2$ and $\int_0^a \exp(-sc\tau)\delta(\tau)c\,\mathrm{d}\tau = 1$ for $f(t)=2\delta(ct)$ so that since $z,t \geqslant 0$

$$L^{-1}\{s^{-1}\exp(sz)\} = 2c\int_0^{ct}\delta(c\tau+z)\,\mathrm{d}\tau = 0. \tag{28b}$$

Then, substituting Eq. (28b) into Eq. (27b) gives

$$\phi_+(z,t) = L^{-1}\{\Phi_+(z,s)\} = U(ct-ct_0+z_0+z), \tag{29}$$

which implies together with Eq. (27a)

$$\phi(z,t) = U(ct-ct_0+z_0+z) + U(ct-ct_0+z_0-z) - U(ct-z). \tag{29a}$$

Substituting finally Eqs. (20) and (29a) into Eq. (23), the total pulse due to the TRM reflection of a rectangular pulse is

$$\psi^*(z,t) = [\psi_i(z,t)+\psi_r(z,t)]/2 + \{U(ct-ct_0+z_0+z)+U(ct-ct_0+z_0-z)-U(ct-z)\}/2. \tag{30}$$

The physical meaning of Eq. (30) valid for $ct_0 > z_0$ is discussed in Section 5.

3.1.1. Remark

It has been stated that there is no TRM reflection for $ct_0 < z_0$ and that the TRM mirror becomes transparent. In this situation, integral equation (11) cannot be used since boundary condition (5) is not fulfilled by $\psi(z,t)$. Instead, the convenient integral equation is defined with the free space Green's function $g_0(\mathbf{u},t;\mathbf{u}',t')$ used in Ref. [8], so that

$$\psi(\mathbf{u},t) = (1/2\mathrm{i}\pi^2)\int_{Br}\mathrm{d}s\exp(sct)\int_{-\infty}^{\infty}\mathrm{d}\beta\exp(\mathrm{i}\beta x - sz)F(\beta s) \tag{31}$$

with $F(\beta,s)$ still given by Eq. (11a). For the rectangular pulse (12), $F(\beta,s)$ is expression (15) divided by two and in Eq. (16) the exponential $\exp(sz)$ does not exist. Then, the same calculations leading from Eqs. (17) to (20) show that Eq. (31) supplies $\psi_i(z,t)$, in agreement with the previous statement.

3.2. Unit-step plane harmonic pulse

A unit-step plane harmonic pulse launched at $t=0$ from $(0,z_0)$ in the z-direction has the form

$$\psi_i(z,t) = \exp[\mathrm{i}\omega(ct-z_0+z)]U(ct-z_0+z), \tag{32}$$

which becomes on the $z'=0$ plane

$$\psi_i(0,t') = \exp[\mathrm{i}\omega(ct'-z_0)]U(ct'-z_0). \tag{32a}$$

On a conventional mirror: $\psi_r(0,t')=\psi_i(0,t')$, so that the total field is

$$\psi(0,t') = 2\exp[\mathrm{i}\omega(ct'-z_0)]U(ct'-z_0) \tag{33}$$

with $\sigma = s-\mathrm{i}\omega$ and the lower bound $a=z_0$ since $U=0$ for $ct'<z_0$, the second integral in Eq. (11a) becomes

$$\begin{aligned}\int_{-\infty}^{\infty}c\,\mathrm{d}t'\exp(-sct')\psi(0,t') &= 2\exp(-\mathrm{i}\omega z_0)\int_a^{\infty}\exp(-\sigma t')c\,\mathrm{d}t' \\ &= 2\exp(-z_0 s)/\sigma.\end{aligned} \tag{34}$$

Substituting Eq. (34) into Eq. (11a) gives

$$F(\beta,s) = 4\pi\delta(\beta)\exp(-sz_0)/\sigma \tag{35}$$

and with Eq. (35), the β-integral of Eq. (11) may be written

$$\psi(z,t) = L^{-1}\{\Psi_-(z,s)\} + L^{-1}\{\Psi_+(z,s)\}, \tag{36}$$

$$\Psi_\pm(z,s) = \exp[-s(z_0 \pm z)]/\sigma. \tag{36a}$$

Then, still using Eq. (18) the inverse Laplace transform of $\Psi_\pm(z,s)$ is since $L^{-1}\{(1/\sigma)\} = \exp(i\omega t)$

$$L^{-1}\{\Psi_\pm(z,s)\} = \exp[i\omega(ct - z_0 \pm z)]U(ct - z_0 \pm z) \tag{37}$$

and as expected from the Descartes–Snell law, ψ_r being obtained by changing z into $-z$ in ψ_i the total field is

$$\psi(z,t) = \psi_i(z,t) + \psi_r(z,t). \tag{37a}$$

Now on a TRM mirror, the reflected field is according to Eqs. (3) and (32a)

$$\begin{aligned}\psi_r^*(0,t') = T\psi_i(0,t') &= \exp[-i\omega(ct' + z_0)]U(-ct' - z_0) \\ &= \exp[-i\omega(ct' + z_0)][1 - U(ct' + z_0)]\end{aligned} \tag{38}$$

and $\psi_r^*(0,t') = 0$ since $U(ct' + z_0) = 1$: a TRM mirror is transparent to a unit step plane harmonic pulse.

