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# Mindlin shear coefficient determination using model comparison

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**Abstract**

This paper derives an analytical expression of the shear coefficient in the Mindlin plate equation for a plate of infinite spatial extent subjected to forced excitation at definite frequency and wavenumber. The displacement transfer function derived from the Mindlin plate equation is set equal to the displacement transfer function derived from the Rayleigh–Lamb thick plate equation, and the result is a closed-form expression of the shear coefficient. A numerical example is included to illustrate the variation of the shear coefficient with respect to the system parameters. It is shown that the shear coefficient is extremely dependent on wavenumber and only slightly dependent on frequency. Shear coefficients derived in other work for free vibration are compared favorably to the values calculated by this new method at the wavenumber and frequency of the flexural wave response of the plate.
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## 1. Introduction

The Mindlin plate equation [1] is a modification of classical thin plate theory that includes rotary inertia and shear effects so that it is applicable to thick plates. Incorporated in the Mindlin plate equation is a shear coefficient, which is an adjustment parameter that is included to compensate for the stress distribution across the sectional area of the object. Mindlin derived two equations for the value of the shear coefficient in his original paper [1], one dependent on the value of Poisson's ratio and the second a constant. The problem of determining the shear coefficient in a Mindlin plate equation was addressed by Hutchinson [2] based on matching a mode of the Mindlin plate theory to the exact Rayleigh–Lamb frequency equation for the flexural wave response at long wavelengths. This expression is dependent only on Poisson's ratio. Later, Stephen [3] re-examined this solution, and called this the "best shear coefficient." Over the years, it has become evident in plate theory that the shear coefficient is theoretically dependent on more than Poisson's ratio. The above theories are all based on excitation of the flexural wave in a structure, and the corresponding shear coefficients are determined at the specific wavenumber and frequency of the flexural wave. They do not account for the shear coefficient as a function of all wavenumbers and frequencies, which is a response and excitation condition that exists in structures that are loaded by turbulent boundary layers or acoustical forces.

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This paper derives an analytical expression of the shear coefficient in the Mindlin plate equation subjected to plane wave excitation at any wavenumber and frequency for a plate of infinite extent. This is accomplished by computing the displacement field of the plate using the Mindlin plate equation and the Rayleigh–Lamb plate equation, and then setting them equal to each other. Because the shear coefficient is explicit in the Mindlin plate equation and implicit in the Rayleigh–Lamb plate equation, it can be solved for as a function of wavenumber, frequency and plate parameters. A numerical example is included to depict the dependence of the shear coefficient on wavenumber and frequency. It is shown that the shear coefficient is only slightly dependent on frequency and extremely dependent on wavenumber of excitation. Comparisons of previous analytical expressions are also included in the numerical example to illustrate how other theories compare to the one derived here.

## 2. System models and shear coefficient

Two system models are developed: one contains the shear coefficient explicitly and the other contains the shear coefficient implicitly. The first system model is that of a Mindlin plate whose governing equation is [1,4]

$$\left(\nabla^2 - \frac{\rho}{\kappa^2\mu}\frac{\partial^2}{\partial t^2}\right)\left(D\nabla^2 - \frac{\rho h^3}{12}\frac{\partial^2}{\partial t^2}\right)u(x,t) + \rho h\frac{\partial^2 u(x,t)}{\partial t^2} = \left(1 - \frac{D\nabla^2}{\kappa^2\mu h} + \frac{\rho h^2}{12\kappa^2\mu}\frac{\partial^2}{\partial t^2}\right)f(x,t), \tag{1}$$

where $\kappa^2$ is the shear coefficient, $u(x,t)$ is the displacement of the plate in the $z$-direction, $f(x,t)$ is the force distribution on the plate, $\rho$ is the density, $\mu$ is the shear modulus, $h$ is the thickness, $x$ is the spatial location, $t$ is the time, $\nabla$ is the spatial gradient operator and $D$ is equal to

$$D = \frac{Eh^3}{12(1-\nu^2)}, \tag{2}$$

where $E$ is Young's modulus and $\nu$ is Poisson's ratio. It is noted that some authors use $k$ as the shear coefficient rather than $\kappa^2$. The system is modeled as infinitely long with a continuous forcing function varying in time and space; thus, the displacement and forcing function terms are written as

$$u(x,t) = U(k,\omega)\exp(\mathrm{i}\omega t)\exp(\mathrm{i}kx) \tag{3}$$

and

$$f(x,t) = F(k,\omega)\exp(\mathrm{i}\omega t)\exp(\mathrm{i}kx), \tag{4}$$

where $\omega$ is angular frequency, $k$ is wavenumber with respect to the $x$-axis and i is the square root of $-1$. Solving the transfer function response of displacement divided by input force yields

