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Short Communication

Solution of multi-delay systems using hybrid of block-pulse functions and Taylor series

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Abstract

A method for finding the solution of linear time-varying multi-delay systems using a hybrid function is proposed. The properties of the hybrid functions which consist of block-pulse functions plus Taylor series are presented. The method is based upon expanding various time functions in the system as their truncated hybrid functions. Operational matrices of integration, delay and product are presented and are utilized to reduce the solution of multi-delay systems to the solution of algebraic equations. Illustrative examples are included to demonstrate the validity and applicability of the technique.

1. Introduction

Delays occur frequently in biological, chemical, transportation, electronic, communication, manufacturing and power systems [1]. Time-delay and multi-delay systems are therefore very important classes of systems whose control and optimization have been of interest to many investigators [2–5].

Orthogonal functions and Taylor series have received considerable attention in dealing with various problems of dynamic systems. Much progress has been made towards the solution of delay systems. The approach is to convert the delay-differential equation to an algebraic form through the use of operational matrices of integration and delay. These matrices can be uniquely determined based on the particular choices of basis functions. Special attention has been given to applications of Walsh functions [6], block pulse functions [7], Laguerre polynomials [8], Legendre polynomials [9], Chebyshev polynomials [10] and Taylor series [11]. To the best of our knowledge, the literature on numerical solution of multi-delay systems by using orthogonal functions and Taylor series, is sparse. Chen [12] used Walsh functions for the solution of multi-delay systems. Due to the nature of these functions, the solution obtained were piecewise constant and Razzaghi and Razzaghi [13] employed Taylor series to derive continuous solution.

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The available sets of orthogonal functions can be divided into three classes. The first includes set of piecewise constant basis functions (PCBFs) (e.g., Walsh, block-pulse, etc.). The second consists of a set of orthogonal polynomials (e.g., Laguerre, Legendre, Chebyshev, etc.). The third is the widely used set of sine–cosine functions in Fourier series. While orthogonal polynomials and sine–cosine functions together form a class of continuous basis functions, PCBFs have inherent discontinuities or jumps. It is worth mentioning that approximating a continuous function with PCBFs results in an approximation that is piecewise constant. On the other hand, if a discontinuous function is approximated by continuous basis functions, the discontinuities are not properly modeled. Signals frequently have mixed features of continuity and jumps. These signals are continuous over certain segments of time, with discontinuities or jump occurring at the transitions of the segments. In such situations, neither the continuous basis functions nor PCBFs taken alone would form an efficient basis in the representation of such signals. In general, the computed response of the delay systems via orthogonal functions and Taylor series is not in good agreement with the exact response of the system [14].

In the present paper, we introduce a new direct computational method to solve multi-delay systems. The method consists of reducing the multi-delay problem to a set of algebraic equations by first expanding the candidate function as a hybrid function with unknown coefficients. These hybrid functions, which consist of block-pulse functions plus Taylor series are first introduced. The operational matrices of integration, delay, and product are given. These matrices are then used to evaluate the coefficients of the hybrid function for the solution of multi-delay systems.

The paper is organized as follows: in Section 2, we describe the basic properties of the hybrid functions of block-pulse and Taylor series required for our subsequent development. Section 3 is devoted to the formulation of linear time-varying multi-delay systems. In Section 4, we apply the proposed numerical method to multi-delay systems, and in Section 5, we report our numerical finding and demonstrate the accuracy of the proposed scheme by considering numerical examples.

2. Properties of hybrid functions

2.1. Hybrid functions of block-pulse and Taylor polynomials

Hybrid functions $b_{nm}(t)$, $n = 1, 2, \ldots, N$, $m = 0, 1, \ldots, M-1$, are defined on the interval $[0, t_f)$ as

$$b_{nm}(t) = \begin{cases} T_m(Nt - (n-1)t_f), & t \in \left[\left(\frac{n-1}{N}\right)t_f, \frac{n}{N}t_f\right), \\ 0, & \text{otherwise}, \end{cases} \tag{1}$$

where n and m are the order of block-pulse functions and Taylor polynomials, respectively, and $T_m(t) = t^m$.