3.3. Rectangular plane harmonic pulse

A rectangular harmonic pulse of duration t_0 launched at $t = 0$ from $(0, z_0)$ and impinging normally on the mirror in the plane $z = 0$ has the form

$$\psi_i(z,t) = \exp[i\omega(ct - z_0 + z)]V(t,z;t_0,z_0), \tag{39}$$

$$V(t,z;t_0,z_0) = U(ct - z_0 + z) - U(ct - ct_0 - z_0 + z), \tag{39a}$$

so that on the $z' = 0$ plane

$$\psi_i(0,t') = \exp[i\omega(ct' - z_0)]V(t',0;t_0,z_0) \tag{40}$$

and, as in the previous two sections, the first step for using the integral equation approach of Section 2 is to obtain the total field $\psi(0,t')$ on the $z' = 0$ plane.

For a conventional mirror, $\psi_r(0,t') = \psi_i(0,t')$ and the total field is

$$\psi(0,t') = 2\exp[i\omega(ct' - z_0)]V(t',0;t_0,z_0), \tag{41}$$

so with $\sigma = s - i\omega$ and $a = z_0$, $b = z_0 + ct_0$ bounds of the interval inside which $V \neq 0$, the second integral in expression (11a) of $F(\beta,s)$ becomes

$$\begin{aligned}\int_{-\infty}^{\infty} c\,dt' \exp(-sct')\psi(0,t') &= 2\exp(-i\omega z_0)\int_a^b \exp(-\sigma t')c\,dt' \\ &= 2\,\mathrm{Exp}_c(-\sigma ct_0)\exp(-z_0 s)/\sigma,\end{aligned} \tag{42}$$

in which Exp_c is the function

$$\mathrm{Exp}_c(x) = 1 - \exp(x). \tag{42a}$$

Substituting Eq. (42) into Eq. (11a) gives at once

$$F(\beta,s) = 4\pi\delta(\beta)\,\mathrm{Exp}_c(-\sigma ct_0)\exp(-z_0 s)/\sigma \tag{43}$$

and the β-integral in Eq. (11) becomes with Eq. (43)

$$\psi(z,t) = L^{-1}\{\Psi_-(z,s)\} + L^{-1}\{\Psi_+(z,s)\}, \tag{44}$$

$$\Psi_\pm(z,s) = \mathrm{Exp}_c(-\sigma ct_0)\exp[-s(z_0 \pm z)]/\sigma. \tag{44a}$$

Using Eq. (42a) and the definition $s - i\omega$ of σ gives

$$\Psi_\pm(z,s) = \exp[-s(z \pm z_0)]/\sigma - \exp(i\omega ct_0)\exp[-s(ct_0 + z \pm z_0)]/\sigma, \tag{44b}$$

while according to Eq. (18) and since $L\{1/\sigma\} = \exp(i\omega ct)$

$$L^{-1}\{\exp[-s(z_0 \pm z)]/\sigma\} = \exp[i\omega\{ct - (z_0 \pm z)\}]U[ct - (z_0 \pm z)] = \chi_\pm(z,t), \tag{45a}$$

$$L^{-1}\{\exp[-s(ct_0 + z \pm z_0)]/\sigma\} = \chi_\pm[z,(t - t_0)]. \tag{45b}$$

Then, taking into account Eqs. (44) and (44a)

$$\begin{aligned}\psi_\pm(z,t) &= \chi_\pm(z,t) - \exp(\mathrm{i}\omega ct_0)\chi_\pm[z,(t-t_0)]\\ &= \exp[\mathrm{i}\omega(ct - z_0 \pm z)]V(t,\pm z;t_0,z_0)\end{aligned} \tag{46}$$

in which V is function (39a), so that $\psi_+ = \psi_i, \psi_- = \psi_r$ still in agreement with the Descartes–Snell law

$$\psi(z,t) = \psi_i(z,t) + \psi_r(z,t). \tag{47}$$

3.3.1. Remark

Due to the properties of the V function, ψ_i exists only for $0 \leqslant z \leqslant z_0$ at times in the interval $0 \leqslant t \leqslant t_1 + t_0$ with $t_1 = z_0/c$ while ψ_r propagates in the region $z \geqslant 0$ at times $t \geqslant t_1$.

Now for a TRM, the reflected field on the $z' = 0$ plane is according to Eq. (40)

$$\psi_r^*(0,t') = T\psi_i(0,t') = \exp[-\mathrm{i}\omega(ct' + z_0)]V(-t',0;-t_0,z_0) \tag{48}$$

with from Eqs. (3) and (39a)

$$V(-t',0,0;-t_0,z_0) = -[U(ct'+z_0) - U(ct' - ct_0 + z_0)] \tag{48a}$$

in which $ct_0 > z_0$, otherwise $V = 0$ and the TRM mirror would be transparent.

Then, with $\psi(0,t')$ given by Eq. (41), the total field on the mirror $\psi^* = \psi_i + \psi_r^*$ may be written

$$\psi^*(0,t') = (\tfrac{1}{2})\psi(0,t') + (\tfrac{1}{2})\phi(0,t') \tag{49}$$

with according to Eqs. (48) and (48a) in which $U(ct' + z_0) = 1$

$$\phi(0,t') = -2\exp[-\mathrm{i}\omega(ct' + z_0)][1 - U(ct' - ct_0 + z_0)], \tag{49a}$$

so that the total pulse outside the mirror takes the form

$$\psi^*(z,t) = \psi(z,t)/2 + \phi(z,t)/2, \tag{50}$$

where $\psi(z,t)$ is expression (47). So, it is just sufficient to look for the contribution of Eq. (49a) to form factor (11a). Introducing the variable $\sigma^\dagger = s + \mathrm{i}\omega$ and substituting Eq. (49a) into the second integral of Eq. (11a) gives with the upper bound $Z = ct_0 - z_0$

$$\begin{aligned}\int_0^\infty c\,\mathrm{d}t' \exp(-sct')\phi(0,t') &= -2\exp(-\mathrm{i}\omega z_0)\left[\int_0^Z \exp(-\sigma^\dagger ct')c\,\mathrm{d}t'\right]\\ &= -2[\exp(-\mathrm{i}\omega z_0) - \exp(-\sigma^\dagger ct_0 + sz_0)]/\sigma^\dagger.\end{aligned} \tag{51}$$

