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Short Communication

Solution of a quadratic nonlinear oscillator by the method of harmonic balance

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Abstract

This paper deals with the quadratic nonlinear oscillator (QNO) $\ddot{x}+x+\varepsilon x^2=0$. This oscillator is compared with the nonlinear oscillators $\ddot{x}+x\pm\varepsilon|x|x=0$, which are solved by the method of harmonic balance. Therefore, we obtain the first approximate solution to the QNO. This solution is more accurate than the second approximate solution obtained by the Lindstedt–Poincaré method. For QNOs, it has been pointed out that the angular frequency for $x\geqslant 0$ should be different from the angular frequency for $x\leqslant 0$ if solutions of higher degree of accuracy are required.

1. Introduction

Mickens [1] has recently examined the periodic solution of quadratic nonlinear oscillators (QNOs). QNOs give useful models for both the testing of perturbation methods and the analysis of various phenomena in the physical and engineering sciences [1,2]. In general, the quadratic nonlinear oscillating differential equations are special cases of the second-order equation [1]

$$\ddot{x}+x+\alpha x^2+\beta x\dot{x}+\gamma(\dot{x})^2=0, \tag{1}$$

where (α, β, γ) are parameters and overdots denote differentiation with respect to time t. An example of Eq. (1) is [2]

$$\ddot{x}+x+\varepsilon x^2=0, \quad x(0)=A>0, \quad \dot{x}(0)=0. \tag{2}$$

The most widely used analytical techniques to solve nonlinear oscillation problems are the perturbation methods [2–5]. Application of the Lindstedt–Poincaré method to Eq. (2) gives the following second approximation [2]:

$$\begin{aligned} x_2=x(\theta,\varepsilon)&=A\cos\theta+\varepsilon\left(\frac{A^2}{6}\right)(-3+2\cos\theta+\cos 2\theta)\\ &+\varepsilon^2\left(\frac{A^3}{3}\right)\left[-1+\left(\frac{29}{48}\right)\cos\theta+\left(\frac{1}{3}\right)\cos 2\theta+\left(\frac{1}{16}\right)\cos 3\theta\right]+O(\varepsilon^3), \end{aligned} \tag{3}$$

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where $\theta = \omega(\varepsilon)t$, and

$$\omega_2 = \omega(\varepsilon) = 1 - \varepsilon^2\left(\frac{5A^2}{12}\right) + O(\varepsilon^3). \tag{4}$$

The corresponding approximate period of the oscillation is

$$T_2 = \frac{2\pi}{\omega_2} = 2\pi\left(1 - \frac{5\varepsilon^2 A^2}{12}\right)^{-1} + O(\varepsilon^3). \tag{5}$$

The Lindstedt–Poincaré method usually applies to weakly nonlinear oscillator problems [2–5]. The method of harmonic balance is capable of producing analytical approximation to the solution to the nonlinear system, valid even for the case where the nonlinear terms are not "small" [2]. But this method cannot be directly applied to Eq. (2) because $f(x) = x + \varepsilon x^2$ is not an odd function of x [6]. The main purpose of the present paper is to use the method of harmonic balance for obtaining solutions to QNOs. Without loss of generality, we will consider Eq. (2) in detail. To this end, in Section 2 Eq. (2) is first compared with the following equation [4]:

$$\ddot{x} + x + \varepsilon|x|x = \ddot{x} + x + \varepsilon x^2\,\mathrm{sgn}(x) = 0, \quad x(0) = A, \quad \dot{x}(0) = 0, \tag{6}$$

where sgn(x) is the sign function, equal to $+1$ if $x>0$, 0 if $x = 0$, and -1 if $x<0$. Some remarks are given in Section 3.

