Available online at www.sciencedirect.com
SCIENCE DIRECT®

Journal of Sound and Vibration 292 (2006) 535–551

JOURNAL OF SOUND AND VIBRATION

www.elsevier.com/locate/jsvi

Frequencies and modes of vibration of a cylindrical shell with attached rigid body

Yu.V. Trotsenko

Institute of Mathematics of the National Academy of Sciences of Ukraine 3 Tereschenkovskaya St., 01601Kiev-4, Ukraine

Received 27 May 2004; received in revised form 5 August 2005; accepted 9 August 2005
Available online 4 November 2005

Abstract

We consider a mechanical system consisting of a circular cylindrical shell and perfectly rigid body attached to one of the shell ends. Starting from the principle of virtual works, we construct a mathematical model of the equilibrium state of our system subjected to stresses of general form. A boundary eigenvalue problem describes free vibrations of the "body – shell" system, and its approximate solution is determined. We construct the exact solution of the above problem by replacing the shell with an equivalent Timoshenko beam. The effect of the rigid body on the system vibrations is estimated, and the accuracy of the beam approximation to shell bending vibrations is studied.

1. Introduction

Thin elastic shells of revolution with attached bodies are widely used in present-day mechanical and civil engineering. Investigation of dynamics of such constructions under action of various non-stationary loads requires solving a rather complicated partial differential initial boundary-value problem. Generally, solutions of this problem are based on reduction to a set of ordinary differential equations with independent time variable. For this purpose the natural modes of free vibrations of the structure are used. A set of ordinary differential equations in generalized coordinates obtained in that way can be investigated with known methods. That is why determination of frequencies and modes of natural vibration of composite mechanical systems is an important first step in investigating their dynamic behaviour under action of lumped and distributed loads.

On the other hand, simple mathematical models are worth developing to meet engineering needs. (Of course, they must adequately describe dynamic behaviour of shell constructions.) Such models can be derived on the basis of various beam theories used to approximate bending vibrations of shells. The problem of determination of applicability limits for such simplified approaches then becomes of crucial importance.

The longitudinal and torsional vibrations of a cylindrical shell with masses attached at its ends were investigated by Breslavsky [1,2]. In [3], Palamarchuk studied interaction between a cylindrical shell and perfectly rigid body attached to the shell inside with rigid bars. The work [4] deals with construction of a mathematical model of interaction of a cylindrical shell with a perfectly rigid body attached to one of its ends.

E-mail address: trots@imath.kiev.ua.

0022-460X/$ - see front matter
doi:10.1016/j.jsv.2005.08.015

Shveiko et al. [5] have derived the exact solution of the problem of natural vibrations of beams joined to each other with a circular cylindrical shell. An approximate solution of the above problem was proposed in Ref. [6]. Rabinovich et al. [7] advanced a theory of vibrations of constructions supporting elastic vessels with liquid. In their calculations, they applied an elastic Euler–Bernoulli beam with a torque shell of revolution (attached to it with elastic braces) filled with an ideal incompressible liquid. Trotsenko and Kladinoga [8] investigated natural vibrations of a prestressed zero-torque shell of revolution (made of a hyperelastic material) with a rigid disk attached to one of the shell ends.

It is a traditional engineering practice to neglect the secondary effects due to shear strain and rotary inertia of the shell cross-section when applying the beam approximation of bending vibrations of shells. However, Forsberg [9] considered free vibrations of a thin cylindrical shell and showed that the above secondary effects become of primary importance for short shells, especially when calculating the higher vibration modes.

There exist a number of works dealing with investigation of vibrations of Timoshenko beams with attached bodies of finite sizes. A sufficiently complete review of these works was given by White and Heppler in Ref. [10]. One should note also the work [11] by Rossi and Laura who obtained comprehensive results (in table and graphic forms) of calculations of frequencies and modes of vibrations of a Timoshenko beam clamped at one end and carrying a finite mass at the other. From the results of the above works one can conclude that vibrations of beams with attached bodies are studied rather well. At present, the researchers concentrate their efforts on refinement of calculation algorithms and detection of new mechanical effects of interaction between a body and beam at their joint vibrations.

This work deals with construction of a mathematical model and solving the problem on free non-axisymmetric vibrations of a circular cylindrical shell with a rigid body of finite size attached to one of the shell ends. The construction of approximate analytical solutions of the stated spectral problems is based on their equivalent variational formulations and Ritz method. We investigate the applicability limits for simplified statements of the problem obtained with application of different beam theories.

2. Statement of the problem

Let us consider a mechanical system consisting of a thin-walled circular cylindrical shell (of radius R and length l) and a perfectly rigid body that is rigidly attached to one of the shell ends. The other shell end is assumed to be rigidly fixed. Let the body have two mutually perpendicular planes of symmetry whose intersection line is the axis Oz that coincides with the longitudinal axis of the shell (Fig. 1). Let the coordinate

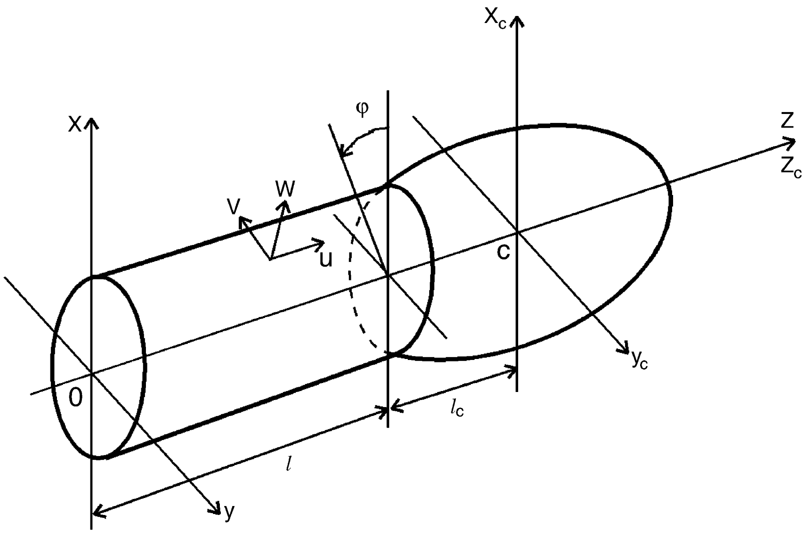

Fig. 1. The general view of the construction and the system of coordinates.

plane Oxz coincide with one of the planes of symmetry of the body and the origin of coordinates lie in the plane of the fixed shell end.

To describe displacements of the rigid body, we introduce a Cartesian system of coordinates $Cx_cy_cz_c$ whose origin lies at the centre of inertia of the rigid body and axes Cx_c and Cy_c are parallel to Ox and Oy, respectively. The unit vectors of the system of coordinates $Cx_cy_cz_c$ will be designated as $\mathbf{i}_c$, $\mathbf{j}_c$ and $\mathbf{k}_c$. We refer the middle surface of the shell to the orthogonal system of curvilinear coordinates z and φ where φ is the polar angle counted from the axis Ox. Local orthogonal bases $\mathbf{e}_1$, $\mathbf{e}_2$ and $\mathbf{e}_3$ are referred to these coordinates. In this basis, $\mathbf{e}_1$ and $\mathbf{e}_2$ are the unit vectors that are tangential to the lines of principal curvatures of the middle surface of the shell and are oriented along increase of coordinates z and φ.

Let us assume that the following loads are applied to the above construction: (i) a small load of general type—a force concentrated at the point C and a moment about the point C

$$\Delta\mathbf{F} = \Delta F_1\mathbf{i}_c + \Delta F_2\mathbf{j}_c + \Delta F_3\mathbf{k}_c, \quad \Delta\mathbf{M} = \Delta M_1\mathbf{i}_c + \Delta M_2\mathbf{j}_c + \Delta M_3\mathbf{k}_c,$$

that act on the rigid body, and (ii) a distributed load

$$\Delta\mathbf{Q} = \Delta Q_1\mathbf{e}_1 + \Delta Q_2\mathbf{e}_2 + \Delta Q_3\mathbf{e}_3,$$

acting on the shell. The system experiences strains and displacements and, as a result, comes to a disturbed state of equilibrium. We shall characterize this state with the displacement vector of the points of the middle surface of the shell,

$$\mathbf{U} = u\mathbf{e}_1 + v\mathbf{e}_2 + w\mathbf{e}_3,$$

the vector of translational displacement of the centre of mass of the rigid body and the vector of its angular displacement about this centre,

$$\mathbf{U}_0 = u_{01}\mathbf{i}_c + u_{02}\mathbf{j}_c + u_{03}\mathbf{k}_c, \quad \theta_0 = \vartheta_{01}\mathbf{i}_c + \vartheta_{02}\mathbf{j}_c + \vartheta_{03}\mathbf{k}_c.$$

Here we assume that the displacements of the rigid body and shell are small, so that linear theory is valid.