With Eq. (51) the form factor (11a) becomes

$$F_\phi(\beta,s) = -4\pi\delta(\beta)[\exp(-\mathrm{i}\omega z_0) - \exp(-\mathrm{i}\omega ct_0)\exp\{-s(ct_0 - z_0)\}]/\sigma^\dagger \tag{52}$$

and the contribution $\phi(z,t)$ to the total pulse is with Eq. (52) substituted to $F(\beta,s)$ in the β-integral of Eq. (11) using the definition of $\cosh z$ in terms of $\exp(\pm z)$

$$\phi(z,t) = L^{-1}\{\Phi_-(z,s)\} + L^{-1}\{\Phi_+(z,s)\}, \tag{53}$$

$$\Phi_\pm(z,s) = -\exp(-\mathrm{i}\omega z_0)\exp(\pm sz)/\sigma^\dagger + \exp(-\sigma^\dagger ct_0)\exp[s(z_0 \pm z)]/\sigma^\dagger. \tag{53a}$$

Then, using Eqs. (18), (28a) and $L^{-1}\{1/\sigma^\dagger\} = \exp(-\mathrm{i}\omega t)$ the expressions of $\phi_\pm(z,t)$ are as in Eqs. (27a) and (29a)

$$\begin{aligned}\phi_-(z,t) &= L^{-1}\{\Phi_-(z,s)\} = -\exp[-\mathrm{i}\omega(ct + z_0 - z)][U(ct - z) - U(ct - ct_0 + z_0 - z)],\\ \phi_+(z,t) &= L^{-1}\{\Phi_+(z,s)\} = \exp[-\mathrm{i}\omega(ct + z_0 + z)]U(ct - ct_0 + z_0 + z).\end{aligned} \tag{54}$$

Substituting Eqs. (47) and (54) into Eq. (50) gives the total pulse due to the TRM reflection of a rectangular plane harmonic field provided that $ct_0 > z_0$

$$\psi^*(z,t) = (\tfrac{1}{2})[\psi_i(z,t) + \psi_r(z,t)] + (\tfrac{1}{2})[\phi_-(z,t) + \phi_+(z,t)]. \tag{55}$$

3.3.2. Remark

A pulse incident from $(0, z_0)$ in the θ-direction has the expression with $Z = x\sin\theta + (z - z_0)\cos\theta$

$$\psi(\mathbf{u},t) = \exp[i\omega(ct - Z)][U(ct - Z) - U(ct - ct_0 - Z)] \tag{56}$$

leading just to change z and z_0 into $z\cos\theta + x\sin\theta$ and $z_0\cos\theta$ in the previous results.

The physical meaning of these results is discussed in Section 5.

4. Reflection of windowed spherical harmonic pulses

The acoustic point source still on the z-axis at the altitude z_0 is now supposed to launch at $t = 0$ a truncated spherical harmonic pulse with duration t_0

$$\psi_i(\rho,z,t) = \exp[i\omega(ct - r_-)]V(t,r_-)/r_-, \quad V(t,r_-) = [U(ct - r_-) - U(ct - ct_0 - r_-)], \tag{57}$$

$$r_\pm \equiv r_\pm(\rho,z) = [\rho^2 + (z \pm z_0)^2]^{1/2}, \quad r_\pm(0,z) = z_0 \pm z, \quad \rho^2 = x^2 + y^2. \tag{57a}$$

3D-integral equation (9) has now to be used and the first step to get form factor (9a) is to define the total field $\psi_i(\rho',0,t')$ on the plane $\Sigma' = \{z' = 0\}$. According to Eq. (57), the incident field is

$$\psi_i(\rho',0,t') = \exp[i\omega(ct' - r')]V(t',r')/r', \tag{58}$$

$$r' = (\rho'^2 + z_0^2)^{1/2}, \quad V(t',r') = U(ct' - r') - U(ct' - ct_0 - r') \tag{58a}$$

and, for the reflected field $\psi_r(\rho',0,t')$, conventional and TRMs are considered apart.

4.1. Reflection on a conventional mirror

The reflected pulse on the $z' = 0$ plane is $\psi_r(\rho',0,t') = \psi_i(\rho',0,t')$ and the total field according to Eq. (58)

$$\psi(\rho',0,t') = 2\exp[i\omega(ct' - r')]V(t',r')/r'. \tag{59}$$

Then, with $\sigma = s - i\omega$ and $a = r'$, $b = r' + ct_0$, bounds of the interval inside which $V \neq 0$, the second integral in Eq. (9a) is

$$\begin{aligned}\int_{-\infty}^{\infty} c\,dt' \exp(-sct')\psi(\rho',0,t') &= (2/r')\exp(-i\omega r')\int_a^b \exp(-\sigma ct')c\,dt' \\ &= 2\,\mathrm{Exp}_c(-\sigma ct_0)\exp(-sr')/\sigma r'\end{aligned} \tag{60}$$

in which Exp_c is function (42a). Substituting Eq. (60) into Eq. (9a) gives

$$F(\beta,\gamma,s) = (1/\sigma)\,\mathrm{Exp}_c(-\sigma ct_0)F_0(\beta,\gamma,s), \tag{61}$$

$$F_0(\beta,\gamma,s) = 2\iint_{-\infty}^{\infty} dx'\,dy' \exp(-i\beta x' - i\gamma y' - sr')/r'. \tag{61a}$$

