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Exact solutions for the free in-plane vibration of rectangular plates with two opposite edges simply supported

D.J. Gorman*

Department of Mechanical Engineering, The University of Ottawa, Ottawa, Ontario, Canada K1N 6N5

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Abstract

Two distinct types of simple support boundary conditions are formulated for rectangular plates undergoing in-plane free vibration. Each type of simple support is shown to be analogous to the well known simple support edge conditions encountered in the free transverse vibration analysis of rectangular plates. It is shown that exact solutions may be obtained for plate free vibration eigenvalues and mode shapes when two opposite plate edges are given either type of simple support, the other two edges being given any combination of classical edge conditions. A complete analysis is provided for plate in-plane vibration with pairs of clamped or free edge conditions imposed on the non-simply supported boundaries and exact eigenvalues are tabulated for a large range of plate aspect ratios. This appears to be the first thorough study of these in-plane vibration problems with exact solutions.

1. Introduction

It is well known that a vast literature exists pertaining to the free lateral vibration of rectangular plates. The situation with regard to the free in-plane vibration of the same plates is quite different. Only a relatively small number of publications have been devoted to this latter subject over the years. This is no doubt due in part to the fact that plate in-plane vibration generally involves much higher natural frequencies which are considered to be beyond the level of available excitation forces. Nevertheless, it is found that in the case of plates subjected to tangential fluid boundary flow, such as on the hulls of ocean-going ships, for example, plate in-plane vibration can be excited. There can also be an interrelation between this in-plane vibration and associated acoustic effects, with generation of noise in the immediate environment. There is therefore a strong incentive to gain an understanding of the entire subject of in-plane plate vibration and the mathematical modeling of such phenomena.

A significant contribution to the subject of rectangular plate in-plane vibration was made by Bardell et al. in 1996 [1]. They also made a valuable survey of much of the related literature available up to that time. In particular, they referred to the pioneering work of Lord Rayleigh dealing with what was referred to as 'simply supported' plates [2].

*Tel: +1 613 562 5800; fax: +1 613 562 5177.
E-mail address: dgorman@genie.uottawa.ca.

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Nomenclature

a, b	quarter-plate edge lengths
a_{11}	$= 1.0$
a_{12}	$= \nu$
a_{66}	$= (1-\nu)/2$
E	Young's modulus of plate material
u, v	plate displacements in ξ and η directions, respectively
U, V	dimensionless displacements, $U = u/a$, $V = v/b$
x, y	quarter-plate rectangular coordinates
ξ, η	dimensionless coordinates, $\xi = x/a$, $\eta = y/b$
ϕ	plate aspect ratio b/a
ϕ^{l}	inverse of plate aspect ratio
$\sigma_x^*, \sigma_y^*, \tau_{xy}^*$	dimensionless in-plane normal and shear stresses, defined in text
ω	circular frequency of plate vibration
ρ	mass density of plate material
ν	Poisson ratio of plate material taken here as 0.3
λ^2	dimensionless frequency of plate vibration, $\lambda^2 = \omega a[\rho(1-\nu^2)/E]^{1/2}$
λ_b^2	alternate formulation of dimensionless frequency, $\lambda_b^2 = \omega b[\rho(1-\nu^2)/E]^{1/2}$

The work undertaken in the present paper is in many respects a counterpart of that undertaken much earlier by Leissa in connection with the free lateral vibration of rectangular plates [3]. In Leissa's paper it was shown that, for rectangular plates with a pair of opposite edges simply supported, exact solutions are obtained for all possible combinations of classical boundary conditions along the other edges.

In the present paper it is demonstrated, in a parallel fashion, that for in-plane vibration there exists two distinct classes of 'simple support' edge conditions. It is further shown that for any rectangular plate, with a pair of opposite edges given either of the above classes of simple support, exact free vibration solutions may be obtained for any combination of classical boundary conditions enforced along the remaining edges. Exact eigenvalues and mode shape information are presented for plates with a pair of opposite edges given either of the two classes of simple support and the non-simply supported edges being either free or clamped.