From the condition that the displacements of the shell and rigid body in the contour L (formed by the shell cross-section at $z = l$) are the same it follows that

$$\mathbf{U} = \mathbf{U}_0 + [\theta_0 \times \mathbf{r}_0], \tag{1}$$

where $\mathbf{r}_0 = (R\cos\varphi)\mathbf{i}_c - (R\sin\varphi)\mathbf{j}_c - l_c\mathbf{k}_c$ is the radius vector of the points of the contour L in the system of coordinates $Cx_cy_cz_c$; l_c is the distance from the point C along the axis Oz to the shell end section at which the rigid body is attached.

From Eq. (1) and continuity condition for the corresponding angles of rotation of the rigid body and shell, with allowance made for interrelation between the unit vectors of the system of coordinates $Cx_cy_cz_c$ and unit vectors $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$

$$\mathbf{i}_c = -\sin\varphi\mathbf{e}_2 + \cos\varphi\mathbf{e}_3, \quad \mathbf{j}_c = \cos\varphi\mathbf{e}_2 - \sin\varphi\mathbf{e}_3, \quad \mathbf{k}_c = \mathbf{e}_1, \tag{2}$$

one obtains the following geometric boundary conditions in the contour L:

$$\begin{aligned}
&u = u_{03} - \vartheta_{01}R\sin\varphi - \vartheta_{02}R\cos\varphi,\\
&v = (\vartheta_{02}l_c - u_{01})\sin\varphi - (\vartheta_{01}l_c + u_{02})\cos\varphi - \vartheta_{03}R,\\
&w = -(\vartheta_{01}l_c + u_{02})\sin\varphi - (\vartheta_{02}l_c - u_{01})\cos\varphi,\\
&\left.\frac{\partial w}{\partial z}\right|_{z=l} = \vartheta_{01}\sin\varphi + \vartheta_{02}\cos\varphi.
\end{aligned} \tag{3}$$

To obtain the equilibrium equations for our system, let us apply the principle of virtual works

$$\delta\Pi = \delta A, \tag{4}$$

where $\delta\Pi$ is the variation of the potential energy of the elastic strain of the shell, δA is the work of external forces in virtual works of the system.

The strain potential energy for a thin cylindrical shell may be presented as [12]

$$\begin{aligned} \Pi = \frac{Eh}{2(1-\nu^2)} \int_\Sigma \int \left[\left(\frac{\partial u}{\partial z} \right)^2 + \frac{1}{R^2} \left(\frac{\partial v}{\partial \varphi} + w \right)^2 + \frac{2\nu}{R} \frac{\partial u}{\partial z} \left(\frac{\partial v}{\partial \varphi} + w \right) \right. \\ \left. + \frac{1-\nu}{2} \left(\frac{1}{R} \frac{\partial u}{\partial \varphi} + \frac{\partial v}{\partial z} \right)^2 \right] \mathrm{d}\Sigma + \frac{D}{2} \int_\Sigma \int \left[\left(\frac{\partial^2 w}{\partial z^2} \right)^2 + \left(\frac{1}{R^2} \frac{\partial^2 w}{\partial \varphi^2} \right)^2 \right. \\ \left. + \frac{2\nu}{R^2} \frac{\partial^2 w}{\partial z^2} \frac{\partial^2 w}{\partial \varphi^2} + 2(1-\nu) \left(\frac{1}{R} \frac{\partial^2 w}{\partial z \partial \varphi} \right)^2 \right] \mathrm{d}\Sigma, \quad D = \frac{Eh^3}{12(1-\nu^2)}, \end{aligned} \tag{5}$$

where E (ν) is the modulus of elasticity (Poisson ratio) of the shell material, Σ (h) is the middle surface (thickness) of the shell.

The first integral in Eq. (5) is the stretching and shear potential energy, while the second one is the bending and torsion potential energy. Here, we applied the known Mushtari–Donnell theory of shells [12,13], according to which one neglects shell tangential displacements in the expressions for curvature and torsion variations.

The work of external forces applied to the body and shell is

$$A = \int_\Sigma \int \Delta \mathbf{Q} \cdot \mathbf{u} \, \mathrm{d}\Sigma + \Delta \mathbf{F} \cdot \mathbf{u}_0 + \Delta \mathbf{M} \cdot \theta_0. \tag{6}$$

Let us denote the variations of displacements of the shell and rigid body and of the angles of rotation of the body by $\delta \mathbf{u}$, $\delta \mathbf{u}_0$ and $\delta \theta_0$, respectively. Then, after substitution of Eqs. (5) and (6) into Eq. (4) and integration by parts of the double integrals, the variational equation (with allowance made for the conditions (3)) may be brought to the following form:

$$\begin{aligned} & -\frac{1}{g} \int_\Sigma \int \{ [L_{11}(u) + L_{12}(v) + L_{13}(w) + g\Delta Q_1] \delta u + [L_{21}(u) + L_{22}(v) + L_{23}(w) + g\Delta Q_2] \delta v \\ & - [L_{31}(u) + L_{32}(v) + L_{33}(w) - g\Delta Q_3] \delta w \} \, \mathrm{d}\Sigma + \left[\oint_L (Q_1^* \cos\varphi - S \sin\varphi) \, \mathrm{d}s - \Delta F_1 \right] \delta u_{01} \\ & + \left[\oint_L (Q_1^* \sin\varphi + S\cos\varphi) \, \mathrm{d}s + \Delta F_2 \right] \delta u_{02} + \left(\oint_L T_1 \, \mathrm{d}s - \Delta F_3 \right) \delta u_{03} \\ & + \left[\oint_L (P_1 \sin\varphi + l_c S \cos\varphi) \, \mathrm{d}s + \Delta M_1 \right] \delta \vartheta_{01} + \left[\oint_L (P_1 \cos\varphi - l_c S \sin\varphi) \, \mathrm{d}s + \Delta M_2 \right] \delta \vartheta_{02} \\ & + \left[\oint_L RS \, \mathrm{d}s + \Delta M_3 \right] \delta \vartheta_{03} = 0. \end{aligned} \tag{7}$$

Here we use the following designations:

$$\begin{aligned} & L_{11} = \frac{\partial^2}{\partial z^2} + \frac{\nu_1}{R^2} \frac{\partial^2}{\partial \varphi^2}, \quad L_{12} = L_{21} = \frac{\nu_2}{R} \frac{\partial^2}{\partial z \partial \varphi}, \quad L_{13} = L_{31} = \frac{\nu}{R} \frac{\partial}{\partial z}, \\ & L_{22} = \frac{1}{R^2} \frac{\partial^2}{\partial \varphi^2} + \nu_1 \frac{\partial^2}{\partial z^2}, \quad L_{23} = L_{32} = \frac{1}{R^2} \frac{\partial}{\partial \varphi}, \quad L_{33} = \frac{1}{R^2} (c^2 \Delta\Delta + 1), \\ & \Delta = R^2 \frac{\partial^2}{\partial z^2} + \frac{\partial^2}{\partial \varphi^2}, \quad \nu_1 = \frac{1-\nu}{2}, \quad \nu_2 = \frac{1+\nu}{2}, \quad c^2 = \frac{h^2}{12R^2}, \quad g = \frac{1-\nu^2}{Eh}, \\ & S = \frac{Eh}{2(1+\nu)} \left(\frac{\partial v}{\partial z} + \frac{1}{R} \frac{\partial u}{\partial \varphi} \right), \quad T_1 = \frac{1}{g} \left[\frac{\partial u}{\partial z} + \frac{\nu}{R} \left(\frac{\partial v}{\partial \varphi} + w \right) \right], \\ & P_1 = RT_1 + M_1 + l_c Q_1^*, \quad M_1 = -D \left(\frac{\partial^2 w}{\partial z^2} + \frac{\nu}{R^2} \frac{\partial^2 w}{\partial \varphi^2} \right), \\ & Q_1^* = -\frac{c^2}{g} \left[R^2 \frac{\partial^3 w}{\partial z^3} + (2-\nu) \frac{\partial^3 w}{\partial z \partial \varphi^2} \right]. \end{aligned}$$

After setting the coefficients at δu, δv and δw in the surface integrals (7) equal to zero, we obtain the known equilibrium equations for the cylindrical shell. And the equilibrium equations for the rigid body follow from the condition that the coefficients at variations of parameters of its motion vanish.

The equations of free vibrations of the "body-shell" system can be obtained from the derived equilibrium equations after applying the d'Alembert's principle and setting

$$\Delta \mathbf{Q} = -\rho h \frac{\partial^2 \mathbf{u}}{\partial t^2}, \quad \Delta \mathbf{F} = -m_0 \frac{\partial^2 \mathbf{u}_0}{\partial t^2},$$
$$\Delta M_1 = -J_{x_c} \frac{\partial^2 \vartheta_{01}}{\partial t^2}, \quad \Delta M_2 = -J_{y_c} \frac{\partial^2 \vartheta_{02}}{\partial t^2}, \quad \Delta M_3 = -J_{z_c} \frac{\partial^2 \vartheta_{03}}{\partial t^2}, \tag{8}$$

where J_{x_c}, J_{y_c} and J_{z_c} are the moments of inertia of the rigid body relative to the axes Cx, Cy and Cz, respectively; m_0 is the mass of the rigid body and ρ is the density of the shell material.

