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Short Communication

A harmonic oscillator having "volleyball damping"

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Abstract

Volleyball damping corresponds to linear damping up to a certain critical velocity, with zero damping above this value. The dynamics of a linear harmonic oscillator is investigated with this damping mechanism.

The harmonic oscillator equation forms the basis for the analysis of many dynamic systems having a linear relationship between the elastic force and the extension from the equilibrium position [1]. The next level of approximation includes the affects of dissipative forces. Such damping terms include those generated by a range of dissipative phenomena [1,2]: Coulomb, cubic, linear, quadratic, and negative feedback. The particular case

$$\ddot{x} + x = -\varepsilon[a|\dot{x}|\dot{x} + b\dot{x} + c\,\mathrm{sign}(\dot{x})] \tag{1}$$

was investigated by Mickens [2]. In this differential equation the parameters (ε, a, b, c) are non-negative parameters and are coefficients of terms representing quadratic, linear, and Coulomb damping. The purpose of this communication is to study the dynamics of the linear oscillator system acted on by so-called "volleyball damping" (VBD). This type of dissipative force appears to play an important role in understanding the behavior of the ball in a game of volleyball [3]. An idealized form for this frictional force is given by the expression

$$F(\dot{x}) = -\varepsilon\dot{x}\theta(a^2 - \dot{x}^2), \tag{2}$$

where a is a constant velocity and the step-function $\theta(z)$ is defined to be

$$\theta(z) = \left\{\begin{matrix} 1 & \text{for } z>0, \\ 0 & \text{for } z<0. \end{matrix}\right\} \tag{3}$$

Note that $F(\dot{x})$ is a piece-wise linear function, i.e., $F(\dot{x}) = -\varepsilon\dot{x}$ for $|\dot{x}|<a$, and $F(\dot{x}) = 0$ for $|\dot{x}|>a$. Thus, overall $F(\dot{x})$ is a nonlinear function of $\dot{x}$. This means that a is the value of the magnitude of a critical velocity: when $|\dot{x}|>a$, the oscillator experiences zero damping force; while for $|\dot{x}|<a$, the oscillator has the usual

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properties of a linear damped system. Putting all of this together, the harmonic oscillator with VBD has the equation of motion

$$\ddot{x} + x = -\varepsilon \dot{x} \theta(a^2 - \dot{x}^2). \tag{4}$$

Inspection of Eq. (4) shows that it is not immediately obvious how to apply the standard perturbation methods [1,4] to calculate analytic approximations to the solution for the case where $0 < \varepsilon \ll 1$. The major difficulty is the term on the right-hand side.

A way out of this dilemma is to examine the behavior of the phase-space trajectories in the $(x, y = \dot{x})$ plane [5,6]. The system equations for Eq. (4) are

$$\frac{\mathrm{d}x}{\mathrm{d}t} = y, \quad \frac{\mathrm{d}y}{\mathrm{d}t} = -x - \varepsilon y \theta(a^2 - y^2). \tag{5}$$

The fixed-point or equilibrium solution [6] is $(\bar{x}, \bar{y}) = (0, 0)$ and the local stability, i.e., $|x| \ll a$, $|y| \ll a$, is determined by the solutions of

$$\frac{\mathrm{d}x}{\mathrm{d}t} = y, \quad \frac{\mathrm{d}y}{\mathrm{d}t} = -x - \varepsilon y \quad \text{or} \quad \ddot{x} + x = -\varepsilon \dot{x}. \tag{6}$$

Since Eq. (6) correspond to a linear damped oscillator, it can be concluded that $(\bar{x}, \bar{y}) = (0, 0)$ is stable [1]. One way of determining the global stability of Eq. (4) or (5) is to examine the behavior of the function $V(x, y)$ defined to be

$$V(x, y) \equiv (\tfrac{1}{2})(x^2 + y^2). \tag{7}$$

Taking the derivative of $V(x, y)$ and using Eq. (6), it follows that

$$\frac{\mathrm{d}V}{\mathrm{d}t} = x\frac{\mathrm{d}x}{\mathrm{d}t} + y\frac{\mathrm{d}y}{\mathrm{d}t} = -\varepsilon y^2 \theta(a^2 - y^2) \leqslant 0. \tag{8}$$

Eqs. (7) and (8) jointly imply that $V(t) \equiv V[x(t), y(t)]$ is a monotonic decreasing function of t. This result implies that the fixed-point, $(\bar{x}, \bar{y}) = (0, 0)$, is globally stable [7].