2. Comparison of Eq. (2) with Eq. (6)

In Eq. (6), the force function is

$$f(x) = x + |x|x = x + \mathrm{sgn}(x)x^2, \tag{7}$$

which is the non-analytic odd function of x [4]. The first approximation to Eq. (6) based on the method of harmonic balance is assumed in the form

$$x = A\cos\omega_A t, \tag{8}$$

where ω_A is the angular frequency of Eq. (6). It can be easily shown that the following Fourier series expansion holds:

$$|A\cos\theta|A\cos\theta = b_1\cos\theta + b_3\cos 3\theta + \cdots. \tag{9}$$

Here,

$$b_1 = \frac{2}{\pi}\int_0^{\pi}|A\cos\theta|A\cos^2\theta\,\mathrm{d}\theta = \frac{4A^2}{\pi}\int_0^{\pi/2}\cos^3\theta\,\mathrm{d}\theta = \frac{8A^2}{3\pi}, \tag{10}$$

which is in agreement with the coefficient of $\cos\omega t$ on the right-hand side of Eq. (4.106) in Ref. [2]. Substituting Eq. (8) into Eq. (6) and taking into account Eq. (9), we have

$$\left(-\omega_A^2 + 1 + \frac{8\varepsilon A}{3\pi}\right)A\cos\omega_A t + \mathrm{HOH} = 0, \tag{11}$$

where HOH stands for higher-order harmonics. Setting the coefficient of $\cos\omega_A t$ equal to zero and solving for ω_A yields

$$\omega_A = \sqrt{1 + \frac{8\varepsilon A}{3\pi}}. \tag{12}$$

Therefore, a first approximation to the periodic solution of Eq. (6) is

$$x = A\cos\sqrt{1 + \frac{8\varepsilon A}{3\pi}}\,t. \tag{13}$$

The corresponding approximate period of oscillation is

$$T_A = \frac{2\pi}{\omega_A} = 2\pi\left(1 + \frac{8}{3\pi}\varepsilon A\right)^{-1/2}. \tag{14}$$

It should be pointed out that if the value of εA is small, Eq. (12) becomes

$$\omega_A = 1 + \frac{4\varepsilon A}{3\pi}. \tag{15}$$

Substituting Eq. (15) into Eq. (8) gives

$$x = A\cos\left(1 + \frac{4\varepsilon A}{3\pi}\right)t, \tag{16}$$

which is identical to the result obtained by the method of multiple scales [4]. Apparently when $0 \leqslant t \leqslant T_A/4$, $x \geqslant 0$. In this case, Eq. (6) can be written as

$$\ddot{x} + x + \varepsilon x^2 = 0, \quad x(0) = A, \quad \dot{x}(0) = 0, \quad 0 \leqslant t \leqslant T_A/4, \tag{17}$$

which is identical to Eq. (2). Therefore, Eq. (13) is also the approximate solution to Eq. (2) for $0 \leqslant t \leqslant T_A/4$. When $T_A/4 \leqslant t \leqslant (T_A/4) + (T_B/2)$ (T_B is given by Eq. (34)), $x \leqslant 0$. In this case, we let

$$x = -y, \tag{18}$$

where $y \geqslant 0$. Substituting $x = -y$ into Eq. (2) yields

$$\ddot{y} + y - \varepsilon y^2 = 0, \quad y(0) = B > 0, \quad \dot{y}(0) = 0. \tag{19}$$

Now we discuss the initial conditions in Eq. (19). Obviously, Eq. (2) can be rewritten as

$$\dot{x}\mathrm{d}\dot{x} + (x + \varepsilon x^2)\mathrm{d}x = 0. \tag{20}$$

Integrating of this equation gives the first integral

$$\frac{\dot{x}^2}{2} + \frac{x^2}{2} + \frac{\varepsilon x^3}{3} = h, \tag{21}$$

where h is a constant of integration. Assuming that the system oscillates between asymmetric limits $[-B, A]$ ($B > 0$), we have from Eq. (21)

$$\frac{B^2}{2} - \frac{\varepsilon B^3}{3} = \frac{A^2}{2} + \frac{\varepsilon A^3}{3}. \tag{22}$$

Solving this equation for B gives

$$B_1 = -A, \tag{23}$$

$$B_{2,3} = \frac{1}{4\varepsilon}\left[3 + 2\varepsilon A \pm 3\sqrt{1 - \frac{4}{3}\varepsilon A(1 + \varepsilon A)}\right]. \tag{24, 25}$$