2.2. Function approximation

A function $f(t)$ defined over the interval 0 to t_f may be expanded as

$$f(t) = \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} c_{nm} b_{nm}(t), \tag{2}$$

where

$$c_{nm} = \frac{1}{N^m m!}\left(\frac{\mathrm{d}^m f(t)}{\mathrm{d}t^m}\right)\bigg|_{t=((n-1)/N)t_f}.$$

If $f(t)$ in Eq. (2) is truncated, then Eq. (2) can be written as

$$f(t) \simeq \sum_{n=1}^{N}\sum_{m=0}^{M-1} c_{nm} b_{nm}(t) = C^{\mathrm{T}} B(t),$$

where

$$C = [c_{10}, \ldots, c_{1M-1}, c_{20}, \ldots, c_{2M-1}, \ldots, c_{N0}, \ldots, c_{NM-1}]^{\mathrm{T}}$$

and

$$B(t) = [b_{10}(t), \ldots, b_{1M-1}(t), b_{20}(t), \ldots, b_{2M-1}(t), \ldots, b_{N0}(t), \ldots, b_{NM-1}(t)]^{\mathrm{T}}. \tag{3}$$

The integration of the vector $B(t)$ defined in Eq. (3) can be approximated by

$$\int_0^t B(t')\,\mathrm{d}t' \simeq PB(t), \tag{4}$$

where P is the $MN \times MN$ operational matrix for integration and is given by

$$P = \begin{pmatrix} E & H & H & \cdots & H \\ 0 & E & H & \cdots & H \\ 0 & 0 & E & \cdots & H \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & E \end{pmatrix} \tag{5}$$

with

$$H = \frac{1}{N} \begin{pmatrix} t_f & 0 & 0 & \cdots & 0 \\ \dfrac{t_f^2}{2} & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ \dfrac{t_f^M}{M} & 0 & 0 & \cdots & 0 \end{pmatrix}$$

and E is the operational matrix of integration for Taylor polynomials on the interval $[((n-1)/N)t_f, (n/N)t_f]$ which is given in Ref. [13] by

$$E = \frac{1}{N} \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & \frac{1}{2} & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & \dfrac{1}{M-1} \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

2.3. The product operational matrix of the hybrid of block-pulse and Taylor polynomials

The following property of the product of two hybrid function vectors will also be used:
Let

$$B(t)B^{\mathrm{T}}(t)C \simeq \tilde{C}B(t), \tag{6}$$

where $\tilde{C}$ is a $MN \times MN$ product operational matrix. To illustrate the calculation procedure we choose $M = 3$ and $N = 4$. Thus we have

$$C = [c_{10}, c_{11}, c_{12}, \ldots, c_{40}, c_{41}, c_{42}]^{\mathrm{T}}, \tag{7}$$

$$B(t) = [b_{10}(t), b_{11}(t), b_{12}(t), \ldots, b_{40}(t), b_{41}(t), b_{42}(t)]^{\mathrm{T}}. \tag{8}$$

In Eq. (8) we have

$$\left.\begin{aligned} b_{10} &= 1 \\ b_{11} &= 4t \\ b_{12} &= 16t^2 \end{aligned}\right\} 0 \leqslant t < \tfrac{1}{4}, \quad \left.\begin{aligned} b_{20} &= 1 \\ b_{21} &= 4t-1 \\ b_{22} &= (4t-1)^2 \end{aligned}\right\} \tfrac{1}{4} \leqslant t < \tfrac{1}{2} \tag{9}$$

and

$$\left.\begin{aligned} b_{30} &= 1 \\ b_{31} &= 4t-2 \\ b_{32} &= (4t-2)^2 \end{aligned}\right\} \tfrac{1}{2} \leqslant t < \tfrac{3}{4}, \quad \left.\begin{aligned} b_{40} &= 1 \\ b_{41} &= 4t-3 \\ b_{42} &= (4t-3)^2 \end{aligned}\right\} \tfrac{3}{4} \leqslant t < 1. \tag{10}$$

Using Eqs. (9) and (10) we have $b_{ij}b_{kl} = 0$ if $i \neq k$, $b_{i0}b_{ij} = b_{ij}$, $b_{i1}b_{i1} = b_{i2}$, $b_{i1}b_{i2} = b_{i3}$, $b_{i2}b_{i2} = b_{i4}$. If we retain only the elements of $B(t)$ in Eq. (8), then we get

$$B(t)B^{\mathrm{T}}(t) = \begin{pmatrix} b_{10} & b_{11} & b_{12} & & & \\ b_{11} & b_{12} & 0 & & \bigcirc & \\ b_{12} & 0 & 0 & & & \\ & & & b_{40} & b_{41} & b_{42} \\ & \bigcirc & & b_{41} & b_{42} & 0 \\ & & & b_{42} & 0 & 0 \end{pmatrix}.$$