When considering steady-state free vibration of our system with frequency ω, we set $\{\mathbf{U}, \mathbf{U}_0, \theta_0\} = \{\tilde{\mathbf{U}}, \tilde{\mathbf{U}}_0, \tilde{\theta}_0\}\mathrm{e}^{\mathrm{i}\omega t}$. (From here on we shall omit the tilde.) As a result, determination of the amplitude values for six parameters of motion of the rigid body and three components of the shell displacement vector is reduced to integration of a set of partial differential equations

$$L_{11}(u) + L_{12}(v) + L_{13}(w) = -\omega^2 \rho h g u,$$
$$L_{21}(u) + L_{22}(v) + L_{23}(w) = -\omega^2 \rho h g v,$$
$$L_{31}(u) + L_{32}(v) + L_{33}(w) = \omega^2 \rho h g w \tag{9}$$

with non-local boundary conditions at $z = l$

$$\oint_L (Q_1^* \cos \varphi - S \sin \varphi)\, \mathrm{d}s = m_0 \omega^2 u_{01},$$
$$\oint_L (Q_1^* \sin \varphi + S \cos \varphi)\, \mathrm{d}s = -m_0 \omega^2 u_{02},$$
$$\oint_L T_1\, \mathrm{d}s = m_0 \omega^2 u_{03}, \quad \oint_L (P_1 \sin \varphi + l_c S \cos \varphi)\, \mathrm{d}s = \omega^2 J_{x_c} \vartheta_{01},$$
$$\oint_L (P_1 \cos \varphi - l_c S \sin \varphi)\, \mathrm{d}s = \omega^2 J_{y_c} \vartheta_{02}, \quad \oint_L RS\, \mathrm{d}s = \omega^2 J_{z_c} \vartheta_{03}. \tag{10}$$

One should add to the boundary conditions (10) the geometric compatibility conditions (3) and the condition of rigid fixation of the shell end at $z = 0$.

It should be noted particularly that the boundary conditions (10) are natural for the corresponding functional on the class of functions satisfying conditions (3) and the geometric conditions of fixation of that shell end which is free of the rigid body.

3. Construction of solution

Smallness of the parameters of motion and system symmetry make it possible to present general motion of the system as superposition of two independent components, along and about the longitudinal axis, as well as in two mutually perpendicular planes Oxz and Oyz. In what follows, we shall consider the transverse vibrations of the construction in one of the planes of symmetry (let it be the plane Oxz). In this case the displacement of the middle surface of the shell may be sought in the following form:

$$u(z, \varphi) = \sum_{n=1}^{\infty} u_n(z) \cos n\varphi, \quad v(z, \varphi) = \sum_{n=1}^{\infty} v_n(z) \sin n\varphi,$$
$$w(z, \varphi) = \sum_{n=1}^{\infty} w_n(z) \cos n\varphi. \tag{11}$$

According to the expressions (11), the forces and moments in the middle surface of the shell will be determined from the expressions

$$T_1 = \sum_{n=1}^{\infty} T_{1(n)} \cos n\varphi, \quad S = \sum_{n=1}^{\infty} S_{(n)} \sin n\varphi,$$
$$M_1 = \sum_{n=1}^{\infty} M_{1(n)} \cos n\varphi, \quad Q_1^* = \sum_{n=1}^{\infty} Q_{1(n)}^* \cos n\varphi. \tag{12}$$

Introduce non-dimensional quantities that are related to the corresponding dimensional ones by the following interrelations:

$$\{u_n, v_n, w_n, u_{01}\} = R\{\bar{u}_n, \bar{v}_n, \bar{w}_n, \bar{u}_{01}\},$$
$$\{T_{1(1)}, Q_{1(1)}^*, S_{(1)}\} = \frac{1}{g}\{\bar{T}_{1(1)}, \bar{Q}_{1(1)}^*, \bar{S}_{(1)}\},$$
$$\omega^2 = \frac{\bar{\omega}^2}{gh\rho R^2}, \quad M_{1(1)} = \frac{R\bar{M}_{1(1)}}{g}, \quad m_0 = \pi\rho h R^2 \bar{m}_0, \quad J_{y_c} = \pi\rho h R^4 \bar{J}_{y_c}. \tag{13}$$

In what follows we shall use non-dimensional quantities omitting bars over them.

Substituting Eqs. (11) and (12) into Eqs. (3), (9) and (10), one obtains the following set of ordinary differential equations for determination of the frequencies and modes of the natural vibrations of our construction:

$$L_{11}^{(n)}(u_n) + L_{12}^{(n)}(v_n) + L_{13}^{(n)}(w_n) + \omega^2 u_n = 0,$$
$$L_{21}^{(n)}(u_n) + L_{22}^{(n)}(v_n) + L_{23}^{(n)}(w_n) + \omega^2 v_n = 0,$$
$$L_{31}^{(n)}(u_n) + L_{32}^{(n)}(v_n) + L_{33}^{(n)}(w_n) - \omega^2 w_n = 0,$$
$$(n = 1, 2 \ldots), \tag{14}$$

where the operators $L_{ij}^{(n)}$ $(i, j = 1, 2, 3)$ are obtained from the operators L_{ij} after separation of the angle variable.

Eq. (14) describes system natural vibrations of two types. Those of the first type correspond to joint motion of the body and shell in the plane Oxz. In this case one should solve the set of Eqs. (14) with $n = 1$ using the following boundary conditions at $z = l$ and 0:

$$(Q_{1(1)}^* - S_{(1)})_{z=l} = \omega^2 m_0 u_{01},$$
$$(T_{1(1)} + M_{1(1)} + Q_{1(1)}^* l_c - S_{(1)} l_c)_{z=l} = \omega^2 J_{y_c} \vartheta_{02}, \tag{15}$$

$$u_1(l) = -\vartheta_{02}, \quad v_1(l) = \vartheta_{02} l_c - u_{01}, \quad w_1(l) = -v_1(l), \quad \left.\frac{\mathrm{d}w_1}{\mathrm{d}z}\right|_{z=l} = \vartheta_{02}, \tag{16}$$

$$u_1(0) = v_1(0) = w_1(0) = \left.\frac{\mathrm{d}w_1}{\mathrm{d}z}\right|_{z=0} = 0. \tag{17}$$

Vibrations of the second type can exist for $u_{01} = \vartheta_{02} \equiv 0$ and $n > 1$ only. In this case the solutions of the set (14) have to meet the following boundary conditions:

$$u_n(l) = v_n(l) = w_n(l) = \left.\frac{\mathrm{d}w_n}{\mathrm{d}z}\right|_{z=l} = 0,$$
$$u_n(0) = v_n(0) = w_n(0) = \left.\frac{\mathrm{d}w_n}{\mathrm{d}z}\right|_{z=0} = 0. \tag{18}$$

The boundary conditions obtained for two types of system vibrations arise from orthogonality of the functions $\sin n\varphi$ and $\cos n\varphi$ in the interval $[0, 2\pi]$.

Thus, when the number of circumferential waves $n > 1$, we have the classical problem on determination of frequencies and modes of non-axisymmetric natural vibrations of a cylindrical shell with two rigidly fixed ends. And when $n = 1$, we have a spectral problem with frequency parameter entering not only Eqs. (14) but the boundary conditions (15) as well. Besides, the boundary conditions (15) contain the generalized coordinates of rigid body motion. They are related to the corresponding shell displacements at $z = l$ with the geometric compatibility conditions (16). The minimal frequency of the system vibrations is the lower of two frequencies corresponding to the first or second type of vibrations. From the above equations and boundary conditions it also follows that for every eigenfrequency the only term remaining in the series (11) for displacements u, v and w is that with the index $n = 1$ (at joint vibrations of the body and shell) or with an arbitrary index $n > 1$ (for the second type of vibrations).

If in the set of Eqs. (14) there are coefficients that do not depend on the longitudinal coordinate z, then one can obtain the exact solution of the boundary value problem (14)–(17) and (14), (18) on the basis of the Euler method. Such approach, however, leads to rather complicated solution algorithms for the formulated problems. That is why we shall construct approximate solutions of the considered spectral problems using their equivalent variational statements. It was noted earlier that the most complicated boundary conditions (15) are natural ones for the corresponding functional obtained from the variational Eq. (4). So minimization of that functional should be made on the class of functions that meet the boundary conditions (16) and (17). For system vibrations of the second type, the class of admissible functions must obey the boundary conditions (18). Therefore let us present the sought functions for both types of vibrations in the following form:

$$u_n(z) = \sum_{j=1}^{N} a_j U_j(z) + \delta_{1n}\vartheta_{02}u_0(z),$$

$$v_n(z) = \sum_{j=1}^{N} b_j V_j(z) + \delta_{1n}(\vartheta_{02}l_c - u_{01})v_0(z),$$

$$w_n(z) = \sum_{j=1}^{N} c_j W_j(z) + \delta_{1n}(u_{01}w_0(z) + \vartheta_{02}f(z)), \tag{19}$$

where a_j, b_j and c_j are some arbitrary constants to be determined later on along with the constants u_{01} and ϑ_{02}, $\delta_{1n} = 1$ when $n = 1$ and $\delta_{1n} = 0$ when $n > 1$.