2. Mathematical procedure

2.1. The dimensionless equilibrium equations and in-plane stress formulations

The governing dynamic equilibrium equations for the in-plane problem were developed in dimensionless form in Ref. [4]. They are reproduced here for the sake of completeness only. They are,

$$a_{11}\frac{\partial^2 U}{\partial \xi^2} + \frac{a_{12}}{\phi}\frac{\partial^2 V}{\partial \xi \partial \eta} + \frac{a_{66}}{\phi}\left\{\frac{\partial^2 V}{\partial \xi \partial \eta} + \frac{1.}{\phi}\frac{\partial^2 U}{\partial \eta^2}\right\} + \lambda^4 U = 0 \tag{1}$$

and

$$a_{66}\left\{\frac{\partial^2 V}{\partial \xi^2} + \frac{1.}{\phi}\frac{\partial^2 U}{\partial \xi \partial \eta}\right\} + \frac{a_{12}}{\phi}\frac{\partial^2 U}{\partial \eta \partial \xi} + \frac{a_{11}}{\phi^2}\frac{\partial^2 V}{\partial \eta^2} + \lambda^4 V = 0. \tag{2}$$

All of the symbols are as defined in the nomenclature with sign conventions as indicated in Fig. 1. The dimensionless frequency is designated by the symbol, λ^2, where $\lambda^2 = \omega a\sqrt{\rho(1-\nu^2)/E}$.

It was also shown in Ref. [4] that in-plane normal and shear stresses are expressed in dimensionless form as

$$\sigma_y^* = \upsilon\frac{\partial U}{\partial \xi} + \frac{1.}{\phi}\frac{\partial V}{\partial \eta}, \quad \sigma_x^* = \frac{\partial U}{\partial \xi} + \frac{\upsilon}{\phi}\frac{\partial V}{\partial \eta}, \quad \tau_{xy}^* = \frac{\partial U}{\partial \eta} + \phi\frac{\partial V}{\partial \xi}.$$

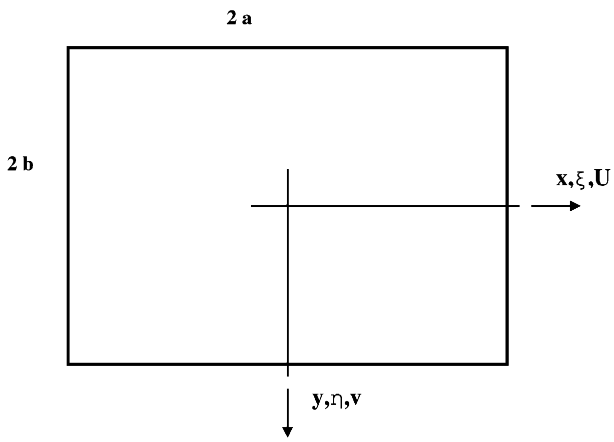

Fig. 1. Schematic view of typical rectangular plate with central coordinate axes.

2.2. The two classes of simple support boundary conditions

It will be recalled that in the study of rectangular plate free lateral vibration a simply supported edge is characterized by the fact that lateral displacement along the edge is forbidden and the same edge is free of bending moment. Such edges are often referred to as 'pinned' edges.

In examining the free lateral vibration of rectangular plates with two opposite edges simply supported the analyst is presented with an immediate significant advantage. The plate vibratory behavior may be expressed in the form of a solution with trigonometric sine functions running between the simply supported edges. This is because each term in the solution satisfies exactly the prescribed simple support edge conditions. This being the case, no series summations are required and exact solutions for the eigenvalues and free vibration mode shapes associated with any number of half sine waves running across the plate are immediately achievable. This is so, provided the other boundary conditions are of the classical (simply supported, clamped, free) type.

