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Short Communication

Coherence function of transverse random vibrations of a rotating shaft

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Abstract

A simple Jeffcott rotor with both external and internal damping is considered. The rotor is subject to a random excitation which results in transverse random vibrations even at rotation speeds below the instability threshold. The random forces in two perpendicular directions are assumed to be uncorrelated white noises which may have different intensities in general. An analytical expression is derived for peak value of coherence function of responses in these directions as a function of ratio of rotation speed to its value at the instability threshold. Numerical simulation results are presented for verification of the corresponding coherence-based method for on-line stability margin evaluation for rotating shafts.

1. Introduction

This brief Note is a sequel to paper [1] where random vibrations of a rotating shaft with internal damping were studied. In particular, the peak coherence function of the shaft's transverse displacements in two perpendicular directions has been obtained as an explicit function of the ratio of the shaft's rotation speed to its value at the instability threshold. This relation has been derived under assumption of equal intensities of white-noise random excitations in the above directions. In the present Note a more general analytical expression is derived and analyzed for the case of unequal excitation intensities. Furthermore, computer simulation tests are performed for the suggested coherence-based procedure for on-line stability margin evaluation for rotating shafts.

Consider a common single-mass/two-degrees-of-freedom rotor consisting of a simply supported weightless shaft of stiffness K with a disk of mass m at its midspan. This model may also be used for the case of flexible supports, as long as only axisymmetric problems will be considered here, with the supports' flexibility regarded

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as connected in series to that of the shaft (including case of a rigid shaft) [2]. The shaft which rotates with speed v possesses both external or "non-rotating" damping and internal or "rotating" damping with corresponding damping factors c_n and c_r respectively. Let $X(t)$ and $Y(t)$ be mutually perpendicular lateral displacements of the disk's centre in the inertial frame with origin at the undeformed shaft's axis; orientation of this frame within the disk's plane is arbitrary as long as gravity forces will be neglected for sufficiently high rotation speeds. The equations of motion with random excitation terms added to the RHSs may then be written as [1,2]

$$\ddot{X} + 2\kappa\dot{X} + \Omega^2 X + 2\beta v Y = \varsigma_X(t), \quad \ddot{Y} + 2\kappa\dot{Y} + \Omega^2 Y - 2\beta v X = \varsigma_Y(t), \tag{1}$$

where $\Omega^2 = K/m,\ \kappa = \alpha + \beta,\ \alpha = c_n/2m,\ \beta = c_r/2m$. The random forces in the RHSs of Eqs. (1) are assumed to be stationary zero-mean uncorrelated Gaussian white noises with power spectral densities (PSDs) $\sigma_X^2/2\pi$ and $\sigma_Y^2/2\pi$. The assumption of zero cross-correlation between $\varsigma_X(t)$ and $\varsigma_Y(t)$ does not restrict generality since directions X and Y may be chosen so as to correspond to the principal axes of the correlation matrix.

In the absence of any external excitations Eqs. (1)—with zero RHSs in this case—clearly have a trivial solution $X(t) \equiv 0,\ Y(t) \equiv 0$. This solution is stable if $v < v^*$ and unstable if $v > v^*$, where $v^* = \Omega(1 + \alpha/\beta)$ is the instability threshold of the shaft; at this rotation speed Eqs. (1) with zero RHS have a neutrally stable periodic solution with period $2\pi/\Omega$.

Consider the PSDs of the responses $X(t)$, $Y(t)$ for the case of stability of the linear shaft's model (1) ($v < v^*$). They can be derived by using the following definition of cross-spectral density of any pair of stationary random processes $Z_k(t)$, $Z_j(t)$:

$$\Phi_{Z_k Z_j}(\omega) = \lim_{T\to\infty} (\pi/T)\langle \tilde{Z}_k(\omega, T)\tilde{Z}_j^*(\omega, T)\rangle, \quad \tilde{Z}(\omega, T) = \int_{-T}^{T} Z(t)\exp(\mathrm{i}\omega t)\,\mathrm{d}t, \quad \mathrm{i} = \sqrt{-1}, \tag{2}$$

where the star superscript denotes a complex conjugate quantity; this definition covers auto-spectral densities as well if $k = j$. Applying to Eqs. (1) the Fourier Transform with finite limits $+T$ and $-T$ as denoted by tilde in Eq. (2) yields two equivalent real algebraic equations in the frequency domain:

$$\tilde{X}(-\omega^2 + 2\mathrm{i}\kappa\omega + \Omega^2) + \tilde{Y}(2\beta v) = \tilde{\xi}_X, \quad \tilde{X}(-2\beta v) + \tilde{Y}(-\omega^2 + 2\mathrm{i}\kappa\omega + \Omega^2) = \tilde{\xi}_Y. \tag{3}$$

The auto- and cross-spectral densities of $X(t)$ and $Y(t)$ can now be obtained by solving Eq. (3) for $\tilde{X}$ and $\tilde{Y}$ and applying the basic definition (2). The result is

$$\begin{aligned} &\Phi_{XX}(\omega) = (1/2\pi\Delta\Delta^*)\left\{\sigma_X^2\left[(\omega^2 - \Omega^2)^2 + 4\kappa^2\omega^2\right] + 4v^2\beta^2\sigma_Y^2\right\}, \\ &\Phi_{XY}(\omega) = (2v\beta/2\pi\Delta\Delta^*)\left[(\sigma_X^2 - \sigma_Y^2)(\Omega^2 - \omega^2) + 2\mathrm{i}\omega\kappa(\sigma_X^2 + \sigma_Y^2)\right], \\ &\Delta = (-\omega^2 + 2\mathrm{i}\kappa\omega + \Omega^2)^2 + (2\beta v)^2 \quad \text{and} \quad \kappa = \alpha + \beta. \end{aligned} \tag{4}$$