$$\frac{U(k,\omega)}{F(k,\omega)} = \frac{-12\mu h\kappa^2 - 12Dk^2 + \rho h^3\omega^2}{(12D\mu hk^4 - \mu\rho h^4\omega^2k^2 - 12\mu\rho h^2\omega^2)\kappa^2 - 12D\rho h\omega^2k^2 + \rho^2h^4\omega^4}, \tag{5}$$

or

$$\frac{U(k,\omega)}{F(k,\omega)} = \frac{a\kappa^2 + b}{c\kappa^2 + d}, \tag{6}$$

where

$$a = -12\mu h, \quad b = -12Dk^2 + \rho h^3\omega^2, \tag{7,8}$$

$$c = 12D\mu hk^4 - \mu\rho h^4\omega^2k^2 - 12\mu\rho h^2\omega^2 \tag{9}$$

and

$$d = -12D\rho h\omega^2k^2 + \rho^2h^4\omega^4. \tag{10}$$

The second model is derived from the equations of motion [5] of a solid medium, governed by

$$\mu\nabla^2\mathbf{u} + (\lambda+\mu)\nabla\nabla\bullet\mathbf{u} = \rho\frac{\partial^2\mathbf{u}}{\partial t^2}, \tag{11}$$

where $\lambda$ and $\mu$ are the Lamé constants, $\bullet$ denotes a vector dot product and $\mathbf{u}$ is the Cartesian coordinate displacement vector of the plate. Assuming harmonic response in space and time, Eq. (11) can be manipulated; the resulting expression [6] is the displacement in the $z$-direction at the mid-plane of the plate divided by the incident normal force and is written as

$$\frac{U_z(k,-h/2,\omega)}{F(k,\omega)} = \frac{\Phi(k,-h/2,\omega)}{\mu\Delta(k,\omega)}, \tag{12}$$

where

$$\Phi\left(k,-\frac{h}{2},\omega\right) = 8\alpha^2\beta k^2(\beta^2-k^2)\sin\left(\frac{\alpha h}{2}\right)\cos^2\left(\frac{\beta h}{2}\right) + 4\alpha k^2(\beta^2-k^2)^2\sin\left(\frac{\beta h}{2}\right)\cos^2\left(\frac{\alpha h}{2}\right) + 16\alpha^2\beta k^4\sin\left(\frac{\alpha h}{2}\right)\cos\left(\frac{\alpha h}{2}\right)\cos\left(\frac{\beta h}{2}\right) + 2\alpha(\beta^2-k^2)^3\sin\left(\frac{\beta h}{2}\right)\cos\left(\frac{\alpha h}{2}\right)\cos\left(\frac{\beta h}{2}\right) \tag{13}$$

and

$$\Delta(k,\omega) = -8\alpha\beta k^2(\beta^2-k^2)^2[\cos(\alpha h)\cos(\beta h)-1] + [(\beta^2-k^2)^4 + 16\alpha^2\beta^2k^4]\sin(\alpha h)\sin(\beta h). \tag{14}$$

In Eqs. (13) and (14), $\alpha$ is the modified wavenumber associated with the dilatational wave and is expressed as

$$\alpha = \sqrt{k_d^2 - k^2}, \tag{15}$$

where $k_d$ is the dilatational wavenumber equal to $\omega/c_d$, where $c_d$ is the dilatational wave speed; $\beta$ is the modified wavenumber associated with the shear wave and is expressed as

$$\beta = \sqrt{k_s^2 - k^2}, \tag{16}$$

where $k_s$ is the shear wavenumber equal to $\omega/c_s$, where $c_s$ is the shear wave speed. The relationships between the wave speeds ($c_d$ and $c_s$) and the plate's Lamé constants ($\lambda$ and $\mu$) are determined by

$$c_d = \sqrt{\frac{\lambda+2\mu}{\rho}} \tag{17}$$

and

$$c_s = \sqrt{\frac{\mu}{\rho}}. \tag{18}$$

The analytical expression for the shear coefficient is now determined by equating Eqs. (6) and (12) and solving for $\kappa^2$, which results in

$$\kappa^2(k,\omega) = \frac{d\Phi - b\mu\Delta}{a\mu\Delta - c\Phi}. \tag{19}$$

When the system response is at the flexural resonance, the denominator of Eq. (12) is zero and the corresponding shear coefficient calculated from Eq. (19) is

$$\kappa_f^2(k,\omega) = \frac{-d}{c} = \frac{12D\rho\omega^2k^2 - \rho^2h^3\omega^4}{12D\mu k^4 - \mu\rho h^3\omega^2k^2 - 12\mu\rho h\omega^2}. \tag{20}$$

It is noted that this shear coefficient is not only a function of wavenumber and frequency but also of Young's modulus, shear modulus, Poisson's ratio and the thickness and density of the plate.

