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Short Communication

Iteration method solutions for conservative and limit-cycle $x^{1/3}$ force oscillators

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Abstract

An iterative technique is used to calculate a higher-order approximation to the periodic solutions of a conservative oscillator for which the elastic force term is proportional to $x^{1/3}$. The related van der Pol-type limit-cycle oscillator is also studied.

The purpose of this Short Communication is to calculate a higher-order approximation to the periodic solutions of the following differential equations [1–3]:

$$\ddot{x} + x^{1/3} = 0, \tag{1}$$

$$\ddot{x} + x^{1/3} = -\varepsilon(1 - x^2)\dot{x}, \tag{2}$$

using an iteration technique derived by Mickens [4]. These equations represent a new class of nonlinear oscillating systems [1]. The work presented here extends previous results given in Mickens [1–3] which relied primarily on the method of harmonic balance [5] as the tool for determining the oscillatory solutions.

The details of the iteration technique are given in Mickens [4]; consequently, only an outline of the method is required here.

The nonlinear oscillator equation is assumed to take the form

$$\ddot{x} + g(x) = \varepsilon f(x, \dot{x}), \quad x(0) = A, \ \ \dot{x}(0) = 0, \tag{3}$$

where ε is a positive parameter and the functions $g(x)$ and $f(x, \dot{x})$ are assumed to satisfy the conditions

$$g(-x) = -g(x), \quad f(-x, -\dot{x}) = -f(x, \dot{x}). \tag{4}$$

Eq. (3) can be rewritten as

$$\ddot{x} + \Omega^2 x = G(x, \dot{x}), \tag{5}$$

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where the constant Ω^2 is to be determined later and $G(x, \dot{x})$ is given by the expression

$$G(x, \dot{x}) \equiv \Omega^2 x - g(x) + \varepsilon f(x, \dot{x}). \tag{6}$$

The iteration scheme defines the $(k+1)$-approximation to the solution of Eq. (5) as

$$\ddot{x}_{k+1} + \Omega^2 x_{k+1} = G(x_k, \dot{x}_k) + G_x(x_k, \dot{x}_k)(x_k - x_{k-1}) + G_{\dot{x}}(x_k, \dot{x}_k)(\dot{x}_k - x_{k-1}), \tag{7}$$

where $k = 0, 1, 2, \ldots,$

$$G_x \equiv \frac{\partial G}{\partial x}, \quad G_{\dot{x}} \equiv \frac{\partial G}{\partial \dot{x}}, \tag{8}$$

and the initiation or starting solutions are

$$x_{-1}(t) = x_0(t) = A\cos(\Omega t), \tag{9}$$

$$x_{k+1}(0) = A, \quad \dot{x}_{k+1}(0) = 0. \tag{10}$$

The angular frequency, Ω, is calculated anew at each stage of the iteration procedure by demanding that the right-hand side of Eq. (7) contains no terms giving rise to secular terms in the complete solution of Eq. (7) with initial conditions stated in Eq. (10).

For the oscillator modeled by Eq. (1), it follows that

$$G(x, \dot{x}) = \Omega^2 x - x^{1/3}. \tag{11}$$

Note that $G_{\dot{x}}(x, \dot{x}) = 0$, and

$$G_x(x, \dot{x}) = \Omega^2 - \left(\frac{1}{3}\right)\frac{1}{x^{2/3}}. \tag{12}$$

Thus, G_x has a singularity at $x = 0$. However, what appears, for $k = 0$, in Eq. (7) is the expression $G_x(x_0, \dot{x}_0)(x_0 - x_{-1})$ which when properly evaluated, using Eq. (9), gives the result

$$G_x(x_0, \dot{x}_0)(x_0 - x_{-1}) = 0. \tag{13}$$

This means that the differential equation to be solved for $x_1(t)$ is

$$\ddot{x}_1 + \Omega^2 x_1 = \Omega^2 A\cos(\Omega t) - [A\cos(\Omega t)]^{1/3}. \tag{14}$$

At this point, there are two things to note. First, the iteration scheme cannot be extended for Eq. (1) to calculate $x_k(t)$ for $k \geqslant 2$. This is because the singularities occurring on the right-hand side of Eq. (7) cannot be eliminated. Second, a Fourier series representation is needed for $(\cos\theta)^{1/3}$ for the calculation of $x_1(t)$ to proceed.