This form of solution is often referred to as a Levy type solution, or Voight solution. Voight was the first to replace the static surface loading with the inertia (body) force for the purposes of free vibration analysis. Such solutions will be referred to in this paper as Voight solutions.

It is well known that three categories of classical boundary conditions exist for rectangular plate lateral vibration as well as static problems. These are clamped, free, and simply supported boundary conditions. A completely analogous set of boundary conditions, also known as clamped, free, and simply supported, also exist for rectangular plate in-plane vibration. Of course, the mathematical formulation of these edge conditions is slightly different. This was recognized by Bardell et al. [1] for example, who have employed the same terminology, as has the present author. It is again, only those plate problems where at least one pair of opposite simply supported edges exist that exact solutions can be obtained for plate free in-plane vibration behavior. Exact solutions for a limited number of in-plane vibration problems with simple support along all edges were obtained by Lord Rayleigh and are referred to in Ref. [1].

It is found that in the case of in-plane rectangular plate free vibration analysis two distinct sets of 'simple support' boundary conditions are physically realizable. We choose to designate these two sets of boundary conditions by the symbols SS1, and SS2.

Edges associated with the first set, SS1, are characterized by the fact that plate displacement parallel to the edge is forbidden as well as normal stress perpendicular to the edge.

The second set of simply supported edges are designated here by the symbol SS2. Such edges are characterized by the fact that shear stress along the edge is forbidden as well as displacement normal to the edge.

This latter type of boundary condition was the one designated by the term 'simple support' in the work of Bardell et al. [1], and Lord Rayleigh [2].

2.3. Analysis of plates with type SS1 boundary conditions

We will analyze the free in-plane vibration of plates of this set with two distinct types of support along the non-simply supported edges. These latter edges will both be free, or both given clamped support.

In view of the nature of the imposed boundary conditions it will be obvious that for all problems analyzed here the free vibration mode displacements must possess a symmetry, or antisymmetry, about the central axes of the plate under analysis (Fig. 1). The reader will find a thorough discussion of this phenomenon in Refs. [4,5]. Suffice it to say that here a mode is said to possess symmetry with respect to a central axis if amplitude of displacement normal to the axis is a maximum and displacement parallel to the axis is zero. Conversely, a mode is said to possess antisymmetry with respect to a central axis if displacement normal to the axis is zero while amplitude of displacement parallel to the axis is a maximum. We define modes as being fully symmetric if their displacement possesses symmetry with respect to both central axes. Conversely, modes whose motion is antisymmetric with respect to both axes are said to be fully antisymmetric. There exists two other families of modes. One family possesses symmetry with respect to the ξ-axis (Fig. 1) and antisymmetry with respect to the η-axis. The other family possesses symmetry with respect to the η-axis and antisymmetry with respect to the ξ-axis. These four families of modes comprise all possible modes for the plate vibration problems discussed here. Each mode family will be analyzed separately.

It will also be observed that, provided appropriate boundary conditions are enforced, only one quarter of the plate of interest need be analyzed [4].

One might be tempted at this point to settle for a single analysis of the entire plate with a view to obtaining eigenvalues and mode shapes for every possible plate free vibration mode. This might seem to require less effort than focusing on the three families of plate vibration modes separately as discussed above, through analysis of the quarter plate. Experience has shown this latter approach to be highly preferable.

First, through separate analysis of the individual mode families, only the eigenvalues associated with the particular family under investigation will be uncovered. The eigenvalue density will therefore be only about one third of that which would be otherwise encountered. The probability of missing an eigenvalue is thereby greatly reduced. Furthermore, one avoids the problem of having to contend with repeated eigenvalues.

Secondly, interpretation of the computed mode shapes with mode family separation becomes a much more manageable task. Repeated eigenvalues were encountered for square plates in the excellent work of Bardell et al. [1, pp. 462–463]. Had they analyzed the mode families separately, as is done here, the presence of repeated eigenvalues, which are easily missed in the eigenvalue search, would have been avoided and it would not have been necessary to try and identify the significance of repeated eigenvalues through mode shape studies of related shapes.