We take the functions $u_0(z)$, $v_0(z)$, $w_0(z)$ and $f(z)$ and coordinate functions $U_j(z)$, $V_j(z)$ and $W_j(z)$ in the expressions (19) to be of the following form:

$$u_0(z) = -\frac{1}{l}z, \quad v_0(z) = -u_0(z), \quad w_0(z) = \left(\frac{3}{l^2} - \frac{2}{l^3}z\right)z^2,$$

$$f(z) = \left(-\frac{l + 3l_c}{l^2} + \frac{2l_c + l}{l^3}z\right)z^2, \quad U_j(z) = V_j(z),$$

$$U_j(z) = z(z - l)P_j\left(\frac{2z}{l} - 1\right), \quad W_j(z) = z^2(z - l)^2 P_j\left(\frac{2z}{l} - 1\right)$$

$$(j = 1, 2 \ldots, N). \tag{20}$$

Here $P_j(z)$ are the Legendre polynomials shifted by unity in the index j. One can calculate them

$$P_{j+2}(z) = \frac{1}{j+1}\left[(2j + 1)zP_{j+1}(z) - jP_j(z)\right] \quad (j = 1, 2, \ldots). \tag{21}$$

The proposed presentations of the sought solutions as Eq. (19) meet the main boundary conditions (16), (17) (for vibrations of the first type) and Eq. (18) (for vibrations of the second type) at any values of the vector

$$\mathbf{X} = [a_1, a_2, \ldots, a_N, b_1, b_2, \ldots, b_N, c_1, c_2, \ldots, c_N, u_{01}, \vartheta_{02}]^{\mathrm{T}}.$$

The components of the vector $\mathbf{X}$ are determined further from the stationary conditions for the above functional. In this case the initial problem is reduced to solving the uniform algebraic set

$$(\mathbf{A} - \omega^2\mathbf{B})\mathbf{X} = 0, \tag{22}$$

where $\mathbf{A}$ and $\mathbf{B}$ are symmetric matrices of the order $3N + 2$ for $n = 1$ and $3N$ for $n > 1$.

One should note that, contrary to the traditional Ritz method, the solutions of the form (19) for the sought functions when $n = 1$ are not independent because they contain general unknown constants u_{01} and ϑ_{02}. As a result, formation of elements of the matrices **A** and **B** on the basis of the standard approach leads to rather cumbersome expressions for them, and the calculation algorithm becomes considerably more complicated. In this connection, the variation of the corresponding functional is presented as

$$\begin{aligned}\delta I = & \int_0^{\gamma} [\Psi_{11}(u_n, \delta u_n) + \Psi_{12}(v_n, \delta u_n) + \Psi_{13}(w_n, \delta u_n) \\ & + \Psi_{21}(u_n, \delta v_n) + \Psi_{22}(v_n, \delta v_n) + \Psi_{23}(w_n, \delta v_n) \\ & + \Psi_{31}(u_n, \delta w_n) + \Psi_{32}(v_n, \delta w_n) + \Psi_{33}(w_n, \delta w_n)]\, \mathrm{d}z \\ & - \omega^2 \int_0^{\gamma} (u_n \delta u_n + v_n \delta v_n + w_n \delta w_n)\, \mathrm{d}z \\ & - \delta_{1n} \omega^2 (m_0 u_{01} \delta u_{01} + J_{y_c} \vartheta_{02} \delta \vartheta_{02}).\end{aligned} \tag{23}$$

The introduced differential operators are of the following form:

$$\Psi_{11}(p,q) = \frac{\mathrm{d}p}{\mathrm{d}z}\frac{\mathrm{d}q}{\mathrm{d}z} + v_1 n^2 pq, \quad \Psi_{12}(p,q) = vnp\,\frac{\mathrm{d}q}{\mathrm{d}z} - v_1 n\,\frac{\mathrm{d}p}{\mathrm{d}z}\,q,$$

$$\Psi_{13}(p,q) = vp\,\frac{\mathrm{d}q}{\mathrm{d}z}, \quad \Psi_{23}(p,q) = npq, \quad \Psi_{22}(p,q) = n^2 pq + v_1\,\frac{\mathrm{d}p}{\mathrm{d}z}\frac{\mathrm{d}q}{\mathrm{d}z},$$

$$\Psi_{33}(p,q) = pq + c^2\left[\left(\frac{\mathrm{d}^2 p}{\mathrm{d}z^2} - vn^2 p\right)\frac{\mathrm{d}^2 q}{\mathrm{d}z^2} + \left(n^4 p - vn^2\,\frac{\mathrm{d}^2 p}{\mathrm{d}z^2}\right)q + 2(1-v)n^2\,\frac{\mathrm{d}p}{\mathrm{d}z}\frac{\mathrm{d}q}{\mathrm{d}z}\right],$$

where p and q are arbitrary functions.

Use of the variation of the functional as Eq. (23) makes it possible to comparatively easily determine, on a general basis, the elements of the matrices **A** and **B**. It also makes much more convenient programming of the proposed algorithm for solving the problem considered. The elements of the upper (over the leading diagonal) part of the matrices **A** and **B** are given in Appendix A.

When $n > 1$, the matrices **A** and **B** are obtained from the constructed ones by removal of two last rows and columns.

Thus the problem of determination of natural frequencies and modes of non-axisymmetric vibrations of a cylindrical shell with a rigid body attached to one of its ends was reduced to calculation of one-dimensional integrals followed by solving the generalized eigenvalue problem (22). With the proper choice of representation of the sought solutions that provide the required accuracy of calculations and stability of calculation procedure, the latter problem is solved easily using the standard software for modern PC. The proposed algorithm of solving the considered problem may be applied (without essential alterations) also for a shell whose elastic-mass characteristics vary along the axis.

4. A simplified statement of the problem and its solution

The initial problem can be essentially simplified in the case of relatively long shells, if one assumes that shell cross-sections remainn plane when being deformed. Then a shell can be replaced by an equivalent beam whose linear mass $m = \rho F = 2\pi Rh\rho$ and bending rigidity $EJ = E\pi R^3 h$ are constant along the beam length. In what follows we shall apply the refined Timoshenko's beam theory which takes into account shear strains and rotary inertia of beam cross-section. Then, in accordance with the results of the works [10,11,14], beam bending vibrations in the plane Oxz are described by a set of partial differential equations

$$\begin{aligned}& \rho F\,\frac{\partial^2 w}{\partial t^2} - \kappa GF\left(\frac{\partial^2 w}{\partial z^2} - \frac{\partial \psi}{\partial z}\right) = 0, \\ & \rho J\,\frac{\partial^2 \psi}{\partial t^2} - EJ\,\frac{\partial^2 \psi}{\partial z^2} - \kappa GF\left(\frac{\partial w}{\partial z} - \psi\right) = 0,\end{aligned} \tag{24}$$

where $w(z,t)$ are the displacements of the points of the neutral axis of elastic beam along the axis Ox; $\psi(z,t)$ is the angle the tangential curve makes with the beam elastic line due to action of bending moments; GF is the beam shear rigidity. The shear coefficient κ will be determined from the expression $\kappa = 2(1-\nu)/(4+3\nu)$ proposed in Ref. [15].

If the beam end is rigidly fixed at $z=0$ and a rigid body is attached to the other end at $z=l$, then the solutions of Eq. (24) must meet the following boundary conditions:

$$\left[\kappa GF\left(\frac{\partial w}{\partial z}-\psi\right)+m_0\frac{\partial^2 w}{\partial t^2}+L_0\frac{\partial^2\psi}{\partial t^2}\right]_{z=l}=0,$$
$$\left[EJ\frac{\partial\psi}{\partial z}+L_0\frac{\partial^2 w}{\partial t^2}+J_{y_1}\frac{\partial^2\psi}{\partial t^2}\right]_{z=l}=0,\quad w(0,t)=\psi(0,t)=0, \tag{25}$$

where $L_0=m_0 l_c$, $J_{y_1}=m_0 l_c^2+J_{y_c}$.

In the case of free harmonic vibrations of the system with frequency ω, we present the functions $w(z,t)$ and $\psi(z,t)$ as

$$w(z,t)=W(z)\mathrm{e}^{\mathrm{i}\omega t},\quad \psi(z,t)=\Psi(z)\mathrm{e}^{\mathrm{i}\omega t}.$$

Let us take the radius of the cylindrical shell R to be the characteristic linear size of the system studied and introduce the following non-dimensional quantities which are related to the corresponding dimensional ones with the following formulae:

$$\beta^2=\frac{\omega^2R^4\rho F}{EJ},\quad r^2=\frac{J}{FR^2},\quad s^2=\frac{EJ}{\kappa GFR^2},$$
$$\bar{m}_0=\frac{m_0}{\rho FR},\quad \bar{J}_{y_c}=\frac{J_{y_c}}{\rho FR^3},\quad \bar{W}=\frac{W}{R}.$$

The interrelation between the non-dimensional quantities introduced according to expressions (13) (in what follows they will be designated by (*)) and those above is of the following form:

$$\bar{\omega}^2=\frac{(1-\nu^2)}{2}\beta^2,\quad \bar{m}_0=\frac{1}{2}m_0^*,\quad \bar{J}_{y_c}=\frac{1}{2}\bar{J}_{y_c}^*.$$

For the sake of simplicity, from now on we shall omit bars over non-dimensional quantities.