The Fourier series for $(\cos\theta)^{1/3}$ has been calculated [6] and is given by

$$(\cos\theta)^{1/3} = \sum_{n=0}^{\infty} a_{2n+1}\cos(2n+1)\theta, \tag{15}$$

$$a_{2n+1} = \frac{3\Gamma(\frac{7}{3})}{2^{4/3}\Gamma(n+\frac{5}{3})\Gamma(\frac{2}{3}-m)}, \tag{16}$$

with $a_1 = 1.159595266963929$. Therefore, the first several terms are

$$(\cos\theta)^{1/3} = a_1\left[\cos\theta - \frac{\cos(3\theta)}{5} + \frac{\cos(5\theta)}{10} - \frac{7\cos(7\theta)}{110} + \frac{\cos(9\theta)}{22} - \frac{13\cos(11\theta)}{374} + \cdots\right]. \tag{17}$$

Substituting Eq. (16) into the right-hand side of Eq. (14) gives

$$\ddot{x}_1 + \Omega^2 x_1 = (\Omega^2 A - A^{1/3}a_1)\cos(\Omega t) - A^{1/3}\sum_{n=1}^{\infty} a_{2n+1}\cos[(2n+1)\Omega t]. \tag{18}$$

Secular terms can be eliminated by requiring the coefficient of the $\cos(\Omega t)$ term to be zero, i.e.,

$$\Omega^2 = \frac{a_1}{A^{2/3}} \quad \text{or} \quad A^{1/3}\Omega = \sqrt{a_1} = 1.076845. \tag{19}$$

Therefore, $x_1(t)$ is the solution to the differential equation

$$\ddot{x}_1 + \Omega^2 x_1 = -A^{1/3} \sum_{n=1}^{\infty} a_{2n+1} \cos[(2n+1)\Omega(A)t], \tag{20}$$

subject to $x_1(0) = A$ and $\dot{x}_1(0) = 0$. Eq. (20) is a second-order, linear, inhomogeneous differential equation with constant coefficients; its solution is [7]

$$x_1(t) = \beta A \cos[\Omega(A)t] + A \sum_{n=1}^{\infty} \left\{ \frac{a_{2n+1}}{a_1[(2n+1)^2 - 1]} \right\} \cos[(2n+1)\Omega(A)t], \tag{21}$$

where $\Omega(A)$ is given by Eq. (19), and β is the constant

$$\beta = 1 - \sum_{n=1}^{\infty} \frac{a_{2n+1}}{a_1[(2n+1)^2 - 1]}. \tag{22}$$

The actual value for β can be determined by numerically summing the right-hand side of Eq. (22) to some large value N, i.e.,

$$\beta \to \beta_N = 1 - \sum_{n=1}^{N} \frac{a_{2n+1}}{a_1[(2n+1)^2 - 1]}. \tag{23}$$

Several techniques are available for doing this in a fast and efficient manner [8]. In any case, the essential point is that β has a definite numerical value that can be calculated as accurately as needed.

Observe that the solution $x_1(t)$ contains only odd harmonics and has contributions from all of them. In Eq. (19) the value of $A^{1/3}\Omega(A)$ is given. For purposes of comparison, the following estimates for this quantity are presented:

$$A^{1/3}\Omega_1(A) = 1.0491, \quad A^{1/3}\Omega_2(A) = 1.0704, \tag{24}$$

$$A^{1/3}\Omega_e(A) = 1.070451. \tag{25}$$

In the above, $\Omega_1(A)$ and $\Omega_2(A)$ are calculated, respectively, from first- and second-order harmonic balance procedures [2]. The exact value, $\Omega_e(A)$, is that given in Eq. (25).

The calculation of $\Omega_e(A)$ proceeds as follows. A first-integral [5] of Eq. (1) is

$$\frac{y^2}{2} + \left(\frac{3}{4}\right) x^{4/3} = \left(\frac{3}{4}\right) A^{4/3}, \quad y \equiv \dot{x} \tag{26}$$

and the corresponding period of the oscillation is

$$T(A) = \sqrt{\frac{32}{3}} \int_0^A \frac{dx}{\sqrt{A^{4/3} - x^{4/3}}}. \tag{27}$$

The substitution $x = Aw^{3/2}$ gives, after some simplification,

$$T(A) = \left(2\sqrt{6}\right) \phi A^{1/3}, \tag{28}$$

where

$$\phi \equiv \int_0^1 \frac{w^{1/2}\, dw}{\sqrt{(1+w)(1-w)}}. \tag{29}$$

Since $\Omega(A) = 2\pi/T(A)$, it follows that

$$A^{1/3}\Omega(A) = \frac{\pi}{\sqrt{6}\,\phi}. \tag{30}$$

Using Gradshteyn and Ryzhik [9], see Section 3.14 (formula 10), the value of ϕ can be determined from the definite integral

$$\int_0^1 \sqrt{\frac{w}{(1+w)(1-w)}}\,\mathrm{d}w = 2\sqrt{2}E\left(\frac{\pi}{2},\frac{1}{\sqrt{2}}\right) - \sqrt{2}F\left(\frac{\pi}{2},\frac{1}{\sqrt{2}}\right), \tag{31}$$

where

$$F\left(\frac{\pi}{2},\frac{1}{\sqrt{2}}\right) = F\left(\frac{1}{\sqrt{2}}\right), \quad E\left(\frac{\pi}{2},\frac{1}{\sqrt{2}}\right) = E\left(\frac{1}{\sqrt{2}}\right), \tag{32}$$

are, respectively, the complete elliptic integrals of the first and second kinds [10]. Their values are

$$F\left(\frac{1}{\sqrt{2}}\right) = 1.854075, \quad E\left(\frac{1}{\sqrt{2}}\right) = 1.350644. \tag{33}$$

With these results, ϕ can be calculated and $\Omega(A)$ found to be

$$A^{1/3}\Omega_e(A) = 1.070451, \tag{34}$$

which is the result reported in Eq. (25).