By using the vector C in Eq. (7) the 12×12 matrix $\tilde{C}$ in Eq. (6) is

$$\tilde{C} = \begin{pmatrix} \tilde{C}_1 & 0 & 0 & 0 \\ 0 & \tilde{C}_2 & 0 & 0 \\ 0 & 0 & \tilde{C}_3 & 0 \\ 0 & 0 & 0 & \tilde{C}_4 \end{pmatrix},$$

where $\tilde{C}_i,\ i = 1,2,3,4$ are 3×3 matrices given by

$$\tilde{C}_i = \begin{pmatrix} c_{i0} & c_{i1} & c_{i2} \\ 0 & c_{i0} & c_{i1} \\ 0 & 0 & c_{i0} \end{pmatrix}.$$

2.4. The multi-delay operational matrix of the hybrid of block-pulse and Taylor polynomials

The delay function $B(t-k_j),\ j = 1,2,\ldots,r$ is the shift of the function $B(t)$ defined in Eq. (3), along the time axis by k_j, where $k_1, k_2, \ldots, k_r$ are rational numbers in $(0,1)$. It is assumed without loss of generality that $k_1 < k_2 < \cdots < k_r$. The general expression is given by

$$B(t-k_j) = D_j B(t), \quad t > k_j, \tag{11}$$

where D_j is the delay operational matrix of hybrid functions corresponding to k_j. To find D_j for $j = 1,2,\ldots,r$, we first choose N the order of block-pulse functions in the following manner:

We define w as the smallest positive integer number for which $wk_j \in \mathbb{Z}$ for $j = 1,2,\ldots,r$. Next we choose λ as the greatest common divisor of the integers $wk_j,\ j = 1,2,\ldots,r$, that is

$$\lambda = \mathrm{g.c.d}(wk_1, wk_2, \ldots, wk_r).$$

Let

$$N=\begin{cases}\dfrac{w}{\lambda} & \text{if } \dfrac{w}{\lambda}\in\mathbb{Z},\\ \left[\dfrac{w}{\lambda}\right]+1 & \text{otherwise,}\end{cases} \tag{12}$$

where [.] denotes greatest integer value.

With the aid of Eq. (1), it is noted that for the case $k_j<t<k_j+\lambda/w$, the only terms with nonzero values are $b_{1m}(t-k_j)$ for $m=0,1,2,\dots,M-1$. If we set $\beta_j=wk_j/\lambda+1$, and expand $b_{1m}(t-k_j)$ in terms of $b_{\beta_j m}(t)$, since $b_{1m}(t-k_j)=b_{\beta_j m}(t)$, then the coefficient (element) of the delay matrix is an $M\times M$ identity matrix.

In a similar manner, for $k_j+\lambda/w<t<k_j+2\lambda/w$, only $b_{2m}(t-k_j)$ for $m=0,1,2,\dots,M-1$ has nonzero values. If we set $\gamma_j=\beta_j+1$, and expand $b_{2m}(t-k_j)$ in terms of $b_{\gamma_j m}(t)$, since $b_{2m}(t-k_j)=b_{\gamma_j m}(t)$, then the element of the delay matrix is $M\times M$ identity matrix. Thus, if we expand $B(t-k_j)$ in terms of $B(t)$ we find $NM\times NM$ matrix D_j as

$$D_j=\begin{pmatrix}0 & 0 & \cdots & 0 & I & 0 & \cdots & 0\\ 0 & 0 & \cdots & 0 & 0 & I & \cdots & 0\\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots\\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & I\\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0\end{pmatrix}.$$

It is noted that the first identity matrix in the first row is located at the β_jth column.

3. Problem statement

Consider the following linear time-varying multi-delay system:

$$\dot{X}(t)=E(t)X(t)+\sum_{j=1}^{r}F_j(t)X(t-k_j)+G(t)U(t),\quad 0\leqslant t\leqslant 1, \tag{13}$$

$$X(0)=X_0, \tag{14}$$

$$X(t)=\Phi(t),\quad t<0, \tag{15}$$

where $X(t)\in\mathbb{R}^l$, $U(t)\in\mathbb{R}^q$, $E(t)$, $G(t)$, and $F_j(t)$, $j=1,2,\dots,r$, are matrices of appropriate dimensions, X_0 is a constant specified vector, and $\Phi(t)$ is an arbitrary known function. The problem is to find $X(t)$, $0\leqslant t\leqslant 1$, satisfying Eqs. (13)–(15).