After separating the variable t in Eqs. (24) and (25) and some simple transformations, the initial problem can be reduced to the following uniform problem concerning the function $W(z)$:

$$\frac{\mathrm{d}^4W}{\mathrm{d}z^4}+b_2\frac{\mathrm{d}^2W}{\mathrm{d}z^2}-\beta^2b_0W=0,$$
$$W(0)=0,\quad \left(b_1\frac{\mathrm{d}W}{\mathrm{d}z}+s^2\frac{\mathrm{d}^3W}{\mathrm{d}z}\right)_{z=0}=0,$$
$$\left(f_1\frac{\mathrm{d}^3W}{\mathrm{d}z^3}+f_2\frac{\mathrm{d}W}{\mathrm{d}z}+f_3W\right)_{z=l}=0,$$
$$\left(f_4\frac{\mathrm{d}^3W}{\mathrm{d}z^3}+f_5\frac{\mathrm{d}^2W}{\mathrm{d}z^2}+f_6\frac{\mathrm{d}W}{\mathrm{d}z}+f_7W\right)_{z=l}=0, \tag{26}$$

where

$$b_0=1-\beta^2r^2s^2,\quad b_1=\beta^2s^4+1,\quad b_2=\beta^2(r^2+s^2),$$
$$f_1=1+\beta^2s^2L_0,\quad f_2=\beta^2b_1L_0+b_2,\quad f_3=\beta^2m_0b_0,\quad f_4=\beta^2s^2J_{y_1},$$
$$f_5=-b_0,\quad f_6=\beta^2b_1J_{y_1},\quad f_7=b_0\beta^2(L_0-s^2).$$

It should be noted that the parameter r^2 is related to the effect of rotary inertia, while the parameter s^2 is related to the effects of shear strain. The equations for an Euler–Bernoulli beam can be obtained from the

Timoshenko equations if one sets $r^2 = s^2 = 0$. Similarly, the equations of beam motion (without regard for the effect of rotary inertia) may be obtained if one sets $r^2 = 0$. To obtain the equations for a Rayleigh beam, one should set $s^2 = 0$.

The general solution of the equation from Eq. (26) at $\mu \geqslant b_2/2$ is

$$W(\beta, z) = A \sinh \gamma_1 z + B \cosh \gamma_1 z + C \sin \gamma_2 z + D \cos \gamma_2 z, \tag{27}$$

where $\gamma_1 = \sqrt{\mu - b_2/2}$, $\gamma_2 = \sqrt{\mu + b_2/2}$. When $\mu < b_2/2$,

$$W(\beta, z) = A' \sin \gamma_1 z + B' \cos \gamma_1 z + C' \sin \gamma_2 z + D' \cos \gamma_2 z, \tag{28}$$

where $\gamma_1 = \sqrt{b_2/2 - \mu}$. Here $\mu = \sqrt{(b_2/2)^2 + b_0 \beta^2}$.

For further use of solutions (27) and (28), it is convenient to represent them as

$$W_i(\beta, z) = C_{1i} S_i(\beta, z) + C_{2i} T_i(\beta, z) + C_{3i} U_i(\beta, z) + C_{4i} V_i(\beta, z). \tag{29}$$

Here and later on $i = 1$ when $\mu \geqslant b_2/2$ and $i = 2$ when $\mu < b_2/2$.

The functions S_i, T_i, U_i and V_i are linear combinations of the functions entering Eqs. (27) and (28). They are of the following form:

$$\begin{aligned}
S_1(\beta, z) &= \frac{1}{2\mu}\left(\gamma_2^2 \cosh \gamma_1 z + \gamma_1^2 \cos \gamma_2 z\right),\\
T_1(\beta, z) &= \frac{1}{2\mu}\left(\frac{\gamma_2^2}{\gamma_1} \sinh \gamma_1 z + \frac{\gamma_1^2}{\gamma_2} \sin \gamma_2 z\right),\\
U_1(\beta, z) &= \frac{1}{2\mu}\left(\cosh \gamma_1 z - \cos \gamma_2 z\right),\\
V_1(\beta, z) &= \frac{1}{2\mu}\left(\frac{1}{\gamma_1} \sinh \gamma_1 z - \frac{1}{\gamma_2} \sin \gamma_2 z\right),\\
S_2(\beta, z) &= \frac{1}{2\mu}\left(\gamma_2^2 \cos \gamma_1 z - \gamma_1^2 \cos \gamma_2 z\right),\\
T_2(\beta, z) &= \frac{1}{2\mu}\left(\frac{\gamma_2^2}{\gamma_1} \sin \gamma_1 z - \frac{\gamma_1^2}{\gamma_2} \sin \gamma_2 z\right),\\
U_2(\beta, z) &= \frac{1}{2\mu}\left(\cos \gamma_1 z - \cos \gamma_2 z\right),\\
V_2(\beta, z) &= \frac{1}{2\mu}\left(\frac{1}{\gamma_1} \sin \gamma_1 z - \frac{1}{\gamma_2} \sin \gamma_2 z\right).
\end{aligned} \tag{30}$$

When representing the solutions in form (29), arbitrary constants are expressed through the values of the functions W_i and their derivatives at the point $z = 0$

$$W_i(\beta, 0) = C_{1i}, \quad W_i'(\beta, 0) = C_{2i}, \quad W_i''(\beta, 0) = C_{3i}, \quad W_i'''(\beta, 0) = C_{4i}.$$

By inserting solution (29) into the boundary conditions from Eq. (26), one obtains a uniform algebraic set in the constants of integration C_{3i} and C_{4i}

$$\begin{aligned}
C_{3i} a_{11}^{(i)} + C_{4i} a_{12}^{(i)} &= 0,\\
C_{3i} a_{21}^{(i)} + C_{4i} a_{22}^{(i)} &= 0.
\end{aligned} \tag{31}$$

Here $C_{1i} = 0$ and $C_{2i} = -K_1 C_{4i}$, where $K_1 = s^2/b_1$. The elements $a_{kj}^{(i)}$ are given in Appendix B.

Thus, solving the initial problem in simplified setting is reduced to solving the set of algebraic equations (31).

5. Numerical results

Following are some results of calculation of frequencies and modes of natural vibrations of our construction performed with the algorithms presented above. From now on we assume that the rigid body attached to the shell is in the form of circular cylinder of radius R and height $H = 2l_c$. In this case the non-dimensional moment of inertia of the rigid body is

$$J^*_{y_c} = \frac{m^*_0}{12}(3 + H^2).$$

In our calculations we took the following values of non-dimensional parameters of the system: $h = 0.01$; $l_c = 0.5$; $v = 0.3$. Both the shell length and body mass were varied.

Table 1 presents the results of calculations of the first five lower frequencies of bending vibrations of a "body–shell" mechanical system, ($n = 1$) at $l = 4$ and $m^*_0 = 100$, depending on the number of terms N in the expansions (19).

The results given in Table 1 indicate a sufficiently fast convergence of the Ritz sequences. As relative length of the shell goes down, the convergence of the computational process is improving as compared to the data given in Table 1. Increase of shell length ($l > 10$) should be accompanied by increasing the number of the coordinate functions. One should note that the chosen form of representation of the sought solutions and their approximation using the Legendre polynomials provide stability of the computational process up to $N \leqslant 40$. This fact enables one to calculate frequencies and modes of natural vibrations with high accuracy for a sufficiently wide range of the starting parameters of the mechanical system considered.

The behaviour of natural frequencies of the construction depending on the rigid body mass m^* is presented in Table 2, where one can see that increase of the rigid body mass leads to decrease of the system frequencies. In the limiting case (denoted as ($*$)) the frequencies are equal to the corresponding frequencies of vibrations of a shell of circumferential form (with $n = 1$) with two rigidly fixed ends.