Introducing the polar coordinates

$$x' = \rho'\cos u', \quad y' = \rho'\sin u', \quad x = \rho\cos u, \quad y = \rho\sin u, \quad \beta = \mu\cos\theta, \quad \gamma = \mu\sin\theta, \tag{62}$$

integral (61a) becomes

$$F_0(\beta,\gamma,s) = 2\int_0^\infty \rho'\,\mathrm{d}\rho' \exp(-sr')/r' \int_0^{2\pi} \mathrm{d}t' \exp[\mathrm{i}\mu r' \cos(u'-\theta)]$$
$$= 4\pi \int_0^\infty \exp(-sr')J_0(\mu\rho')\rho'\,\mathrm{d}\rho'/r' \tag{63}$$

in which J_0 is the Bessel function of the first kind of order zero. But, taking into account the definition (58a) of r' implies that Eq. (63) is a Sonine–Gegenbauer integral and according to Watson [15]

$$F_0(\beta,\gamma,s) = 4\pi(s^2+\mu^2)^{-1/2}\exp[-(s^2+\mu^2)^{1/2}z_0] \tag{63a}$$

and form factor (61) becomes

$$F(\mu,s) = 4\pi\sigma^{-1}(s^2+\ \mu^2)^{-1/2}\,\mathrm{Exp}_c(-\sigma ct_0)\exp[-(s^2+\mu^2)^{1/2}z_0]. \tag{64}$$

Now, with coordinates (62), since $s_z = (s^2+\mu^2)^{1/2}$ and $\cosh(s_z z) = [\exp(s_z z)+\exp(-s_z z)]/2$ integral (9) may be written

$$\psi(\rho,z,t) = \psi_+(\rho,z,t) + \psi_-(\rho,z,t), \tag{65}$$

$$\psi_\pm(\rho,z,t) = (1/16\mathrm{i}\pi^3)\int_{Br}\mathrm{d}s\exp(sct)\int_0^\infty \mu\,\mathrm{d}\mu\int_0^{2\pi}\mathrm{d}\theta\exp[\mathrm{i}\mu\rho\cos(u-\theta)]\exp[\pm z(s^2+\mu^2)^{1/2}]F(\mu,s)$$
$$= (1/8\mathrm{i}\pi^2)\int_{Br}\mathrm{d}s\exp(sct)\int_0^\infty \mu\,\mathrm{d}\mu J_0(\mu\rho)\exp[\pm z(s^2+\mu^2)^{1/2}]F(\mu,s) \tag{65a}$$

and substituting Eq. (64) into Eq. (65a) gives

$$\psi_\pm(\rho,z,t) = (1/2\mathrm{i}\pi)\int_{Br}\sigma^{-1}\,\mathrm{d}s\exp(sct)\,\mathrm{Exp}_c(-\sigma ct_0)\Psi_\pm(\rho,z,s), \tag{66}$$

$$\Psi_\pm(\rho,z,s) = \int_0^\infty \mu\,\mathrm{d}\mu(s^2+\mu^2)^{-1/2}J_0(\mu\rho)\exp[-(s^2+\mu^2)^{1/2}(z_0\pm z)], \tag{66a}$$

which is still a Sonine–Gegenbauer integral with according to Watson [15]

$$\Psi_\pm(\rho,z,s) = \exp(-sr_\pm)/r_\pm, \quad r_\pm = [\rho^2+(z\pm z_0)^2]^{1/2} \tag{67}$$

and with Eq. (67) integral (66) becomes

$$\psi_\pm(\rho,z,t) = (1/r_\pm)L^{-1}\{\sigma^{-1}\exp(-sr_\pm)\,\mathrm{Exp}_c(-\sigma ct_0)\} \tag{68}$$

in which

$$\mathrm{Exp}_c(-\sigma ct_0) = 1-\exp(-sct_0)\exp(\mathrm{i}\omega ct_0). \tag{68a}$$

Then, according to Eq. (18) and to $L^{-1}\{1/\sigma\} = \exp(\mathrm{i}\omega ct)$

$$L^{-1}\{\exp(-sr_\pm)/\sigma r_\pm\} = \exp[\mathrm{i}\omega(ct-r_\pm)]U(ct-r_\pm)/r_\pm \equiv \chi_\pm(r,z,t),$$

$$L^{-1}\{\exp(-sr_\pm)\exp(-sct_0)/\sigma r_\pm\} = \chi_\pm(r,z,t-t_0). \tag{69}$$

Taking into account Eqs. (68a) and (69) relation (68) has the final form

$$\psi_\pm(\rho,z,t) = \chi_\pm(\rho,z,t)-\exp(\mathrm{i}\omega ct_0)\chi_\pm(\rho,z,t-t_0). \tag{70}$$

Using definition (69) of the $\chi_\pm$ functions it is easily checked that $\psi_- = \psi_i, \psi_+ = \psi_r$, so that substituting Eq. (70) into Eq. (65) gives, for the total field generated by a windowed harmonic pulse impinging on a conventional mirror

$$\psi(\rho, z, t) = \psi_i(\rho, z, t) + \psi_r(\rho, z, t) \tag{71}$$

sum of the incident and reflected waves as expected from the Descartes–Snell law, a result justifying the consistency of integral equation (9).