4. Approximation using hybrid functions

Let

$$X(t)=[x_1(t),x_2(t),\dots,x_l(t)]^{\mathrm{T}},\quad U(t)=[u_1(t),u_2(t),\dots,u_q(t)]^{\mathrm{T}}, \tag{16}$$

$$\hat{B}(t)=I_l\otimes B(t),\quad \hat{B}_1(t)=I_q\otimes B(t), \tag{17}$$

where I_l and I_q are the l- and q-dimensional identity matrices, $B(t)$ is $MN\times 1$ vector and $\otimes$ denotes the Kronecker product [15]. Using the property of the Kronecker product, $\hat{B}(t)$ and $\hat{B}_1(t)$ are matrices of order $lMN\times l$ and $qMN\times q$, respectively. Assume that each $x_i(t)$ and each of $u_j(t)$, $i=1,2,\dots,l$, $j=1,2,\dots,q$, can be written in terms of hybrid functions as

$$x_i(t)=B^{\mathrm{T}}(t)X_i,\quad u_j(t)=B^{\mathrm{T}}(t)U_j.$$

Then, using Eqs. (16) and (17), we have

$$X(t) = \hat{B}^{\mathrm{T}}(t)X, \quad U(t) = \hat{B}_1^{\mathrm{T}}(t)U, \tag{18}$$

where X and U are vectors of order $lMN \times 1$ and $qMN \times 1$, respectively, given by

$$X = [X_1, X_2, \ldots, X_l]^{\mathrm{T}}, \quad U = [U_1, U_2, \ldots, U_q]^{\mathrm{T}}.$$

Similarly we have

$$X(0) = \hat{B}^{\mathrm{T}}(t)d, \quad \Phi(t-k_j) = \hat{B}^{\mathrm{T}}(t)R_j, \tag{19}$$

where d and R_j, $j = 1, 2, \ldots, r$, are vectors of order $lMN \times 1$ given by

$$d = [d_1, d_2, \ldots, d_l]^{\mathrm{T}}, \quad R_j = [\alpha_{j1}, \alpha_{j2}, \ldots, \alpha_{jl}]^{\mathrm{T}}.$$

We now expand $E(t)$, $F_j(t)$, $j = 1, 2, \ldots, r$, and $G(t)$ by hybrid functions as follows:

$$E(t) = [E_{10}, E_{11}, \ldots, E_{1M-1}, \ldots, E_{N0}, E_{N1}, \ldots, E_{NM-1}]^{\mathrm{T}}\hat{B}(t) = E^{\mathrm{T}}\hat{B}(t),$$

$$F_j(t) = [F_{j10}, F_{j11}, \ldots, F_{j1(M-1)}, \ldots, F_{jN0}, F_{jN1}, \ldots, F_{jN(M-1)}]^{\mathrm{T}}\hat{B}(t) = F_j^{\mathrm{T}}\hat{B}(t),$$

$$G(t) = [G_{10}, G_{11}, \ldots, G_{1M-1}, \ldots, G_{N0}, G_{N1}, \ldots, G_{NM-1}]^{\mathrm{T}}\hat{B}_1(t) = G^{\mathrm{T}}\hat{B}_1(t),$$

where E^{T}, F_j^{T}, $j = 1, 2, \ldots, r$, and G^{T} are of dimensions $l \times lMN$, $l \times lMN$ and $l \times qMN$, respectively.

We can also write $X(t-k_j)$, $j = 1, 2, \ldots, r$, in terms of hybrid functions as

$$X(t-k_j) = \begin{cases} \hat{B}^{\mathrm{T}}(t)R_j, & 0 \leqslant t \leqslant k_j, \\ \hat{B}^{\mathrm{T}}(t)\hat{D}_j^{\mathrm{T}}X, & k_j < t \leqslant 1, \end{cases}$$