Many of the advantages of the quarter-plate approach, where boundary conditions permit it to be exploited, are already elaborated upon in Section 3 of Ref. [4]. The reader is encouraged to review this section. It is also pointed out that one of the symmetric–antisymmetric modes reported by the present author appears to have been missed in the study of Ref. [1].

For these, and a number of other reasons, the author continues with exploitation of the well established quarter-plate approach when it is applicable such as in the present problem under study.

2.3.1. Fully symmetric modes

Referring briefly to Voight type series solutions utilized in Ref. [5], when analyzing fully symmetric modes of fully clamped plates, and focusing on individual terms of these series, it is found that we can write for displacements related to the quarter plate of interest (Fig. 2),

$$V(\xi,\eta) = V_m(\eta)\sin m\pi\xi \tag{3}$$

and

$$U(\xi,\eta) = U_m(\eta)\cos m\pi\xi, \tag{4}$$

where m takes on values of $0, 1, 2$, etc. The case where m equals zero is a special case to which we will return shortly.

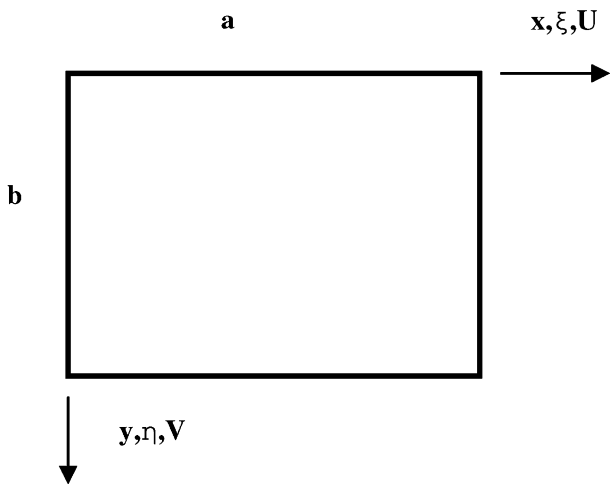

Fig. 2. Schematic view of quarter plate whose vibratory behavior is analyzed.

We focus first on the case where $m \geqslant 1$. It is important to note that all of the required boundary conditions at the extremities of the trigonometric functions, as discussed above, are satisfied exactly by expressing the displacements as given by Eqs. (3) and (4).

The next step in the analysis is to substitute Eqs. (3) and (4) into the governing differential equations. The procedure is described in detail in Ref. [4]. Upon substitution, the variable ξ is eliminated and the governing differential equations become

$$a_{m1} U_m^{\|}(\eta) + b_{m1} V_m^{|}(\eta) + c_{m1} U_m(\eta) = 0 \tag{5}$$

and

$$a_{m2} V_m^{\|}(\eta) + b_{m2} U_m^{|}(\eta) + c_{m2} V_m(\eta) = 0, \tag{6}$$

where

$$a_{m1} = \frac{a_{66}}{\phi^2}, \quad b_{m1} = \mathrm{EMP}(a_{12} + a_{66})/\phi, \quad c_{m1} = -a_{11}\mathrm{EMPS} + \lambda^4, \quad a_{m2} = \frac{a_{11}}{\phi^2},$$
$$b_{m2} = -\frac{\mathrm{EMP}}{\phi}[a_{66} + a_{12}] \quad \text{and} \quad c_{m2} = \lambda^4 - a_{66}\mathrm{EMPS}.$$

Here, superscripts indicate differentiation with respect to the variable η, and the symbols EMP and EMPS represent the product, $m\pi$, and $(m\pi)^2$, respectively.