4.2. Reflection on a time-reversal mirror

Now the reflected pulse on the $z' = 0$ plane $\psi_r^*(\rho', 0, t') = T\psi_i(\rho', 0, t')$ is according to Eq. (58)

$$\psi_r^*(\rho', 0, t') = \exp[-\mathrm{i}\omega(ct' + r')]V(-ct', r')/r' \tag{72}$$

with Eqs. (3) and (58a)

$$V(-ct', r') = -[U(ct' + r') - U(ct' - ct_0 + r')], \tag{72a}$$

so that since $\psi(\rho', 0, t') = 2\psi_i(\rho', 0, t')$ the total field on the mirror is

$$\psi^*(\rho', 0, t') = (\tfrac{1}{2})\psi(\rho', 0, t') + (\tfrac{1}{2})\phi(\rho', 0, t') \tag{73}$$

with $\psi(\rho', 0, t')$ given by Eq. (59) while using Eq. (70a) in which $U(ct' + r') = 1$

$$\phi(\rho', 0, t') = -(2/r')\exp[-\mathrm{i}\omega(ct' + r')][1 - U(ct' - ct_0 + r')]. \tag{73a}$$

In these relations $ct_0 > z_0$, otherwise $V(-t', r') = 0$ for any x', y': there is no reflection and the TRM is transparent to the spherical pulse.

According to Eq. (73), the total pulse outside the mirror has the form

$$\psi^*(\rho, z, t) = \psi(\rho, z, t)/2 + \phi(\rho, z, t)/2 \tag{74}$$

with $\psi(\rho, z, t)$ given by Eq. (71) so that it is sufficient to look for the contribution $\phi(\rho, z, t)$ due to $\phi(\rho', 0, t')$. Using Eq. (73a) where $\sigma^\dagger = s + \mathrm{i}\omega$ and the bounds a, b of the interval inside which $\phi \neq 0$, with $a = 0$ since $U(ct' + r') = 1$ for $t' > 0$ and $b = ct_0 - r'$ provided that $ct_0 > r'$, the second integral in the form factor (9a) becomes

$$\begin{aligned}\int_{-\infty}^{\infty} c\,\mathrm{d}t' \exp(-sct')\phi(\rho', 0, t') &= -(2/r')\exp(-\mathrm{i}\omega r')\left\{\int_0^\infty c\,\mathrm{d}t' \exp(-\sigma^\dagger ct') - \int_b^\infty c\,\mathrm{d}t' \exp(-\sigma^\dagger ct')U(b)\right\} \\ &= -(2/\sigma^\dagger r')\exp(-\mathrm{i}\omega r')[1 - \exp(-\sigma^\dagger b)U(b)].\end{aligned} \tag{75}$$

Substituting Eq. (75) into Eq. (9a) gives the ϕ-contribution to the form factor noted $F(\beta, \gamma, s;\ \phi)$

$$F(\beta, \gamma, s;\ \phi) = F_0(\beta, \gamma, s;\ \phi) + F_1(\beta, \gamma, s;\ \phi). \tag{76}$$

$F_0(\beta, \gamma, s;\ \phi)$ is the contribution of the first term in Eq. (75)

$$F_0(\beta, \gamma, s;\ \phi) = -2/\sigma^\dagger \iint_{-\infty}^{\infty} \mathrm{d}x'\,\mathrm{d}y' \exp(-\mathrm{i}\beta x' - \mathrm{i}\gamma y' - \mathrm{i}\omega r')/r', \tag{77}$$

which is Eq. (61a) with s changed into $\mathrm{i}\omega$ and divided by $-\sigma^\dagger$ so that according to Eq. (63a)

$$F_0(\beta, \gamma, s;\ \phi) = -4\pi(\mu^2 - \omega^2)^{-1/2} \exp[-z_0(\mu^2 - \omega^2)^{1/2}]/\sigma^\dagger. \tag{78}$$

Now taking into account the definition of the lower bound b, the second term of Eq. (75) is

$$F_1(\beta, \gamma, s;\ \phi) = 2\exp(-\sigma^\dagger ct_0)/\sigma^\dagger \iint_{-\infty}^{\infty} \mathrm{d}x'\,\mathrm{d}y' \exp(-\mathrm{i}\beta x' - \mathrm{i}\gamma y' + sr')/r'\, U(ct_0 - r'). \tag{79}$$

The integrations in Eq. (79) may be performed with cylindrical coordinates (62) but the unit function $U(ct_0 - r')$ imposes an upper bound p on p'-integration and $p = (c^2 t_0^2 - z_0^2)^{1/2}$ according to definition (58a) of r' so that integral (79) becomes

$$F_1(\beta,\gamma,s;\ \phi) = -4\pi^2 \exp(-\sigma^\dagger ct_0)/\sigma^\dagger \int_0^p \rho'\,\mathrm{d}\rho' \exp(sr')J_0(\mu\rho')/r'. \tag{79a}$$

Assuming p/z_0 small, the $O(p^2/z_0^2)$ approximation of $F_1(\beta,\gamma,s;\ \phi)$, where O is the Landau symbol is obtained in Appendix B

$$F_1(\beta,\gamma,s;\ \phi) = -4\pi^2/s\sigma^\dagger \exp(-\mathrm{i}\omega ct_0)\,\mathrm{Exp}_c[-s(ct-z_0)]J_0(\mu p) \tag{80}$$

in which $\mathrm{Exp}_c(x)$ is function (42a).