where

$$\hat{D}_j = I_l \otimes D_j$$

and D_j is the delay operational matrix given in Eq. (11). Now we have

$$E(t)X(t) = E^{\mathrm{T}}\hat{B}(t)\hat{B}^{\mathrm{T}}(t)X = \hat{B}^{\mathrm{T}}(t)\tilde{E}^{\mathrm{T}}X, \quad G(t)U(t) = G^{\mathrm{T}}\hat{B}_1(t)\hat{B}_1^{\mathrm{T}}(t)U = \hat{B}^{\mathrm{T}}(t)\tilde{G}^{\mathrm{T}}U, \tag{20}$$

where $\tilde{E}$ and $\tilde{G}$ can be calculated similarly to matrix $\tilde{C}$ in Eq. (6). Moreover,

$$\int_0^t \hat{B}^{\mathrm{T}}(t')\,\mathrm{d}t' = (I_l \otimes B^{\mathrm{T}}(t))(I_l \otimes P^{\mathrm{T}}) = \hat{B}^{\mathrm{T}}(t)\hat{P}^{\mathrm{T}}, \tag{21}$$

$$\int_0^t F_j(t')X(t'-k_j)\,\mathrm{d}t' = \begin{cases} \hat{B}^{\mathrm{T}}(t)\hat{P}^{\mathrm{T}}\tilde{F}_j^{\mathrm{T}}R_j, & 0 \leqslant t \leqslant k_j, \\ \hat{B}^{\mathrm{T}}(t)Z_j\tilde{F}_j^{\mathrm{T}}R_j + \hat{B}^{\mathrm{T}}(t)\hat{P}^{\mathrm{T}}\tilde{F}_j^{\mathrm{T}}\hat{D}_j^{\mathrm{T}}X, & k_j < t \leqslant 1, \end{cases} \tag{22}$$

where

$$\hat{P} = I_l \otimes P$$

and P is the operational matrix of integration given in Eq. (5), and

$$\int_0^{k_j} \hat{B}^{\mathrm{T}}(t)\,\mathrm{d}t = \hat{B}^{\mathrm{T}}(t)Z_j,$$

where Z_j, $j = 1, 2, \ldots, r$, is a constant matrix of order $lMN \times lMN$.

By integrating Eq. (13) from 0 to t and using Eqs. (18)–(22), we have

$$\begin{aligned} \hat{B}^{\mathrm{T}}(t)X - \hat{B}^{\mathrm{T}}(t)d = {} & \hat{B}^{\mathrm{T}}(t)\hat{P}^{\mathrm{T}}\tilde{E}^{\mathrm{T}}X + \sum_{j=1}^{r}[\hat{B}^{\mathrm{T}}(t)\hat{P}^{\mathrm{T}}\tilde{F}_j^{\mathrm{T}}R_j + \hat{B}^{\mathrm{T}}(t)Z_j\tilde{F}_j^{\mathrm{T}}R_j + \hat{B}^{\mathrm{T}}(t)\hat{P}^{\mathrm{T}}\tilde{F}_j^{\mathrm{T}}\hat{D}_j^{\mathrm{T}}X] \\ & + \hat{B}^{\mathrm{T}}(t)\hat{P}^{\mathrm{T}}\tilde{G}^{\mathrm{T}}U. \end{aligned} \tag{23}$$

Using Eq. (23) we obtain

$$X = \left[I - \hat{P}^{\mathrm{T}} \tilde{E}^{\mathrm{T}} - \sum_{j=1}^{r} \hat{P}^{\mathrm{T}} \tilde{F}_j^{\mathrm{T}} \hat{D}_j^{\mathrm{T}} \right]^{-1} \left[d + \sum_{j=1}^{r} (\hat{P}^{\mathrm{T}} \tilde{F}_j^{\mathrm{T}} R_j + Z_j \tilde{F}_j^{\mathrm{T}} R_j) + \hat{P}^{\mathrm{T}} \tilde{G}^{\mathrm{T}} U \right].$$

5. Illustrative examples

In this section three examples are given to demonstrate the applicability, efficiency, and accuracy of our proposed method. First by using Eq. (12), we determine N, which gives the number of intervals for a specific problem. Thus we have different intervals given by

$$\left[0, \frac{\lambda}{w}\right], \left[\frac{\lambda}{w}, \frac{2\lambda}{w}\right], \ldots, \left[(N-1)\frac{\lambda}{w}, N\frac{\lambda}{w}\right].$$

To define $x(t)$ for t in the interval $[0, \lambda/w]$ we map $[0, \lambda/w]$ into $[0, N\lambda/w]$ by mapping t into Nt, and for t in the interval $[\lambda/w, 2\lambda/w]$ we map this interval into $[0, N\lambda/w]$ by mapping t into $Nt - N(\lambda/w)$, and similarly for the other intervals. When selecting M, we first choose an arbitrary number depending on the problem. Since we are approximating by using Taylor series only in each subinterval, if the exact solutions in each subinterval are polynomials, we can increase the value of M by 1 until two consecutive results are the same in each subinterval. When the exact solutions in each subinterval are not polynomials, we evaluate the results for two consecutive M's for different t in $[0, 1]$ until the results are similar up to a required number of decimal places for each subinterval.