Based on the above analysis, the harmonic oscillator with VBD has the following properties:

(1) For points in the (x, y) phase-space with

$$y^2 < a^2, \tag{9}$$

the equation of motion is

$$\ddot{x} + x = -\varepsilon \dot{x}. \tag{10}$$

For this case the trajectories correspond to the standard linearly damped harmonic oscillator, i.e., all trajectories spiral into the stable fixed-point.

(2) For points in the (x, y) phase-space with

$$y^2 > a^2, \tag{11}$$

the nonlinear damping term is zero because it depends on $\theta(a^2 - y^2)$. In this region of phase-space the equation of motion is

$$\ddot{x} + x = 0. \tag{12}$$

(3) Thus, from comments (1) and (2), the general behavior of a trajectory in phase-space, $(x(t), y(t))$, can be determined for the case where $|y(0)| > a$. Without loss of generality, for this argument, the initial conditions can be chosen to be

$$x(0) = 0, \quad y(0) > a. \tag{13}$$

For this selection, the trajectory starts out on the vertical y-axis and has $x(t)$ increasing, with $y(t)$ decreasing. At some time t_1, the value of y becomes

$$y(t_1) = a, \tag{14}$$

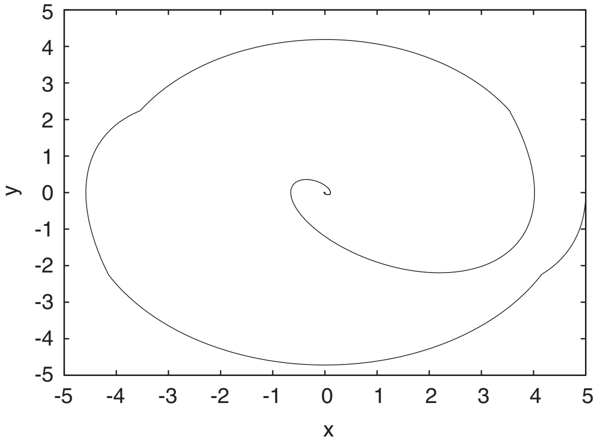

Fig. 1. Plot of $y(x)$ vs x for Eqs. (5). This graph corresponds to $x(0) = 5$, $y(0) = 0$, $\varepsilon = 10$, $a^2 = 5$.

and from Eq. (5), the slope dy/dx develops a kink, i.e., y as a function of x is continuous, but dy/dx is discontinuous. After $t = t_1$, the trajectory spirals around the origin (fixed-point) and, for sufficiently large $y(0)$, may experience several other points on its trajectory where dy/dx is discontinuous. Fig. 1 illustrates this behavior for the set of parameters

$$x(0) = 5, \quad y(0) = 0, \quad \varepsilon = 10, \quad a^2 = 5. \tag{15}$$

Note that the trajectory is given for the time interval $0 \leqslant t \leqslant 500$, and four kinks occur when $|y| = \sqrt{5}$.

Trajectories for which $|y| < \sqrt{5}$ correspond, as explained above, to the standard trajectories for the linear damped oscillator and, as a consequence, are not displaced.

A general observation based on the numerical integration of Eq. (5), is that the linear harmonic oscillator with VBD takes a longer time to approach within a given distance from the origin/fixed-point in phase-space than the standard linear damped oscillator. This is a consequence of the fact that this oscillator spends time in a region of phase-space, where it is essentially an undamped oscillator; see Eq. (12) and the discussion before it.

Future problems to be studied include whether an analytic approximation to the solution of Eq. (4) can be calculated using the standard perturbation procedures [1,4]? However, an exact solution can, in principle, be obtained for Eq. (4) since the nonlinear damping term is piece-wise continuous.

In summary, a new functional form for damping has been introduced. This dissipative term is based on the observed dynamics of the ball behavior for the game of volleyball [3]. The modified oscillations of a harmonic oscillator was studied using this discontinuous, nonlinear damping function and the general properties of the trajectories in phase-space were discussed.

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