Through a simple process of differentiating, and adding and subtracting Eqs. (5) and (6), an ordinary homogeneous differential equation involving the quantity $V_m(\eta)$ only is obtained [4]. This equation is written as

$$V_m^{IV}(\eta) + bV_m^{\|}(\eta) + cV_m(\eta) = 0, \tag{7}$$

where

$$b = [a_{m1}c_{m2} - b_{m1}b_{m2} + c_{m1}a_{m2}]/(a_{m1}a_{m2}) \quad \text{and} \quad c = c_{m1}c_{m2}/(a_{m1}a_{m2}).$$

The characteristic equation associated with Eq. (7) can be written as $D^4 + bD^2 + c = 0$, and setting $D^2 = \varepsilon$, we write the quadratic equation $\varepsilon^2 + b\varepsilon + c = 0$. Thus we obtain the roots $\varepsilon_{1,2} = -b + \sqrt{b^2 - 4c}/2$, and $-b - \sqrt{b^2 - 4c}/2$. Since the quantity $b^2 - 4c \geqslant 0$ for all problems encountered here, both of the above quantities are real. Denoting ε_1 as root1, and ε_2 as root2, the exact solution for Eq. (7) is available from any text dealing with elementary theory of differential equations and depends on whether root1, root2, etc., are positive or negative.

Introducing the quantities $\beta_m = \sqrt{|\text{root1}|}$, and $\gamma_m = \sqrt{|\text{root2}|}$, three possible forms of solution exist for Eq. (7). They are as follows.

Solution 1: root1 $\geqslant 0.0$, root2 $\leqslant 0.0$.

$$V_m(\eta) = A_m \sinh \beta_m \eta + B_m \cosh \beta_m \eta + C_m \sin \gamma_m \eta + D_m \cos \gamma_m \eta, \tag{8}$$

where $A_m, B_{m,\geqslant}$ etc. are constants to be determined.

Solution 2: root1 $\leqslant 0.0$, root2 $\leqslant 0.0$.

$$V_m(\eta) = A_m \sin \beta_m \eta + B_m \cos \beta_m \eta + C_m \sin \gamma_m \eta + D_m \cos \gamma_m \eta. \tag{9}$$

Solution 3: root1 $\geqslant 0.0$, root2 $\geqslant 0.0$.

$$V_m(\eta) = A_m \sinh \beta_m \eta + B_m \cosh \beta_m \eta + C_m \sinh \gamma_m \eta + D_m \cosh \gamma_m \eta. \tag{10}$$

For any of the above solutions the corresponding form of solution for the quantity $U_m(\eta)$ is readily obtained through manipulation of Eqs. (5) and (6) [4].

Turning next to enforcement of appropriate boundary conditions along the edge, $\eta = 0$, it follows that all terms in the above solutions for $V_m(\eta)$ which are antisymmetric with respect to the ξ-axis must be deleted. The final step is to enforce appropriate boundary conditions along the edge, $\eta = 1$. We proceed as follows:

Case 1: Solution 1 applicable.

$$V_m(\eta) = B_m \cosh \beta_m \eta + D_m \cos \gamma_m \eta, \tag{11}$$

with

$$U_m(\eta) = B_m \alpha_{2m} \sinh \beta_m \eta + D_m \alpha_{4m} \sin \gamma_m \eta, \tag{12}$$

where $\alpha_{2m} = \beta_m [a_{m1} a_{m2} \beta_m^2 + a_{m1} c_{m2} - b_{m1} b_{m2}]/c_{m1} b_{m2}$ and $\alpha_{4m} = \gamma_m [a_{m1} a_{m2} \gamma_m^2 - a_{m1} c_{m2} + b_{m1} b_{m2}]/c_{m1} c_{m2}$.

(1) Simply supported plate with other edges free.

Here, free edge conditions must be enforced along the boundary, $\eta = 1$.

This involves enforcing conditions of zero shear stress and zero normal stress along the boundary. Expressions for shear stress and normal stress are provided in Section 2.1.