5.1. Example 1

Consider the multi-delay system described by

$$\dot{x}(t) = x(t - 0.35) + x(t - 0.7) + 1, \quad 0 \leqslant t \leqslant 1, \tag{24}$$

$$x(t) = 0, \quad t \leqslant 0. \tag{25}$$

Although the above system is time invariant, the method described here can be used. The exact solution is [13]

$$x(t) = \begin{cases} t, & 0 \leqslant t \leqslant 0.35, \\ t + \frac{1}{2}(t - 0.35)^2, & 0.35 \leqslant t \leqslant 0.7, \\ \frac{609}{800} + \frac{27}{20}(t - 0.7) + (t - 0.7)^2 + \frac{1}{6}(t - 0.7)^3, & 0.7 \leqslant t \leqslant 1. \end{cases}$$

By using Eq. (12), since $k_1 = 35/100 = 7/20$ and $k_2 = 7/10$, we get $w = 20$ and $\lambda = \text{g.c.d}(7, 14) = 7$, hence we select $N = [20/7] + 1 = 3$, we also choose $M = 4$.

Let

$$x(t) = C^{\mathrm{T}} B(t), \tag{26}$$

where

$$C = [c_{10}, \ldots, c_{13}, c_{20}, \ldots, c_{23}, c_{30}, \ldots, c_{33}, c_{40}, \ldots, c_{43}]^{\mathrm{T}} \tag{27}$$

and

$$B(t) = [b_{10}(t), \ldots, b_{13}(t), b_{20}(t), \ldots, b_{23}(t), b_{30}(t), \ldots, b_{33}(t), b_{40}(t), \ldots, b_{43}(t)]^{\mathrm{T}}. \tag{28}$$

By expanding t in terms of hybrid functions we get

$$t = e^{\mathrm{T}} B(t), \tag{29}$$

where

$$e = [0, \tfrac{1}{3}, 0, 0, \tfrac{7}{20}, \tfrac{1}{3}, 0, 0, \tfrac{7}{10}, \tfrac{1}{3}, 0, 0]^{\mathrm{T}}.$$

We also have

$$x(t-0.35) = C^T D_1 B(t), \quad t > 0.35, \tag{30}$$

$$x(t-0.7) = C^T D_2 B(t), \quad t > 0.7, \tag{31}$$

where D_1 and D_2 are the delay operational matrices given by

$$D_1 = \begin{pmatrix} 0 & I_4 & 0 \\ 0 & 0 & I_4 \\ 0 & 0 & 0 \end{pmatrix}, \quad D_2 = \begin{pmatrix} 0 & 0 & I_4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{32}$$

where I_4 is a 4-dimensional identity matrix.

Integrating Eq. (24) from 0 to t and using Eqs. (25)–(32) we obtain

$$C^T = e^T[I_{12} - DP]^{-1},$$

where $D = D_1 + D_2$ and P is the operational matrix of integration given in Eq. (5). Using Eq. (32) the vector C can be found as

$$C = [0, \tfrac{1}{3}, 0, 0, \tfrac{7}{20}, \tfrac{1}{3}, \tfrac{1}{18}, 0, \tfrac{609}{800}, \tfrac{9}{20}, \tfrac{1}{9}, \tfrac{1}{162}]^T.$$

Further, to define $x(t)$ for t in the interval $[0, 0.35]$ we map $[0, 0.35]$ into $[0, 1.05]$ by mapping t into $3t$ and similarly for the other intervals. From Eq. (26) we get

$$x(t) = \begin{cases} \frac{1}{3}T_1(3t), & 0 \leqslant t \leqslant 0.35, \\ \frac{7}{20}T_0(3t-1.05) + \frac{1}{3}T_1(3t-1.05) + \frac{1}{18}T_2(3t-1.05)^2, & 0.35 \leqslant t \leqslant 0.7, \\ \frac{609}{800}T_0(3t-2.1) + \frac{9}{20}T_1(3t-2.1) + \frac{1}{9}T_2(3t-2.1)^2 + \frac{1}{162}T_3(3t-2.1)^3, & 0.7 \leqslant t \leqslant 1, \end{cases}$$

where $T_m(t) = t^m,\ m = 0, 1, \ldots, M-1$. After simplifying the same value as the exact $x(t)$ would be obtained.