Table 3 presents some results of calculation of minimal frequencies $\omega_{1(n)}$ for a shell with rigidly fixed ends in the parameter range $l/R = 2$–14 and $\chi = R/h = 200$–1000 ($n > 1$ is the second type of system vibrations). Given in parentheses are the numbers n of circular waves of shell surface corresponding to these frequencies (whose values are multiplied by ten). One can see that increase of shell length is accompanied by decreasing of the minimal frequencies of the system, concurrently with decreasing of the corresponding number n. Decrease

Table 1
Five lower frequencies of non-axisymmetric vibrations of a "body–shell" mechanical system depending on the number of terms in expansions (19) at $l = 4$, $m^*_0 = 100$

N	ω_1	ω_2	ω_3	ω_4	ω_5
1	0.02460	0.13472	0.36141	0.99067	1.44920
2	0.01632	0.13185	0.35650	0.61950	0.93549
3	0.01574	0.12695	0.32523	0.61877	0.81494
4	0.01500	0.12681	0.32491	0.57754	0.81054
5	0.01495	0.12637	0.32272	0.57739	0.76240
6	0.01480	0.12636	0.32271	0.57617	0.76237
7	0.01479	0.12631	0.32201	0.57616	0.76122
8	0.01474	0.12630	0.32200	0.57588	0.76121
9	0.01474	0.12630	0.32175	0.57588	0.76106
10	0.01474	0.12630	0.32175	0.57582	0.76105

Table 2
Five lower frequencies of non-axisymmetric vibrations of a "body–shell" mechanical system depending on the rigid body mass at $l = 4$

m^*_0	ω_1	ω_2	ω_3	ω_4	ω_5
0	0.10799	0.34989	0.62869	0.73160	0.81765
10^2	0.01474	0.12629	0.32175	0.57582	0.76106
10^3	0.00469	0.04226	0.30743	0.57360	0.76019
10^4	0.00149	0.01344	0.30614	0.57339	0.76010
10^5	0.00047	0.00425	0.30602	0.57337	0.76010
($*$)	—	—	0.30600	0.57336	0.76010

Table 3
The minimal frequencies ω_{1n} for a cylindrical shell with rigidly fixed ends, $n>1$—the body is stationary (the real frequency values are multiplied by 10)

l	$\chi = 200$	$\chi = 400$	$\chi = 600$	$\chi = 800$	$\chi = 1000$
2	$1.12969_{(7)}$	$0.82296_{(9)}$	$0.67886_{(10)}$	$0.59474_{(11)}$	$0.53114_{(11)}$
4	$0.58626_{(5)}$	$0.42330_{(6)}$	$0.34532_{(7)}$	$0.30348_{(8)}$	$0.26967_{(8)}$
6	$0.39910_{(4)}$	$0.28260_{(5)}$	$0.23213_{(6)}$	$0.20214_{(6)}$	$0.18249_{(7)}$
8	$0.29661_{(4)}$	$0.21836_{(5)}$	$0.17340_{(5)}$	$0.15440_{(5)}$	$0.13619_{(6)}$
10	$0.24526_{(3)}$	$0.16821_{(4)}$	$0.14396_{(5)}$	$0.12075_{(5)}$	$0.10834_{(5)}$
12	$0.19431_{(3)}$	$0.14264_{(4)}$	$0.11534_{(4)}$	$0.10411_{(4)}$	$0.09115_{(5)}$
14	$0.16645_{(3)}$	$0.12714_{(3)}$	$0.09907_{(4)}$	$0.08577_{(4)}$	$0.07886_{(4)}$

Table 4
The frequencies of system vibrations obtained with different calculation techniques (*I*—the shell technique, *II*—the Timoshenko's beam technique, *III* and *IV*—the beam technique with allowance made for either shear strains or rotary inertia only, *V*—the Euler–Bernoulli beam technique)

l	2	4	6	8	10
ω_1					
I	0.04104	0.02062	0.01266	0.00868	0.00640
II	0.04152	0.02073	0.01269	0.00868	0.00640
III	0.04156	0.02076	0.01271	0.00869	0.00640
IV	0.05582	0.02372	0.01367	0.00909	0.00660
V	0.05600	0.02376	0.01369	0.00910	0.00660
ω_2					
I	0.21491	0.16684	0.12665	0.09380	0.07040
II	0.21560	0.16707	0.12666	0.09380	0.07040
III	0.21856	0.16971	0.12842	0.09490	0.07110
IV	0.45942	0.24775	0.16542	0.11600	0.08380
V	0.46749	0.25257	0.16903	0.11840	0.08530
ω_3					
I	0.59670	0.34020	0.25126	0.20393	0.16650
II	0.68582	0.35616	0.25673	0.20530	0.16650
III	0.68635	0.35722	0.25836	0.20749	0.16890
IV	2.39765	0.86431	0.46155	0.30342	0.22220
V	3.86339	1.02308	0.501883	0.32016	0.23180
ω_4					
I	0.84825	0.57846	0.39471	0.30000	0.24788
II	1.29893	0.62354	0.40681	0.30614	0.25096
III	1.33409	0.64521	0.41891	0.31307	0.25542
IV	3.99919	1.69206	0.94024	0.60134	0.42342
V	10.47193	2.65297	1.20558	0.70117	0.46959

of shell thickness also leads to decreasing of the minimal frequencies, but accompanied with increasing of n. A comparison of the data presented in this table with the results of the work [5] (obtained on the basis of the exact solution of the eigenvalue problem considered) demonstrates that they coincide completely.

Table 4 presents the results of calculations of the first four frequencies of bending vibrations of shell and body depending on the shell length. The calculations were performed on the basis of the technical shell theory—(*I*), using the Timoshenko's beam theory—(*II*), the beam theory with allowance made for shear strains only ($r^2 = 0$)—(*III*), with allowance made for rotary inertia of the beam cross-section only ($s^2 = 0$)—(*IV*) and using the Euler–Bernoulli beam technique ($r^2 = s^2 = 0$)—(*V*).

The results presented in Table 4 demonstrate that, at the chosen body mass, the elementary beam theory gives good results when calculating the first frequency for long shells only ($l \geqslant 10$). Taking into account shear strains and rotary inertia in the beam equations considerably improve accuracy of the beam approximation when dealing with the construction considered. To illustrate, when $l > 6$, the first two frequencies calculated using the shell theory and Timoshenko's beam theory are practically the same, while for the third and fourth frequencies the discrepancies are no more than 3%. In this case, allowance for shear strains is of crucial importance. Rotary inertia may be of importance when calculating the higher frequencies of the system.

Fig. 2 gives spatial presentation of the surface of relative error δ_i (in per cent) at determination of the first three frequencies of system vibrations with the Timoshenko's beam theory as a function of the attached body

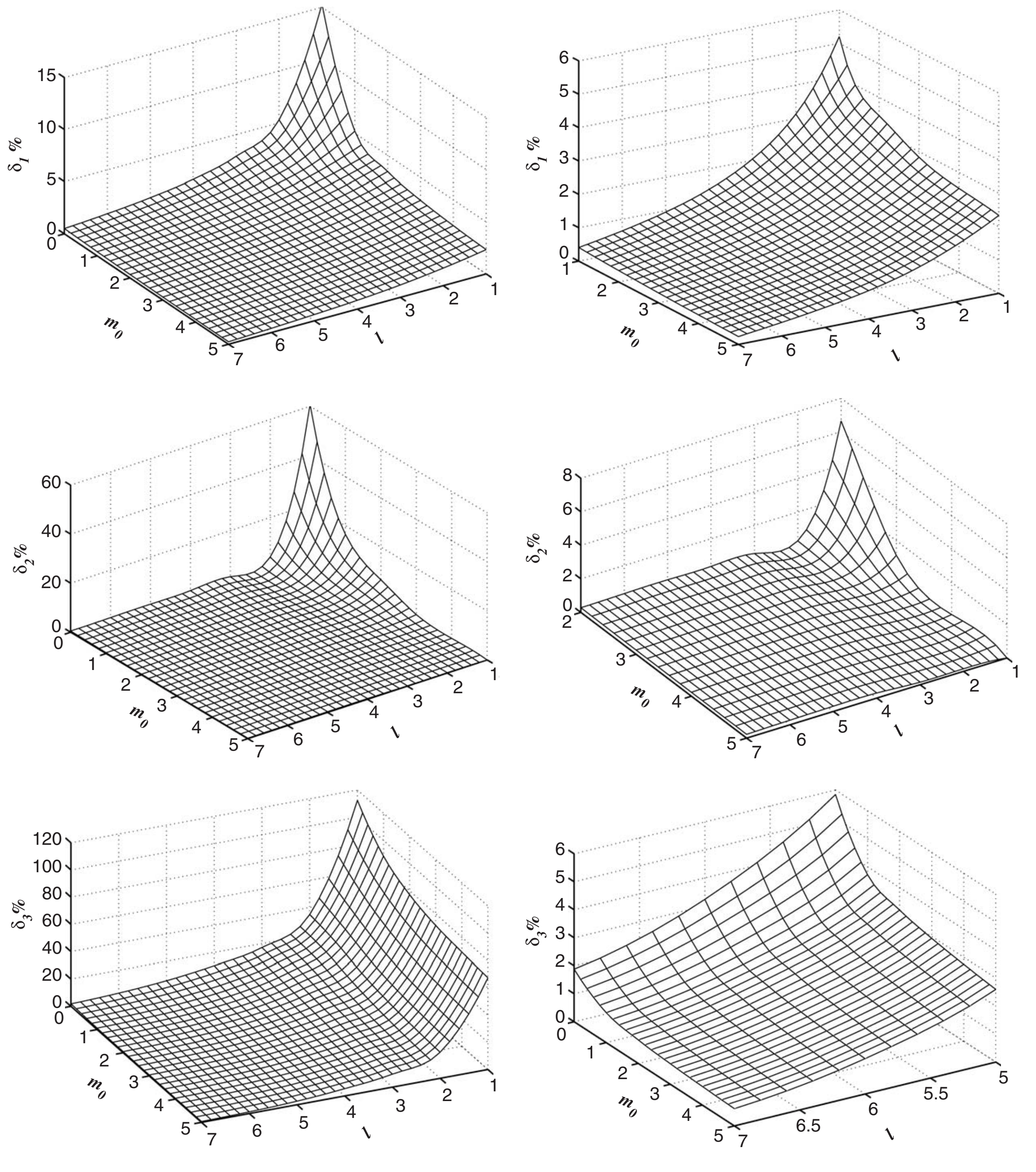

Fig. 2. The relative error δ_i% at calculation of the first three system frequencies with the Timoshenko's beam theory depending on the rigid body mass m_0 and shell length l.