## 3. A numerical example

A numerical example is now analyzed to investigate the behavior of the shear coefficient calculated using Eqs. (19) and (20). The parameters of the plate are listed in Table 1. Using these parameters, the calculated

Table 1
Plate parameters used for numerical example

| | |
|---|---|
| Thickness, $h$ | 0.01 m |
| Young's modulus, $E$ | 7.0e8 N/m$^2$ |
| Shear modulus, $\mu$ | 2.5e8 N/m$^2$ |
| Poisson's ratio, $v$ | 0.4 (dimensionless) |
| Density, $\rho$ | 1200 kg/m$^3$ |

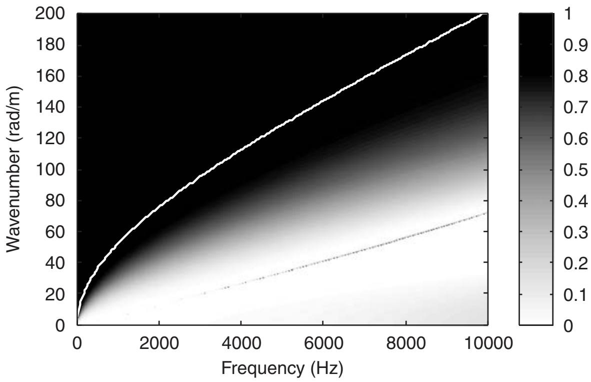


Fig. 1. Shear coefficient versus wavenumber and frequency for the numerical example.

plate wave speed $c_p$ is 833 m/s and the shear wave speed $c_s$ is 456 m/s. Other authors [2,7] have considered the (frequency) range of accuracy of the Mindlin plate equations, so no attempt is made in this paper to investigate this issue. Achenbach [7] uses the equation

$$\omega < 1.2\pi \frac{c_s}{h}, \tag{21}$$

which he states "yield(s) very good results" while Hutchinson [2] uses the equation

$$\omega < 3\frac{c_s}{h}, \tag{22}$$

which he calls "a good practical limit." Using these two formulas applied to the parameters gives an upper frequency range of 27,400 and 21,800 Hz, respectively, for this problem. Fig. 1 is a plot of the shear coefficient versus wavenumber and frequency and was determined using Eq. (19). This figure illustrates the dependency of the shear coefficient on wavenumber and frequency. The parabolic line on the plot is the flexural wavenumber location calculated by finding the maximum value of the displacement in wavenumber at each analysis frequency. The weak line originating at the origin and ending at $f = 10,000$ Hz and $k = 75.4$ rad/m is the plate wave response of the Rayleigh–Lamb model. This plate wave wavenumber can be predicted by

$$k_p = \frac{\omega}{c_p}, \tag{23}$$

where

$$c_p = \sqrt{\frac{E}{\rho(1 - v^2)}}. \tag{24}$$

The Rayleigh–Lamb equation of motion contains the plate wave dynamics while the Mindlin plate equation does not; thus, there is a modeling mismatch around the plate wavenumber that produces a singularity in the analysis. The result of this model mismatch is that the theoretical shear coefficient factor will not be accurate in the region around the plate wavenumber. Fig. 2 is a cut of Fig. 1 in wavenumber at constant frequencies of