Now, using Eq. (76), integral equation (9) which supplies $\phi(r,z,t)$ can be written

$$\phi(\rho,z,t) = \phi_0(\rho,z,t) + \phi_j(\rho,z,t), \tag{81}$$

$$\phi_j(\rho,z,t) = L^{-1}\{\Phi_{j,-}(\rho,z,s)\} + L^{-1}\{\Phi_{j,+}(\rho,z,s)\}, \quad j=0,1. \tag{81a}$$

Then according to Eqs. (9) and (78), still using coordinates (62)

$$\begin{aligned}\Phi_{0,\pm}(\rho,z,s) &= 1/\sigma^\dagger \int_0^\infty \mu\,\mathrm{d}\mu(\mu^2-\omega^2)^{-1/2}J_0(\mu\rho)\exp[-(\mu^2-\omega^2)^{1/2}(z_0\pm z)]\\ &= \exp(-\mathrm{i}\omega r_\pm)/r_\pm\sigma^\dagger, \quad r_\pm = [\rho^2+(z_0\pm z)^2]^{1/2}\end{aligned} \tag{82}$$

obtained from Eqs. (66a) and (67) with s changed into $\mathrm{i}\omega$ so that according to Eqs. (81a) and (82)

$$\phi_0(\rho,z,t) = \exp[-\mathrm{i}\omega(ct+r_-)]/r_- + \exp[-\mathrm{i}\omega(ct+r_+)/r_+]. \tag{83}$$

Now, substituting Eq. (80) into Eq. (9) gives with coordinates (62)

$$\Phi_{1,-}(\rho,z,s) = (1/\sigma^\dagger s)b(s;z_0,t_0)K(s,z), \quad b(s;z_0,t_0) = \exp(-\mathrm{i}\omega ct_0) - \exp[-s(ct_0-z_0)], \tag{84}$$

$$K(s,z) = \int_0^\infty \mu\,\mathrm{d}\mu J_0(\mu\rho)J_0(\mu p)\exp[-(s^2+\mu^2)^{1/2}z]. \tag{84a}$$

Unfortunately, approximation (80) does not make possible to define $\Phi_{1,+}(\rho,z,s)$ since the exponential in the integrand of $K(s,z)$ would have a positive sign. So numerical computations are required to get $\Phi_{1,+}(\rho,z,s)$ and ultimately $\phi_{1,+}(\rho,z,t)$; this problem is now left aside.

Conversely, the inverse Laplace transform $\phi_{1,-}(\rho,z,t) = L^{-1}\{\Phi_{1,-}(\rho,z,s)\}$ of Eq. (84) requires an approximation of Eq. (84a) supplied by the Laplace approximating technique of integrals developed by Olver [16]. Succinctly, let $I(z)$ be the following integral in which $w'(\alpha)>0$ and $q(\alpha)\neq 0$:

$$I(z) = \int_a^b \mathrm{d}\alpha \exp[-zw(\alpha)]q(\alpha). \tag{85}$$

If the peak value of the factor $\exp[-zw(\alpha)]$ occurs at $\alpha = a$ then

$$I(z) \sim q(a)\exp[-zw(a)]/zw'(a). \tag{85a}$$

Now, $K(s,z)$ is an integral of type (85) with $\alpha=\mu^2$, $w(\alpha) = (s^2+\alpha)^{1/2}$, $q(\alpha) = J_0(\rho\surd\alpha)J_0(p\surd\alpha)$ and since in Eq. (82a) the lower bound of the integral is $a=0$ and since $J_0(0)=1$, approximation (85a) applied to Eq. (84) gives

$$K(s,z) \sim sz^{-1}\exp(-sz). \tag{86}$$

Then, substituting Eq. (86) into Eq. (84) and making explicit $b(s,t_0,z_0)$, the approximation of $\Phi_{1,-}(\rho,z,s)$ is

$$\Phi_{1,-}(\rho,z,s) \sim (1/\sigma^\dagger z)\exp(-\mathrm{i}\omega ct_0)\{\exp(-sz) - \exp[-s(ct_0-z_0+z)]\} \tag{87}$$

and the inverse Laplace transform of Eq. (87) is obtained at once using Eq. (18) and $L^{-1}\{1/\sigma^\dagger\} = \exp(-\mathrm{i}\omega ct)$

$$\phi_{1,-}(\rho,z,t) \sim z^{-1}\{\exp[-\mathrm{i}\omega(ct+ct_0-z)]U(ct-z) - \exp[-\mathrm{i}\omega(ct-z_0+z)]U(ct-ct_0+z_0-z)\} \tag{88}$$

an approximation valid for small $(c^2t_0^2-z_0^2)^{1/2}/z_0$ and large z.

To sum up, the contribution $\phi(\rho, z, t)$ to total field (74) outside the mirror, due to component (73a) of the reflected field $\phi(\rho', 0, t')$ on the TRM, is made according to Eq. (81) of two parts: the first one supplied by Eq. (83) does not require any unit step function while only the approximation (88) of the second part could be obtained. Finally, according to Eq. (74), to get the total TRM pulse the field $\psi(\rho, z, t)$ given by Eq. (71) must be added to $\phi(\rho, z, t)$.

5. Discussion

Using the notations $\psi_+ = \psi_i$ and $\psi_- = \psi_{\mathrm{re}}$, to denote the incident and reflected fields, incident rectangular pulse (12) impinging on a conventional mirror generates the total pulse

$$\psi(z, t) = \psi_+(z, t) + \psi_-(z, t) \tag{89}$$

with according to Eq. (19)

$$\psi_\pm(z, t) = U(ct - z_0 \pm z) - U(ct - ct_0 - z_0 \pm z). \tag{89a}$$

Conversely, on a TRM, there is no reflection at all when $ct_0 < z_0$ while if $ct_0 > z_0$ the total field after reflection is given by Eq. (23)

$$\psi^*(z, t) = [\psi(z, t) + \phi(z, t)]/2, \tag{90}$$

in which $\psi(z, t)$ is pulse (89) and

$$\phi(z, t) = \phi_+(z, t) + \phi_-(z, t) \tag{91}$$

with from Eqs. (27a) and (27b)

$$\phi_\pm(z, t) = -[U(ct \pm z) - U(ct - ct_0 + z_0 \pm z)]. \tag{91a}$$

Then, it is natural to compare $\psi_i(z_0, t)$ and $\phi(0, t)$ with Eqs. (89a) and (91a)

$$\psi_i(z, t) = U(ct) - U(ct - ct_0),$$

$$\phi_\pm(0, t) = -[U(ct) - U(ct - ct_0 + z_0)], \quad ct_0 > z_0, \tag{92}$$

so that the TRM acts as a source, launching at $t = 0$ a rectangular pulse with a reduced duration $t_0 - z_0/c$ in the opposite direction to the incident pulse. Exchanging the roles of z_0 and zero in Eq. (92) and assuming $ct_0 < 2z_0$ so that $\phi_+(z_0, t) = 0$ since in this case the argument of both unit step functions in Eq. (91a) is positive, we have

$$\begin{aligned} \psi_i(0, t) &= U(ct - z_0) - U(ct - ct_0 - z_0), \\ \phi_-(z_0, t) &= -[U(ct - z_0) - U(ct - ct_0)]. \end{aligned} \tag{93}$$