Since it is assumed that $B > 0$ and $B \to 0$ if $A \to 0$, we obtain

$$B = \frac{1}{4\varepsilon}\left[3 + 2\varepsilon A - 3\sqrt{1 - \frac{4}{3}\varepsilon A(1 + \varepsilon A)}\right], \tag{26}$$

where

$$\frac{4}{3}\varepsilon A(1 + \varepsilon A) < 1. \tag{27}$$

Solving this inequality results in

$$(0<)\varepsilon A<\frac{1}{2}. \tag{28}$$

If $\varepsilon=1$, then $A<1/2$, which is identical to the result in Ref. [7]. Based on the above discussion, we see that in order to obtain the approximate periodic solution $x(t)(\geqslant 0)$ to Eq. (2), it is necessary to resort to Eq. (6). Similarly, for Eq. (19) we first consider the following equation:

$$\ddot{y}+y-\varepsilon|y|y=\ddot{y}+y-\varepsilon\,\mathrm{sgn}(y)y^2=0,\quad y(0)=B,\quad \dot{y}(0)=0. \tag{29}$$

Now the first approximation to Eq. (29) is assumed to be

$$y=B\cos\omega_B t, \tag{30}$$

where ω_B is the angular frequency of Eq. (29). Substituting Eq. (30) into Eq. (29) and taking into account Eq. (9), we have

$$\left(-\omega_B^2+1-\frac{8\varepsilon B}{3\pi}\right)B\cos\omega_B t+\mathrm{HOH}=0. \tag{31}$$

Setting the coefficient of $\cos\omega_B t$ equal to zero and solving for ω_B yields

$$\omega_B=\sqrt{1-\frac{8\varepsilon B}{3\pi}}. \tag{32}$$

Then from Eq. (30) we obtain

$$y=B\cos\sqrt{1-\frac{8\varepsilon B}{3\pi}}\,t. \tag{33}$$

The corresponding approximate period of the oscillation to Eq. (29) is

$$T_B=\frac{2\pi}{\omega_B}=2\pi\left(1-\frac{8}{3\pi}\,\varepsilon B\right)^{-1/2}. \tag{34}$$

Therefore, the first approximate period T_1 and the corresponding periodic solution $x_1(t)$ to Eq. (2) are, respectively,

$$T_1=\frac{T_A+T_B}{2}=\pi\left[\left(1+\frac{8\varepsilon A}{3\pi}\right)^{-1/2}+\left(1-\frac{8\varepsilon B}{3\pi}\right)^{-1/2}\right] \tag{35}$$

and

$$x_1=x(t)=A\cos\omega_A t,\quad 0\leqslant t\leqslant\frac{T_A}{4}, \tag{36a}$$

$$x_1=x(t)=B\cos\omega_B\left(t-\frac{T_A}{4}+\frac{T_B}{4}\right),\quad \frac{T_A}{4}\leqslant t\leqslant\frac{T_A}{4}+\frac{T_B}{2}, \tag{36b}$$

$$x_1=x(t)=A\cos\omega_A\left(t+\frac{T_A}{2}-\frac{T_B}{2}\right),\quad \frac{T_A}{4}+\frac{T_B}{2}\leqslant t\leqslant T_1. \tag{36c}$$

The exact period T_e to Eq. (2) is

$$T_e=\int_0^A\frac{2\,\mathrm{d}x}{\sqrt{A^2-x^2+\frac{2}{3}\varepsilon(A^3-x^3)}}+\int_0^B\frac{2\,\mathrm{d}x}{\sqrt{B^2-x^2-\frac{2}{3}\varepsilon(B^3-x^3)}}, \tag{37}$$

Table 1
Comparison of approximate periods with the corresponding exact period to Eq. (2) for $\varepsilon = 1$

A	T_e (Eq. (37))	T_2 (Eq. (5))	T_1 (Eq. (35))
0.10	6.3116	6.3095	6.3112
0.20	6.4114	6.3897	6.4095
0.30	6.6294	6.5280	6.6226
0.40	7.1246	6.7320	7.0962
0.45	7.7065	6.8622	7.6277
0.46	7.9052	6.8907	7.8014
0.47	8.1672	6.9201	8.0233
0.48	8.5452	6.9504	8.3278
0.49	9.2080	6.9816	8.8118