5.2. Example 2

Consider the following multi-delay system:

$$\dot{x}(t) = tx(t-0.4) + x(t-0.8) + 1, \quad 0 \leqslant t \leqslant 1, \tag{33}$$

$$x(t) = 0, \quad t \leqslant 0. \tag{34}$$

The exact solution is [13]

$$x(t) = \begin{cases} t, & 0 \leqslant t \leqslant 0.4, \\ \frac{2}{5} + (t-0.4) + \frac{1}{5}(t-0.4)^2 + \frac{1}{3}(t-0.4)^3, & 0.4 \leqslant t \leqslant 0.8, \\ \frac{64}{75} + \frac{33}{25}(t-0.8) + \frac{11}{10}(t-0.8)^2 + \frac{29}{75}(t-0.8)^3 + \frac{7}{60}(t-0.8)^4 + \frac{1}{15}(t-0.8)^5, & 0.8 \leqslant t \leqslant 1. \end{cases}$$

Here, we solve this problem with hybrid functions by choosing $N = 3$ and $M = 6$.

Let

$$x(t) = C^T B(t), \tag{35}$$

where C and $B(t)$ can be obtained similarly to Eqs. (27) and (28). We also have

$$x(t-0.4) = C^T D_1 B(t), \quad t > 0.4, \tag{36}$$

$$x(t-0.8) = C^T D_2 B(t), \quad t > 0.8, \tag{37}$$

$$t = [0, \tfrac{1}{3}, 0, 0, 0, 0, \tfrac{2}{5}, \tfrac{1}{3}, 0, 0, 0, 0, \tfrac{4}{5}, \tfrac{1}{3}, 0, 0, 0, 0]B(t) = K^T B(t), \tag{38}$$

where D_1 and D_2 are the delay operational matrices given in Eq. (32).

Integrating Eq. (33) from 0 to t and using Eqs. (34)–(38) we get

$$C^{\mathrm{T}} = K^{\mathrm{T}}[I_{18} - D_1\tilde{K}P - D_2P]^{-1}, \tag{39}$$

where I_{18} is a 18-dimensional identity matrix, $\tilde{K}$ can be obtained similarly to matrix $\tilde{C}$ in Eq. (6), and P is the operational matrix of integration given in Eq. (5).

From Eq. (39) the vector C can be found as

$$C = [0, \tfrac{1}{3}, 0, 0, 0, 0, \tfrac{2}{5}, \tfrac{1}{3}, \tfrac{1}{45}, \tfrac{1}{81}, 0, 0, \tfrac{64}{75}, \tfrac{11}{25}, \tfrac{11}{90}, \tfrac{29}{2025}, \tfrac{7}{4860}, \tfrac{1}{3645}]^{\mathrm{T}}.$$

Using Eq. (35) the same value as the exact $x(t)$ would be obtained.

5.3. Example 3

Consider the time-varying multi-delay system described by

$$\begin{pmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{pmatrix} = \begin{pmatrix} t & 1 \\ t & 2t \end{pmatrix}\begin{pmatrix} x_1(t-\frac{1}{3}) \\ x_2(t-\frac{1}{3}) \end{pmatrix} + \begin{pmatrix} 2 & t \\ t^2 & 0 \end{pmatrix}\begin{pmatrix} x_1(t-\frac{2}{3}) \\ x_2(t-\frac{2}{3}) \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}u(t) \tag{40}$$

with

$$x_1(t) = x_2(t) = u(t) = 0, \quad t \in [-\tfrac{2}{3}, 0] \tag{41}$$

and

$$u(t) = 2t + 1, \quad t > 0. \tag{42}$$

The exact solutions are [14]