Relations (92) and (93) display the dual character of both sources.

For a rectangular plane harmonic source, the fields $\psi_\pm$, $\phi_\pm$ in Eqs. (89) and (91) are now supplied by Eqs. (39) and (54) giving instead of Eqs. (92) and (93)

$$\psi_i(z_0, t) = \exp(\mathrm{i}\omega ct)[U(ct) - U(ct - ct_0)],$$

$$\phi_\pm(0, t) = -\exp[-\mathrm{i}\omega(ct + z_0)][U(ct) - U(ct - ct_0 + z_0)] \tag{94}$$

and still assuming $ct_0 < 2z_0$

$$\begin{aligned} \psi_i(0, t) &= \exp[\mathrm{i}\omega(ct - z_0)][U(ct - z_0) - U(ct - ct_0 - z_0)], \\ \phi_-(z_0, t) &= -\exp(-\mathrm{i}\omega ct)[U(ct - z_0) - U(ct - ct_0)]. \end{aligned} \tag{95}$$

In this case also, the TRM acts as a dual source launching at $t = 0$ a rectangular harmonic pulse with reduced duration in the opposite direction to the incident pulse.

No doubt that a similar situation prevails for truncated spherical harmonic pulses, but it is difficult to assess the exact form of the pulse launched by the TRM dual source since rather drastic approximations had to be made to get analytical expressions such as Eqs. (83) and (88).

To sum up, the Stokes-like gedanken experiment suggests that TRMs behave as a dual source of pulses in agreement with real experiments analysed by Finch [1] but that they may also become transparent when the pulse duration is smaller that the time needed to go from the source to the mirror. It is not known whether this last property has been observed.

It is assumed in this work that the total field on the mirror satisfies the Neumann boundary condition, similar calculations can be performed with Dirichlet boundary conditions but they are less simple because the form factor $F(\beta, \gamma, s)$ requires the derivative of the total field on the plane $\Sigma' = \{z' = 0\}$.

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Appendix A

Here, to simplify calculations the sound speed $c = 1$, the following result is proved:

Lemma. $\exp(as)$ *with* $a>0$ *has the inverse Laplace transform*

$$L^{-1}\{\exp(as)\} = 2\delta(t+a), \quad a>0, \ t\geqslant 0. \tag{A.1}$$

Proof. The following relations given by Erdelyi [13], valid for $a>0$

$$\begin{aligned} L^{-1}\{s^{-1/2}\cosh(a/s)\} &= (4\pi t)^{-1/2}\left[\cos\left(2\sqrt{at}\right) + \cosh\left(2\sqrt{at}\right)\right], \\ L^{-1}\{s^{-1/2}\sinh(a/s)\} &= (4\pi t)^{-1/2}\left[\cosh\left(2\sqrt{at}\right) - \cos\left(2\sqrt{at}\right)\right] \end{aligned} \tag{A.2}$$

supply the inverse Laplace transforms

$$\begin{aligned} \mathsf{L}^{-1}\{s^{-1/2}\exp(a/s)\} &= (\pi t)^{-1/2}\cosh\left(2\sqrt{at}\right), \\ \mathsf{L}^{-1}\{s^{-1/2}\exp(-a/s)\} &= (\pi t)^{-1/2}\cos\left(2\sqrt{at}\right). \end{aligned} \tag{A.3}$$

But, if $L^{-1}\{F(s)\} = f(t)$ then according to Doetsch [12]

$$L^{-1}\{s^{-1/2}F(s^{-1})\} = (\pi t)^{-1/2}\int_0^\infty \cos\left(2\sqrt{t\tau}\right)f(\tau)\,\mathrm{d}\tau. \tag{A.4}$$

Applying Eq. (A.4) to Eq. (A.3) gives for $a>0$

$$L^{-1}\{\exp(as)\} = (\pi^2 t)^{-1/2}\int_0^\infty \tau^{-1/2}\cos\left(2\sqrt{t\tau}\right)\cosh\left(2\sqrt{a\tau}\right)\mathrm{d}\tau, \tag{A.5}$$

$$L^{-1}\{\exp(-as)\} = (\pi^2 t)^{-1/2}\int_0^\infty \tau^{-1/2}\cos\left(2\sqrt{t\tau}\right)\cos\left(2\sqrt{a\tau}\right)\mathrm{d}\tau. \tag{A.6}$$

Then, using the variables $\tau = x^2$, $t = \xi^2$, $a = b^2$, Eq. (A.6) becomes

$$\begin{aligned} \int_0^\infty \tau^{-1/2}\cos\left(2\sqrt{t\tau}\right)\cos\left(2\sqrt{a\tau}\right)\mathrm{d}\tau &= \int_0^\infty \mathrm{d}x\{\cos[2x(b+\xi)] + \cos[2x(b-\xi)]\} \\ &= 2^{-1}\int_{-\infty}^\infty \mathrm{d}x\{\exp[2\mathrm{i}x(b+\xi)] + \exp[2\mathrm{i}x(b-\xi)]\} \end{aligned} \tag{A.7}$$

and

$$\begin{aligned} (\text{A.7}) &= \pi\{\delta[2(b+\xi)] + \delta[2(b-\xi)]\} \\ &= \pi/2\{\delta(b+\xi) + \delta(b-\xi)\} = \pi\xi\delta(b^2-\xi^2). \end{aligned} \tag{A.8}$$