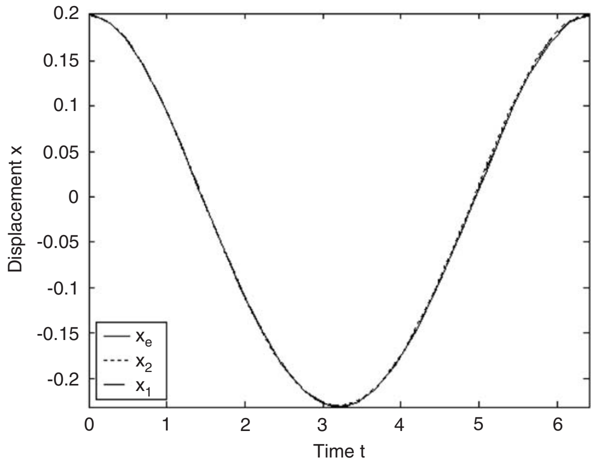

Fig. 1. Comparison of the approximate solutions with the numerical solution for $\varepsilon = 1$, $A = 0.20$.

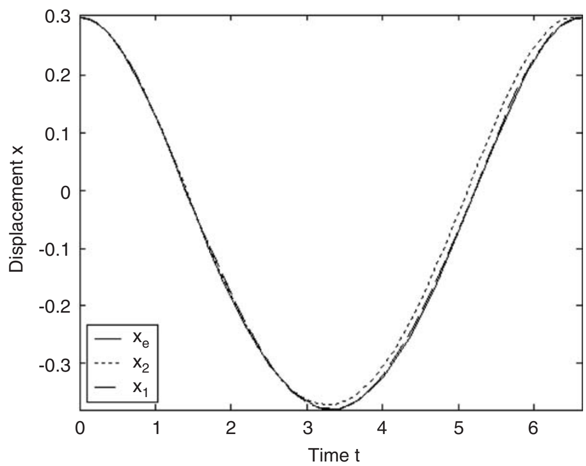

Fig. 2. Comparison of the approximate solutions with the numerical solution for $\varepsilon = 1$, $A = 0.30$.

where B is given, in terms of A, in Eq. (26). For comparison, the exact period T_e obtained by integrating Eq. (37) and the approximate periods T_2 and T_1 computed, respectively, by Eqs. (5) and (35) are listed in Table 1. Table 1 indicates that T_1 is more accurate than T_2. Even when $\varepsilon = 1$ and $A = 0.49$, the relative error of T_1 with respect to T_e is less than 4.31%.

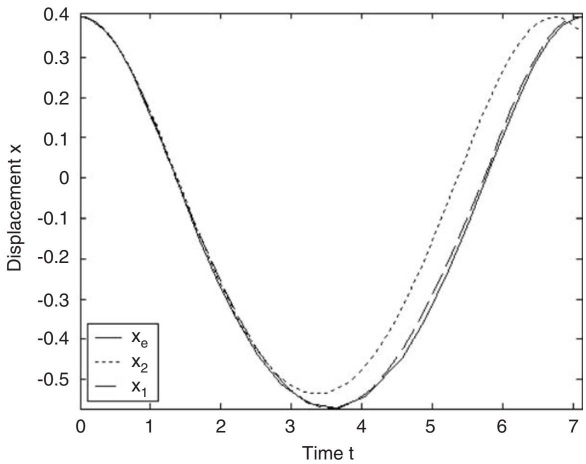

Fig. 3. Comparison of the approximate solutions with the numerical solution for $\varepsilon = 1$, $A = 0.40$.

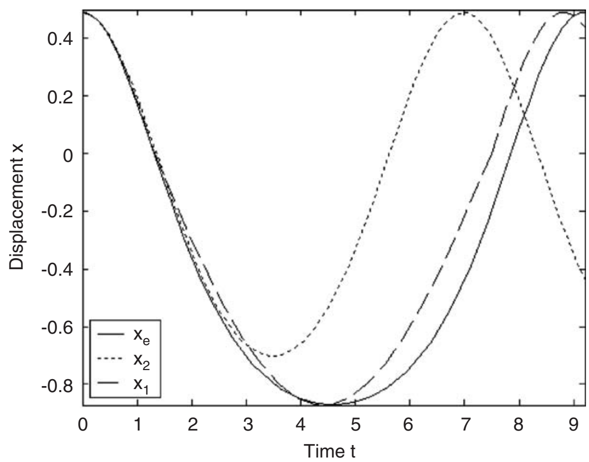

Fig. 4. Comparison of the approximate solutions with the numerical solution for $\varepsilon = 1$, $A = 0.49$.