Enforcing a condition of zero shear stress along this boundary we obtain

$$V_m(\eta) = B_m [\cosh \beta_m \eta + \theta_{1m} \cos \gamma_m \eta] \tag{13}$$

and

$$U_m(\eta) = B_m [\alpha_{2m} \sinh \beta_m \eta + \theta_{1m} \alpha_{4m} \sin \gamma_m \eta], \tag{14}$$

where

$$\theta_{1m} = -\frac{[\alpha_{2m} \beta_m + \phi \text{EMP}] \cosh \beta_m}{[\alpha_{4m} \gamma_m + \phi \text{EMP}] \cos \gamma_m}.$$

Next, enforcing the condition of zero normal stress perpendicular to the boundary, $\eta = 1$, we obtain

$$B_m \theta_{11m} = 0, \tag{15}$$

where

$$\theta_{11m} = \left\{ \left[\frac{\beta_m}{\phi} - \upsilon \text{EMP} \alpha_{2m} \right] \sinh \beta_m - \theta_{1m} \left[\frac{\gamma_m}{\phi} + \upsilon \text{EMP} \alpha_{4m} \right] \sin \gamma_m \right\}.$$

It will be apparent that a non-trivial solution can exist for the coefficient B_m only if the quantity θ_{11m} is equal to zero. Accordingly, for the symmetric–symmetric modes under study, having chosen a desired value for '*m*', the number of half-waves in displacement V running across the quarter plate in the ξ direction (see Eq. (3)), the related eigenvalues are those values of the parameter λ^2 which cause the quantity $\theta_{11m,}$ to vanish. Mode shape displacements associated with any eigenvalue are obtained from Eqs. (13) and (14) by assigning an arbitrary non-zero value to the coefficient B_m.

The simple analytical procedure described above is common to all problems solved here where the plate undergoes two-dimensional in-plane motion. Of course, for plates with clamped edges different boundary conditions will be enforced along the non-simply supported edges. For the remaining problems considered here a detailed description of the analytical procedure will therefore not be provided. Only expressions for the plate displacements and the quantities θ_{1m} and θ_{11m} will be provided for the benefit of the reader.

(2) Simply supported plate with other edges clamped.

Enforcing the boundary conditions, displacement U equals zero, and displacement V equals zero, respectively, along the edge, $\eta = 1$, find

$$\theta_{1m} = -\frac{\alpha_{2m} \sinh \beta_m}{\alpha_{4m} \sin \gamma_m}$$

and

$$\theta_{11m} = \cosh \beta_m + \theta_{1m} \cos \gamma_m,$$

with displacements as given by Eqs. (13) and (14).

Case 2: Solution 2 applicable.

$$V_m(\eta) = B_m \cos \beta_m \eta + D_m \cos \gamma_m \eta, \tag{16}$$

with

$$U_m(\eta) = B_m \alpha_{2m} \sin \beta_m \eta + D_m \alpha_{4m} \sin \gamma_m(\eta), \tag{17}$$

where

$$\alpha_{2m} = \beta_m [a_{m1} a_{m2} \beta_m^2 - a_{m1} c_{m2} + b_{m1} b_{m2}] / c_{m1} b_{m2}$$

and

$$\alpha_{4m} = \gamma_m [a_{m1} a_{m2} \gamma_m^2 - a_{m1} c_{m2} + b_{m1} b_{m2}] / c_{m1} b_{m2}.$$

(1) Simply supported plate with other edges free.