Comparison of the various estimates for $A^{1/3}\Omega(A)$ indicates that the calculation presented here, see Eq. (19), and from the second-order harmonic balance method are in excellent agreement with the exact value of Eq. (34). These results provide confirmation for the validity of the iteration method as applied to Eq. (1). This holds in spite of the fact that this method only holds for one step of iteration.

Turning to the van der Pol-type modification, i.e., Eq. (2), the function $G(x,\dot{x})$ is

$$G(x,\dot{x}) = \Omega^2 x - x^{1/3} + \varepsilon(1-x^2)\dot{x} \tag{35}$$

and the equation for $x_1(t)$ is, see Eqs. (7), (8), and (9),

$$\ddot{x}_1 + \Omega^2 x_1 = \Omega^2 x_0 - x_0^{1/3} + \varepsilon(1-x_0^2)\dot{x}_0. \tag{36}$$

Note that the terms involving G_x and $G_{\dot{x}}$ when placed in the iteration scheme are equal to zero. Using $x_0(t) = A\cos(\Omega t)$ and the Fourier expansion for $(\cos\theta)^{1/3}$, it follows that Eq. (36) becomes

$$\begin{aligned}\ddot{x}_1 + \Omega^2 x_1 = {} & [\Omega^2 A - A^{1/3}a_1]\cos\theta - A^{1/3}\sum_{n=1}^{\infty} a_{2n+1}\cos[2(n+1)\Omega t] \\ & + (\varepsilon A\Omega)\left(\frac{A^2}{4} - 1\right)\sin(\Omega t) + \left(\frac{\varepsilon A^3\Omega}{4}\right)\sin(3\Omega t).\end{aligned} \tag{37}$$

The condition that $x_1(t)$ contains no secular terms gives the two relations

$$\Omega^2 A - a_1 A^{1/3} = 0 \quad \text{or} \quad A^{1/3}\Omega = \sqrt{a_1}, \tag{38}$$

$$\frac{A^2}{4} - 1 = 0 \quad \text{or} \quad A = 2. \tag{39}$$

Therefore,

$$A = 2, \quad \Omega = \frac{\sqrt{a_1}}{2^{1/3}} = 0.8547. \tag{40}$$

Since Eq. (37) is a second-order, linear, inhomogeneous differential equation, its solution can be readily obtained [7] and the result is

$$x_1(t) = \beta A \cos[\Omega(A)t] + \left(\frac{\varepsilon}{2^{4/3}}\right)\{3\sin[\Omega(A)t] - \sin[3\Omega(A)t]\} + A\sum_{n=1}^{\infty}\left\{\frac{a_{2n+1}}{a_1[(2n+1)^2-1]}\right\}\cos[(2n+1)\Omega(A)t], \tag{41}$$

where β is given by Eq. (22). Note that all odd harmonics involving "cosine" terms appear, while only the first two odd harmonics occur for the "sine" terms. Also, observe that while the standard van der Pol equation

$$\ddot{x} + x = \varepsilon(1-x^2)\dot{x}, \quad 0<\varepsilon \ll 1, \tag{42}$$

has the angular frequency

$$\Omega = 1 + O(\varepsilon^2), \tag{43}$$

the modified van der Pol equation has an angular frequency approximately 15% less; see Eq. (40). As was the case for Eq. (1), the iteration scheme cannot be extended beyond the first stage. It should be obvious that only the stationary oscillatory periodic motion can be calculated using this method.

In summary, an iteration technique [4] has been used to calculate approximations to the periodic solutions for two oscillators for which the elastic force terms are proportional to $x^{1/3}$. None of the standard perturbation methods [5] can be applied to these equations. This fact is emphasized in Mickens [2,3]. However, the method of harmonic balance can and has been applied to these equations [2]. Comparison between the iteration procedure and the harmonic balance methods shows that the two techniques are in excellent agreement, especially with regard to the calculated values of the angular frequency. The major conclusion is that the iteration scheme [4] provides excellent approximations to the solutions of Eqs. (1) and (2) even though the iteration can only be done to the first stage. For oscillatory systems, modeled by equations such as those expressed by Eqs. (1) and (2), a future research issue is to see if an alternative iteration method can be devised such that higher-order iterations can be done. It is also of great interest to see, for systems having limit-cycles, if iteration methods exist allowing the calculation of the transitory behavior of the solutions to the limit-cycle.

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