The numerical solution x_e of Eq. (2) obtained by using Runge–Kutta (R–K) method, and the approximate analytical periodic solutions x_2 and x_1 computed, respectively, by Eqs. (3) and (36) are plotted in Figs. 1–4 for the time in one exact period. These figures show that the first approximate solution x_1 (Eq. (36)) obtained by the method of harmonic balance is more accurate than the second approximate solution x_2 (Eq. (3)) by the Lindstedt–Poincaré method.

3. Concluding remarks

A QNO given in Eq. (2) has been attacked by the method of harmonic balance. The method is applied to the two auxiliary equations (6) and (29), where the force functions are odd functions. It is well known that the frequency of nonlinear oscillators is influenced by the amplitude of the oscillation [2,3]. Since the amplitudes of the QNO described by Eq. (2) are not the same when $x \geqslant 0$ and $x \leqslant 0$, the frequency ω_A for $x \geqslant 0$ should be different from ω_B for $x \leqslant 0$. The frequency of QNOs $\omega_A \neq \omega_B$, if solutions of higher degree of accuracy are required, which is the important difference between the QNO and the cubic nonlinear oscillator. This conclusion can be also reached if we note that there are two terms on the right-hand side of Eq. (37). Because $\omega_A \neq \omega_B$, in this sense the QNO is more complicated than the cubic nonlinear oscillator. Obviously the equivalent linear equation corresponding to Eq. (2) can be written as

$$\ddot{x} + (1 + \omega_A)x = 0 \quad \text{for } x \geqslant 0, \tag{38}$$

$$\ddot{x} + (1 + \omega_B)x = 0 \quad \text{for } x \leqslant 0. \tag{39}$$

Similarly, if Eq. (1) has periodic solutions under certain conditions, we can apply the method of harmonic balance to the following two auxiliary equations:

$$\ddot{x} + x + \alpha x^2 \operatorname{sgn}(x) + \beta x\dot{x} \operatorname{sgn}(x) + \gamma(\dot{x})^2 \operatorname{sgn}(x) = 0 \quad \text{for } x \geqslant 0, \tag{40}$$

$$\ddot{x} + x - \alpha x^2 \operatorname{sgn}(x) - \beta x\dot{x} \operatorname{sgn}(x) - \gamma(\dot{x})^2 \operatorname{sgn}(x) = 0 \quad \text{for } x \leqslant 0. \tag{41}$$

Future work will center on applying the method presented in this paper to an oscillator with quadratic and cubic terms described by [2–4]

$$\ddot{x} + x + \alpha x^2 + \beta x^3 = 0. \tag{42}$$

The two auxiliary equations of this equation are

$$\ddot{x} + x + \alpha x^2 \operatorname{sgn}(x) + \beta x^3 = 0 \quad \text{for } x \geqslant 0, \tag{43}$$

$$\ddot{x} + x - \alpha x^2 \operatorname{sgn}(x) + \beta x^3 = 0 \quad \text{for } x \leqslant 0. \tag{44}$$

For the nonlinear oscillator modeled by Eq. (2), the values of εA should satisfy $\varepsilon A < 0.5$ (Eq. (28)). If $\varepsilon A = 0.5$, Eq. (2) has a homoclinic orbit with period $+\infty$ (see, for instance, Ref. [7]). Therefore, the relative errors of approximate periods and approximate periodic solutions are increased when $A \to 0.5$ as shown in Table 1 and Figs. 1–4. But this situation can be improved if the second approximate solutions are obtained. Currently, the author is examining the use of an iteration technique to obtain second approximate solutions to Eq. (2). Preliminary results are very encouraging.

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