(The expression for the PSD of $Y(t)$ is obtained from that of $X(t)$ by swapping excitation intensities σ_X^2 and σ_Y^2.) These response PSDs have their peaks in the immediate vicinity of the rotor's natural frequency Ω which is known to be also the frequency of forward whirl at the instability boundary v^*. This is true in particular for the most interesting case of the lightly damped shaft as long as the shift of the peak is found to be of the order of the total damping ratio κ/Ω; this shift also diminishes with approaching the instability threshold speed $v^* = \kappa\Omega/\beta = \Omega(1 + c_n/c_r)$. Thus, the above solution for the response PSDs can be used to obtain the coherence function of the responses in two perpendicular directions; its value at $\omega = \Omega$ is found to be

$$\gamma_{XY}^2(\Omega) = \frac{\left|\Phi_{XY}^2(\Omega)\right|}{\Phi_{XX}(\Omega)\Phi_{YY}(\Omega)} = \frac{(v/v^*)^2}{\left\{\frac{1}{2}\left[1 + (v/v^*)^2\right]\right\}^2 - \lambda^2\left[1 - (v/v^*)^2\right]^2},$$
$$\text{where} \quad \lambda = \frac{1}{2}\frac{\sigma_X^2 - \sigma_Y^2}{\sigma_X^2 + \sigma_Y^2} \quad \text{so that } \lambda \in \left[0, \frac{1}{2}\right]. \tag{5}$$

This peak value of the coherence function can be measured on line for a shaft rotating with constant speed v. It is seen to be a monotonously increasing function of v/v^* and approaches its potentially maximal value (unity) when approaching the instability threshold. Thus, relation (5) can be used to estimate the shaft's margin with respect to its instability threshold speed by the relevant processing response signals $X(t)$ and $Y(t)$ measured *on-line* during operation at a *constant* rotation speed. The resulting procedure clearly illustrates the advantage of using the stochastic component of the shaft response. Indeed, it would be impossible in principle to obtain such an estimate from the shaft's steady-state (harmonic) response to unbalance and/or misalignment since the internal damping is not involved in this synchronous forward-whirl response [2].

The peak value of coherence at any given rotation speed is seen to be smallest if $\lambda = 0$—that is, in case of the same intensity of excitations in directions X and Y. This important special case, for which solution (5) had been obtained earlier in Ref. [1], should be therefore most convenient for evaluating the stability margin since the peak coherence is found to be most sensitive to variations in v/v^* in this case. In the opposite extreme case $\lambda = \frac{1}{2}$, with the excitation applied only in one direction, the peak coherence is found to be unity at *any* rotation speed. The procedure for evaluating the stability margin would then be impractical in this case because of the permanent "false alarm". Anyway, the relation (5) demonstrates the potential use of the procedure for different intensities of excitations in directions X and Y. Whilst the most favourable condition for this use is that of equal intensities ($\lambda = 0$), cases of other values $\lambda \neq \frac{1}{2}$ can also be handled. This is seen from Fig. 1 illustrating relation (5) for peak coherence as a function of speed ratio v/v^* for several selected values of λ; as could be expected all these peak values approach unity when $v/v^* \to 1$.

Results of direct numerical simulations of the stochastic equations (1) for the case $\lambda = 0,\ \Omega = 1,\ \alpha = \beta = 0.01$ are also presented in Fig. 1 by star symbols. Here the coherence function has been estimated as in simulated on-line tests—directly from "measured" numerically generated samples of $X(t)$ and $Y(t)$. The estimates were obtained in MatLab using built-in function "*cohere*", with a rectangular window providing the most reasonable numerical results; the total number of points used to estimate the coherence function was 262144. As can be seen from the figure, the accuracy of numerical estimates is good, particularly for larger values of v/v^*.

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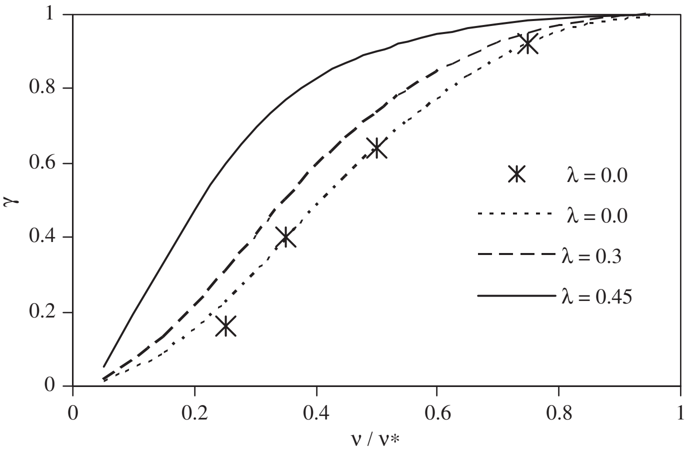

Fig. 1. Peak value of the coherence function of the responses $X(t)$ and $Y(t)$ vs. rotation speed ratio v/v^*: the analytical solution (5) for several different values of λ (—, $\lambda = 0.45$; - - - -, $\lambda = 0.3$; . . . , $\lambda = 0.0$) and results of processing the numerically generated solution to stochastic equations (1) for $\lambda = 0,\ \Omega = 1,\ \alpha = \beta = 0.01$ (*).

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