mass and shell length. One can see that the errors δ_i depend essentially on the attached body mass and shell length. Increase of shell length (at a fixed body mass), as well as increase of body mass (at a fixed shell length), result in decrease of errors δ_i. For instance, the error in determination of the first frequency of system vibrations does not exceed 5%, even when $m_0 > 1$ and $l > 1$. The errors in determination of the first three frequencies of system vibrations do not exceed 1% when $m_0 > 1$ and $l > 6$.

The amplitude values of the first four radial modes of shell vibrations (divided by their maximal values W_i) are presented in Fig. 3 ($z^* = z/l$). Solid lines show the vibration modes determined with the shell theory, dashed lines—those determined with the Timoshenko's beam theory and dash-dotted lines—those determined with the elementary beam theory. The initial data were those from Table 4 at $l = 4$. One can see that the Timoshenko's beam model makes it possible to determine not only the lower frequencies of the mechanical system considered but the corresponding vibration modes as well. The biggest distinctions in vibration modes calculated using the mentioned theories are observed in the vicinity of the shell end cross-sections. These distinctions are of local character. They are due to the edge effects in shell deformation that grow as the relative shell thickness decreases. At the chosen values of system parameters, the elementary beam theory, in its turn, gives satisfactory results when calculating the first vibration mode only.

In conclusion it should be noted that, when determining the dynamic characteristics of the mechanical system considered according to the proposed calculation procedure using the Timoshenko's beam theory, we made a comparison with the corresponding table and graphic data obtained earlier in Ref. [11]. Our results were found to be in full agreement with those determined in Ref. [11].

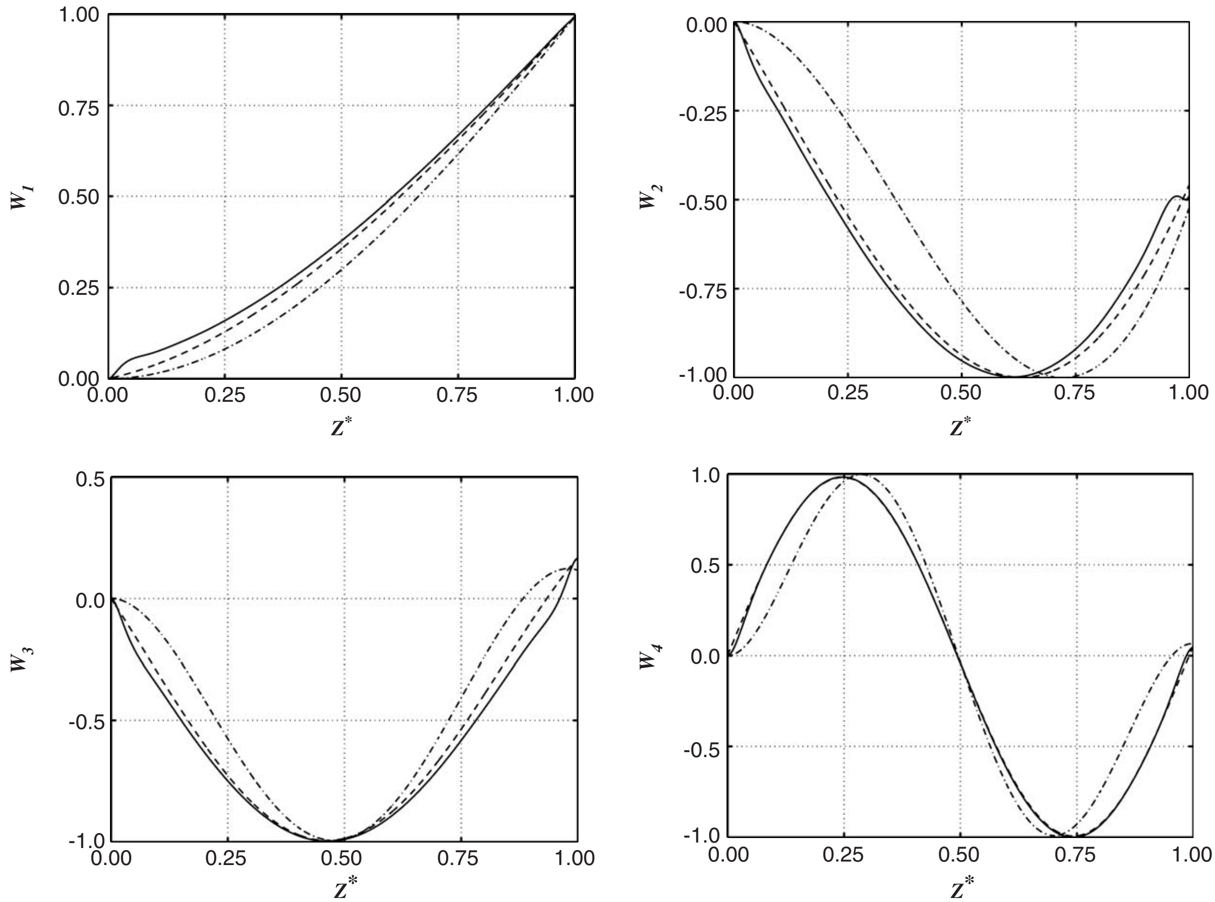

Fig. 3. A comparison between the amplitude values of the first four modes of system vibrations determined with the shell theory (solid lines), the Timoshenko's beam theory (dashed lines) and the Euler–Bernoulli beam theory (dash-dotted lines).

6. Conclusion

We constructed a mathematical model of coupled non-axisymmetric vibrations of a circular cylindrical shell and a perfectly rigid body attached to one of the shell ends. It is shown in the context of linear theory that the natural vibrations of the above system are separated into two types. Those of the first type are due to coupled vibrations of body and shell in one of the planes of symmetry of the system when the number of circumferential shell waves is unity. At vibrations of the second type, the shell executes spatial non-axisymmetric vibrations with the number of circumferential waves over unity; in this case the body remains stationary. The minimal frequency of the elastic system considered is the smallest of the lowest frequencies of vibrations of the first and second types.

We propose approximate solutions of the spectral problems obtained. They are determined on the basis of equivalent variational statements of the problems. We performed an analysis of these solutions, as well compared them with the (existing in the literature) exact ones for vibrations of the second type. The results of comparison evidence that our technique of solving the problems considered makes it possible to obtain rather accurate results when calculating the lower vibration modes in a wide range of the initial system parameters.

We constructed the exact solution of the problem (in its approximate setting) by replacing the shell with an equivalent Timoshenko beam. The obtained algorithm of calculation of the dynamic characteristics of the system enables one to perform calculations on the general basis for simpler beam theories too. It is shown that insertion of shear strains and rotary inertia into the equations of the beam theory results in essential improvement of the accuracy of approximating the shell with a beam. In this case the error of such approximation depends, to a great extent, on the shell length and mass of the attached body.

Acknowledgement

The author gratefully acknowledges partial support from DFG.