$$x_1(t) = \begin{cases} 0, & 0 \leqslant t < \frac{1}{3}, \\ \frac{7}{162} - \frac{2}{9}t + \frac{1}{6}t^2 + \frac{1}{3}t^3, & \frac{1}{3} \leqslant t < \frac{2}{3}, \\ \frac{11}{162} - \frac{58}{243}t + \frac{31}{162}t^2 + \frac{1}{9}t^3 + \frac{7}{72}t^4 + \frac{1}{6}t^5, & \frac{2}{3} \leqslant t \leqslant 1 \end{cases}$$

and

$$x_2(t) = \begin{cases} t + t^2, & 0 \leqslant t < \frac{1}{3}, \\ \frac{5}{486} + t + \frac{7}{9}t^2 + \frac{2}{9}t^3 + \frac{1}{2}t^4, & \frac{1}{3} \leqslant t < \frac{2}{3}, \\ \frac{1}{486} + t + \frac{200}{243}t^2 + \frac{20}{81}t^3 + \frac{29}{72}t^4 - \frac{1}{9}t^5 + \frac{1}{6}t^6, & \frac{2}{3} \leqslant t \leqslant 1. \end{cases}$$

Here, we solve this problem by choosing $N = 3$ and $M = 7$. Let

$$x_1(t) = C_1^{\mathrm{T}}B(t), \quad x_2(t) = C_2^{\mathrm{T}}B(t), \tag{43}$$

where C_1, C_2 and $B(t)$ can be obtained similarly to Eqs. (27) and (28). By expanding t and t^2 in terms of hybrid functions we obtain

$$t = [0, \tfrac{1}{3}, 0, \ldots, 0, \tfrac{1}{3}, \tfrac{1}{3}, 0, \ldots, 0, \tfrac{2}{3}, \tfrac{1}{3}, 0, \ldots, 0]B(t) = K_1^{\mathrm{T}}B(t), \tag{44}$$

$$t^2 = [0, 0, \tfrac{1}{9}, 0, \ldots, 0, \tfrac{1}{9}, \tfrac{2}{9}, \tfrac{1}{9}, 0, \ldots, 0, \tfrac{4}{9}, \tfrac{4}{9}, \tfrac{1}{9}, 0, \ldots, 0]B(t) = K_2^{\mathrm{T}}B(t). \tag{45}$$

We also have

$$tx_1(t-\tfrac{1}{3}) = C_1^{\mathrm{T}}D_1\tilde{K}_1B(t), \quad tx_2(t-\tfrac{1}{3}) = C_2^{\mathrm{T}}D_1\tilde{K}_1B(t), \tag{46}$$

$$t^2x_1(t-\tfrac{2}{3}) = C_1^{\mathrm{T}}D_2\tilde{K}_2B(t), \quad tx_2(t-\tfrac{2}{3}) = C_2^{\mathrm{T}}D_2\tilde{K}_1B(t), \tag{47}$$

where $\tilde{K}_1$ and $\tilde{K}_2$ can be calculated similarly to matrix $\tilde{C}$ in Eq. (6). By integrating Eq. (40) from 0 to t and using Eqs. (41)–(47) we get

$$C_1^{\mathrm{T}} = C_1^{\mathrm{T}}D_1\tilde{K}_1P + C_2^{\mathrm{T}}D_1P + 2C_1^{\mathrm{T}}D_2P + C_2^{\mathrm{T}}D_2\tilde{K}_1P, \tag{48}$$

$$C_2^{\mathrm{T}} = C_1^{\mathrm{T}} D_1 \tilde{K}_1 P + 2C_2^{\mathrm{T}} D_1 \tilde{K}_1 P + C_1^{\mathrm{T}} D_2 \tilde{K}_2 P + K_1^{\mathrm{T}} + K_2^{\mathrm{T}}. \tag{49}$$

Solving Eqs. (48) and (49) and using Eq. (43) the same values as the exact $x_1(t)$ and $x_2(t)$ would be obtained.

6. Conclusion

The hybrid of block-pulse functions and Taylor series and the associated operational matrices of integration P, product $\tilde{C}$, and delay D are applied to solve the linear time-varying multi-delay systems. The method is based upon reducing the system into a set of algebraic equations. The matrices P, $\tilde{C}$, and D have many zeros; hence, the method is computationally very attractive. It is also shown that the hybrid of block-pulse functions and Taylor series provides an exact solution for Examples 1–3. It is noted that exact solutions obtained in Examples 1–3 cannot be obtained either with Taylor series nor with orthogonal functions alone.

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