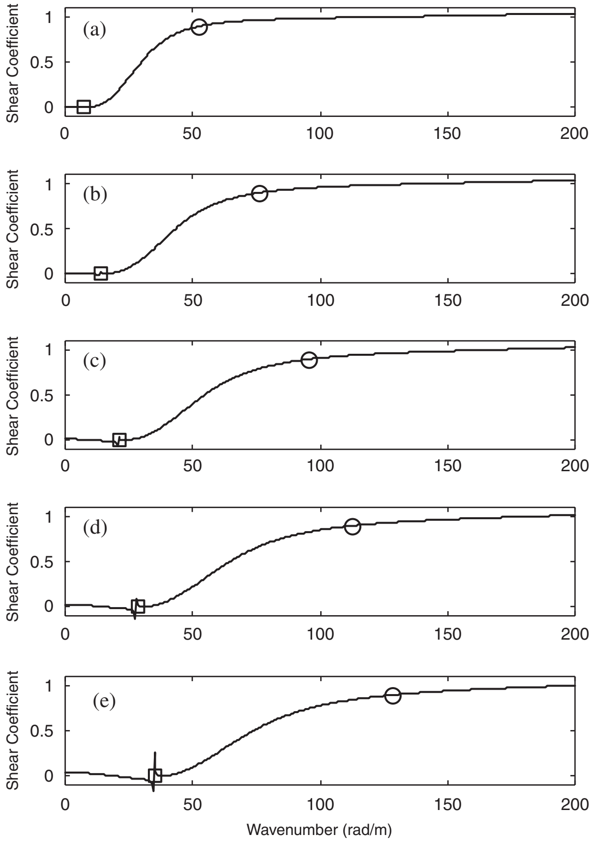


Fig. 2. Shear coefficient versus wavenumber for the numerical example. Plate wavenumber (□) and flexural wavenumber (○) are denoted on the plot. (a) 1000 Hz, (b) 2000 Hz, (c) 3000 Hz, (d) 4000 Hz and (e) 5000 Hz.

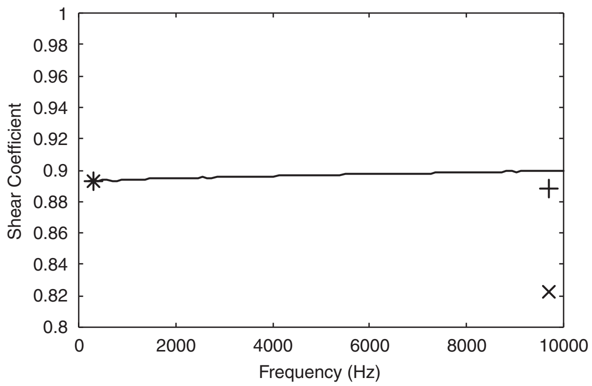


Fig. 3. Shear coefficient versus frequency at flexural wave response for the numerical example. Hutchinson estimate (*), Mindlin estimate one ( × ) and Mindlin estimate two (+) are denoted on the plot.

1000, 2000, 3000, 4000 and 5000 Hz, which illustrates the dependence of the shear coefficient on wavenumber at five different frequencies. The square marker (□) is the plate wavenumber and the round marker (○) is the flexural wavenumber. Fig. 3 is a cut of Fig. 1 in frequency at the flexural wavenumber. The star symbol (*) is the shear coefficient derived by Hutchinson [2] (and later Stephen [3]) and is equal to

$$\kappa^2 = \frac{5}{6-\nu}, \tag{25}$$

while the × symbol is one shear coefficient derived by Mindlin [1] and is

$$\kappa^2 = \frac{\pi^2}{12} \tag{26}$$

and the plus (+) symbol is a second shear coefficient derived by Mindlin [1] and is equal to the root of

$$4\sqrt{(1-\delta\kappa^2)(1-\kappa^2)} - (2-\kappa^2)^2 = 0, \tag{27}$$

where

$$\delta = \frac{1-2v}{2(1-v)}. \tag{28}$$

It is clear that these estimates are close to the solution derived in Eq. (20) for wave propagation at (or near) the flexural wave propagation wavenumber and frequency. They do not account for wave propagation at other frequencies and wavenumbers.

The most interesting aspect of this analysis is that the shear coefficient is highly dependent on the wavenumber. This dependency was previously unknown; however, the fact that the dynamics of the system are dependent on the wavenumber of the excitation is not a new concept. In this type of analysis, low wavenumber excitation generally results in a structural response that is composed of primarily dilatational waves that contain the majority of the energy. Shear effects are secondary until the wavenumber of the excitation becomes moderate. Previous work has typically consisted of modeling the flexural wave in a plate, calculating the response, and then back-calculating the shear coefficient. It does not consist of structural excitation at all wavenumbers and then determining the shear coefficient.

## 4. Conclusions

The theoretical shear coefficient for a Mindlin plate has been derived as an analytical expression based on forced vibration equations of motion. It is shown that this term is dependent on wavenumber, frequency, Young's modulus, shear modulus, Poisson's ratio, density and thickness of the plate. Numerical simulations showed that the shear coefficient is extremely dependent on wavenumber but only slightly dependent on frequency. Previous shear coefficient expressions are close to the analytical expression derived here at the flexural wave frequency and wavenumber of the plate.

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