Appendix A. The elements of the upper (over the principal diagonal) part of the matrices A and B in the algebraic set (22)

$$a_{i,j} = \int_0^l \Psi_{11}(U_j, U_i)\,\mathrm{d}z, \quad a_{i,j+N} = \int_0^l \Psi_{12}(V_j, V_i)\,\mathrm{d}z, \quad a_{i,j+2N} = \int_0^l \Psi_{13}(W_j, U_i)\,\mathrm{d}z,$$

$$a_{i,3N+1} = \int_0^l [\Psi_{13}(w_0, U_i) - \Psi_{12}(v_0, U_i)]\,\mathrm{d}z,$$

$$a_{i,3N+2} = \int_0^l [\Psi_{11}(u_0, U_i) + l_c\Psi_{12}(v_0, U_i) + \Psi_{13}(f, U_i)]\,\mathrm{d}z,$$

$$a_{i+N,j+N} = \int_0^l \Psi_{22}(V_j, V_i)\,\mathrm{d}z, a_{i+N,j+2N} = \int_0^l \Psi_{23}(W_j, V_i)\,\mathrm{d}z,$$

$$a_{i+N,3N+1} = \int_0^l [\Psi_{23}(w_0, V_i) - \Psi_{22}(v_0, V_i)]\,\mathrm{d}z,$$

$$a_{i+N,3N+2} = \int_0^l [\Psi_{12}(V_i, u_0) + l_c\Psi_{22}(v_0, V_i) + \Psi_{23}(f, V_i)]\,\mathrm{d}z,$$

$$a_{i+2N,j+2N} = \int_0^l \Psi_{33}(W_j, W_i)\,\mathrm{d}z,$$

$$a_{i+2N,3N+1} = \int_0^l [\Psi_{33}(w_0, W_i) - \Psi_{23}(W_i, v_0)]\,\mathrm{d}z,$$

$$a_{i+2N,3N+2} = \int_0^l [\Psi_{13}(W_i, u_0) + l_c\Psi_{23}(W_i, v_0) + \Psi_{33}(f, W_i)]\,\mathrm{d}z,$$

$$a_{3N+1,3N+1} = \int_0^l [\Psi_{22}(v_0, v_0) - 2\Psi_{23}(w_0, v_0) + \Psi_{33}(w_0, w_0)]\,\mathrm{d}z,$$

$$a_{3N+1,3N+2} = \int_0^l [\Psi_{33}(f, w_0) - \Psi_{12}(v_0, u_0) - l_c\Psi_{22}(v_0, v_0)$$

$$- \Psi_{23}(f, v_0) + \Psi_{13}(w_0, u_0) + l_c\Psi_{23}(w_0, v_0)]\,\mathrm{d}z,$$

$$a_{3N+2,3N+2} = \int_0^l [\Psi_{11}(u_0, u_0) + 2l_c\Psi_{12}(v_0, u_0) + 2\Psi_{13}(f, u_0)$$

$$+ l_c^2\Psi_{22}(v_0, v_0) + 2l_c\Psi_{23}(f, v_0) + \Psi_{33}(f, f)]\,\mathrm{d}z,$$

$$b_{i,j} = \int_0^l U_i U_j\,\mathrm{d}z, \quad b_{i,j+N} = b_{i,j+2N} = b_{i,3N+1} = 0,$$

$$b_{i,3N+2} = \int_0^l u_0 U_i\,\mathrm{d}z, \quad b_{i+N,j+N} = \int_0^l V_i V_j\,\mathrm{d}z,$$

$$b_{i+N,j+2N} = 0, \quad b_{i+N,3N+1} = -\int_0^l v_0 V_i\,\mathrm{d}z, \quad b_{i+N,3N+2} = l_c \int_0^l v_0 V_i\,\mathrm{d}z,$$

$$b_{i+2N,j+2N} = \int_0^l W_j W_i\,\mathrm{d}z, \quad b_{i+2N,3N+1} = \int_0^l w_0 W_i\,\mathrm{d}z,$$

$$b_{i+2N,3N+2} = \int_0^l f W_i\,\mathrm{d}z, \quad b_{3N+1,3N+1} = \int_0^l (v_0^2 + w_0^2)\,\mathrm{d}z + m_0,$$

$$b_{3N+1,3N+2} = \int_0^l (f w_0 - l_c v_0^2)\,\mathrm{d}z, \quad b_{3N+2,3N+2} = \int_0^l (u_0^2 + l_c^2 v_0^2 + f^2)\,\mathrm{d}z + J_{y_c}.$$

Appendix B. The elements $a_{kj}^{(i)}$ in the algebraic set (31)

$$a_{11}^{(1)} = f_1 Q_{31} + f_2 Q_{11} + f_3 U_1,$$
$$a_{12}^{(1)} = f_1(Q_{21} - K_1\zeta U_1) + f_2(U_1 - K_1 S_1) + f_3(V_1 - K_1 T_1),$$
$$a_{21}^{(1)} = f_4 Q_{31} + f_5 Q_{21} + f_6 Q_{11} + f_7 U_1,$$

$$a_{22}^{(1)} = f_4(Q_{21} - K_1\zeta U_1) + f_5(Q_{11} - K_1\zeta V_1) + f_6(U_1 - K_1 S_1) + f_7(V_1 - K_1 T_1),$$

$$a_{11}^{(2)} = f_1 Q_{32} + f_2 Q_{12} + f_3 U_2,$$
$$a_{12}^{(2)} = f_1(Q_{22} + K_1\zeta U_2) + f_2(U_2 - K_1 S_2) + f_3(V_2 - K_1 T_2),$$
$$a_{21}^{(2)} = f_4 Q_{32} + f_5 Q_{22} + f_6 Q_{12} + f_7 U_2,$$

$$a_{22}^{(2)} = f_4(Q_{22} + K_1\zeta U_2) + f_5(Q_{12} + K_1\zeta V_2) + f_6(U_2 - K_1 S_2) + f_7(V_2 - K_1 T_2).$$

Here the following designations have been introduced into consideration: S_i, T_i, U_i, V_i, $(i = 1, 2)$ *are the functions* (30) *calculated at* $z = l$; $\zeta = \gamma_1^2\gamma_2^2$;

$$Q_{11} = \frac{1}{2\mu}(\gamma_1 \sinh \gamma_1 l + \gamma_2 \sin \gamma_2 l),$$

$$Q_{21} = \frac{1}{2\mu}(\gamma_1^2 \cosh \gamma_1 l + \gamma_2^2 \cos \gamma_2 l),$$

$$Q_{31} = \frac{1}{2\mu}(\gamma_1^3 \sinh \gamma_1 l - \gamma_2^3 \sin \gamma_2 l),$$

$$Q_{12} = \frac{1}{2\mu}(\gamma_2 \sin \gamma_2 l - \gamma_1 \sin \gamma_1 l),$$

$$Q_{22} = \frac{1}{2\mu}(\gamma_2^2 \cos \gamma_2 l - \gamma_1^2 \cos \gamma_1 l),$$

$$Q_{32} = \frac{1}{2\mu}(\gamma_1^3 \sin \gamma_1 l - \gamma_2^3 \sin \gamma_2 l).$$

References

[1] V.E. Breslavsky, Investigation of vibrations of thin shells bound to a loading material, in: *Proceedings of the VIII All-union Conference on the Theory of Shells and Plates*, Leningrad, May 21–28, 1973, Nauka, Moscow, 1973, pp. 271–276 (in Russian).

[2] V.E. Breslavsky, Longitudinal vibrations of a cylindrical shell bound to elasto-viscous filler and lumped masses, *Problemy Mashinostroeniya* 14 (1981) 27–32 (in Russian).

[3] V.G. Palamarchuk, Free vibrations of a system consisting of a ribbed cylindrical shell and perfectly rigid body, *Prikladnaya Mekhanika* 14 (4) (1978) 56–62 (in Russian).

[4] Yu.V. Trotsenko, On equilibrium equations of cylindrical shell with attached rigid body, *Nonlinear Oscillations* 4 (2001) 422–431.

[5] Yu.Yu. Shveiko, Yu.V. Gavrilov, A.D. Brusilovsky, On the effect of boundary conditions on the eigenfrequency spectra of cylindrical shells, *Reports of the MEI Scientific—Technical Conference*, 1965, pp. 131–148.

[6] Yu.V. Trotsenko, Free vibrations of a cylindrical shell joining two elastic beams, *Akustychny Visnyk* 5 (2002) 54–72 (in Russian).

[7] B.I. Rabinovich, V.P. Shmakov, V.S. Kobychkin, On the theory of vibrations of constructions supporting elastic vessels with liquid, *Issledovaniya po Teorii Sooruzheniy* 18 (1970) 68–84 (in Russian).

[8] V.A. Trotsenko, V.S. Kladinoga, Non-axisymmetric vibrations of a prestressed shell of revolution with attached rigid disk, *Prikladnaya Mekhanika* 30 (1994) 17–24 (in Russian).

[9] K. Forsberg, Axisymmetric and beam-type vibrations of thin cylindrical shells, *AIAA Journal* 2 (1969) 221–227.

[10] M.W.D. White, G.R. Heppler, Vibration modes and frequencies of Timoshenko beams with attached rigid bodies, *ASME Journal of Applied Mechanics* 62 (1995) 193–199.

[11] R.E. Rossi, P.A.A. Laura, Vibrations of a Timoshenko beam clamped at one end and carrying a finite mass at the other, *Applied Acoustics* 30 (1990) 293–301.

[12] V. Novozhilov, in: J.R.M. Radok, P. Noordhoft Ltd., (Eds.), *Thin Shell Theory*, second augmented and revised ed., (translated from the second Russian edition by P. G. Lowe) Groningen, the Netherlands, 1964 (Go to Math Reviews number 34:8694).

[13] A.W. Leissa, Vibration of Shells, NASA SP-288, Washington, DC, US Government Printing Office, 1973.

[14] W. Weaver Jr., S.P. Timoshenko, D.H. Young, *Vibration Problems in Engineering*, fifth ed., Wiley, New York, 1990.

[15] G.R. Cowper, The shear coefficient in Timoshenko's beam theory, *Journal of Applied Mechanics* 33 (1966) 335–340.