Substituting Eq. (A.8) into Eq. (A.6) and coming back to the variables τ, t, a, give

$$L^{-1}\{\exp(-as)\} = \delta(t-a)U(t-a), \quad a>0, \tag{A.9}$$

which is the usual result (see, for instance, Ref. [12]). Changing a into $-a$, that is b into ib transforms Eq. (A.6) into Eq. (A.5) leading instead of Eq. (A.7) to

$$\int_0^\infty \tau^{-1/2}\cos\left(2\sqrt{t\tau}\right)\cosh\left(2\sqrt{a\tau}\right)\mathrm{d}\tau = \left\{(1/2)\int_0^\infty \mathrm{d}x[\exp\{\mathrm{i}x(2\xi+2\mathrm{i}b)\}+\exp\{-\mathrm{i}x(2\xi+2\mathrm{i}b)\}]+\{^*\}\right\}, \quad (A.10)$$

where $\{^*\}$ denotes the complex conjugate term. Then, using the relation given by Bremermann [14] for the asymmetrical Dirac distribution δ_+

$$\int_0^\infty \mathrm{d}z\exp(\mathrm{i}tz) = 2\pi\delta_+(z), \quad \mathrm{Im}\, z>0 \quad (A.11)$$

and the similar expression for $\delta_-(z)$, relation (A.10) becomes

$$\int_0^\infty \tau^{-1/2}\cos\left(2\sqrt{t\tau}\right)\cosh\left(2\sqrt{a\tau}\right)\mathrm{d}\tau = 2\pi\{\delta[2(\mathrm{i}b+\xi)]+\delta[2(\mathrm{i}b-\xi)]\}$$
$$= 2\pi\xi\delta(b^2+\xi^2). \quad (A.12)$$

Substituting Eq. (A.11) into Eq. (A.4) and coming back to the variables τ, t, a, give Eq. (A.1).

Appendix B

With $p^2 = c^2t_0^2 - z_0^2$ integral (79a) is written

$$F_1(\beta,\gamma,s;\,\phi) = -4\pi^2\exp(-\sigma^\dagger ct_0)/\sigma^\dagger I(\mu,s), \quad (B.1)$$

$$I(\mu,s) = \int_0^p \rho'\,\mathrm{d}\rho'\exp(sr')J_0(\mu\rho')/r'. \quad (B.2)$$

Lemma. *The integral $F_1(\beta,\gamma,s;\,\phi)$ has the $O(p^2/z_0^2)$ approximation*

$$F_1(\beta,\gamma,s;\,\phi) = -4\pi^2/s\sigma^\dagger\exp(-\mathrm{i}\omega ct_0)\,\mathrm{Exp}_c[-s(ct_0-z_0)]J_0(\mu p), \quad (B.3)$$

$$\mathrm{Exp}_c[-s(ct_0-z_0)] = \exp(-\mathrm{i}\omega ct_0)[1-\exp[-s(ct_0-z_0)]] \quad (B.4)$$

in which O denotes the Landau symbol.

Proof. Introducing the ρ' variable $\rho' = z_0\sinh\alpha$ and since $r' = z_0\cosh\alpha$, $I(\mu,s)$ becomes

$$I(\mu,s) = -z_0\int_0^\eta \sinh\alpha\,\mathrm{d}\alpha\exp(sz_0\cosh\alpha)J_0(\mu z_0\sinh\alpha)$$
$$= -s^{-1}\int_0^\eta \mathrm{d}\{\exp(sz_0\cosh\alpha)\}J_0(\mu z_0\sinh\alpha) \quad (B.5)$$

with

$$\sinh\eta = p/z_0, \quad \cosh\eta = ct_0/z_0. \quad (B.6)$$

Integrating by parts and taking into account Eq. (B.6) give

$$I(\mu,s) = s^{-1}\{\exp(sct_0)J_0(\mu p)-\exp(sz_0)\}+I_1(\mu,s), \quad (B.7)$$

$$I_1(\mu,s) = \mu z_0/s\int_0^\eta \cosh\alpha\,\mathrm{d}\alpha\exp(sz_0\cosh\alpha)J_1(\mu z_0\sinh\alpha) \quad (B.8)$$

Assuming η small, we may use the $O(\alpha^2)$ approximations $\sinh\alpha\sim\alpha$, $\cosh\alpha\sim 1$, in the integrand of Eq. (B.8) so that

$$I_1(\mu,s) = \mu z_0 s^{-1}\exp(sz_0)\int_0^\eta \mathrm{d}\alpha J_1(\mu z_0\alpha)+O(\eta^2). \quad (B.9)$$

And using the integral $\int_0^a J_1(x)\,\mathrm{d}x = 1 - J_0(a)$ (see, for instance, Ref. [15]) reduces Eq. (B.9) to

$$I_1(\mu, s) = s^{-1}\exp(sz_0)[1 - J_0(\mu p)]. \tag{B.10}$$

Taking into account Eq. (B.10), relation (B.7) becomes

$$I(\mu, s) = -s^{-1}J_0(\mu p)\{\exp(sct_0) - \exp(sz_0)\} \tag{B.11}$$

and substituting Eq. (B.11) into Eq. (B.1) gives finally Eq. (B.3).

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