$$V_m(\eta) = B_m [\cos \beta_m \eta + \theta_{1m} \cos \gamma_m \eta] \tag{18}$$

and

$$U_m(\eta) = B_m [\alpha_{2m} \sin \beta_m \eta + \theta_{1m} \alpha_{4m} \sin \gamma_m \eta], \tag{19}$$

where

$$\theta_{1m} = -\frac{[\alpha_{2m} \beta_m + \phi \mathrm{EMP}] \cos \beta_m}{[\alpha_{4m} \gamma_m + \phi \mathrm{EMP}] \cos \gamma_m}$$

and

$$\theta_{11m} = -\left[\frac{\beta_m}{\phi} + \upsilon \mathrm{EMP} \alpha_{2m}\right] \sin \beta_m - \theta_{1m} \left[\frac{\gamma_m}{\phi} + \upsilon \mathrm{EMP} \alpha_{4m}\right] \sin \gamma_m.$$

(2) Simply supported plate with other edges clamped.

Displacements as given by Eqs. (18) and (19) with

$$\theta_{1m} = -\frac{\alpha_{2m} \sin \beta_m}{\alpha_{4m} \sin \gamma_m}, \quad \theta_{11m} = \cos \beta_m + \theta_{1m} \cos \gamma_m.$$

Case 3: Solution 3 applicable.

$$V_m(\eta) = B_m \cosh \beta_m \eta + D_m \cosh \gamma_m \eta, \tag{20}$$

with

$$U_m(\eta) = B_m \alpha_{2m} \sinh \beta_m \eta + D_m \alpha_{4m} \sinh \gamma_m \eta, \tag{21}$$

where $\alpha_{2m} = \beta_m [a_{m1} a_{m2} \beta_m^2 + a_{m1} c_{m2} - b_{m1} b_{m2}] / c_{m1} b_{m2}$ and $\alpha_{4m} = \gamma_m [a_{m1} a_{m2} \gamma_m^2 + a_{m1} c_{m2} - b_{m1} b_{m2}] / c_{m1} b_{m2}$.

(1) Simply supported plate with other edges free.

$$V_m(\eta) = B_m [\cosh \beta_m \eta + \theta_{1m} \cosh \gamma_m \eta] \tag{22}$$

and

$$U_m(\eta) = B_m[\alpha_{2m} \sinh \beta_m\eta + \theta_{1m}\alpha_{4m} \sinh \gamma_m\eta], \tag{23}$$

where

$$\theta_{1m} = -\frac{[\alpha_{2m}\beta_m + \phi\mathrm{EMP}]\cosh \beta_m}{[\alpha_{4m}\gamma_m + \phi\mathrm{EMP}]\cosh \gamma_m}$$

and

$$\theta_{11m} = \left[\frac{\beta_m}{\phi} - \upsilon\mathrm{EMP}\alpha_{2m}\right] \sinh \beta_m + \theta_{1m}\left[\frac{\gamma_m}{\phi} - \upsilon\mathrm{EMP}\alpha_{4m}\right] \sinh \gamma_m.$$

(2) Simply supported plate with other edges clamped.
Displacements as given by Eqs. (22) and (23) with

$$\theta_{1m} = -\frac{\alpha_{2m} \sinh \beta_m}{\alpha_{4m} \sinh \gamma_m}$$

and $\theta_{11m} = \cosh \beta_m + \theta_{1m} \cosh \gamma_m$.

Finally, we must examine the one-dimensional mode ($m = 0$). Returning to Eqs. (3) and (4) it is seen that for this mode displacement V will be zero and displacement U will be a function of η, only. Examining the governing differential equations it is found that only one differential equation, in revised form, is applicable and it may be written as

$$\frac{\partial^2 U(\eta)}{\partial\eta^2} + \alpha^2 U(\eta) = 0, \tag{24}$$

where $U(\eta)$ represents plate in-plane displacement and

$$\alpha^2 = \lambda^4\phi^2/a_{66}.$$

The solution for Eq. (24) is well known and is expressed as

$$U(\eta) = A \sin \alpha\eta + B \cos \alpha\eta, \tag{25}$$

where A and B are constants to be determined.

In view of the antisymmetry of displacement $U(\eta)$ about the ξ axis the constant B must be set equal to zero. We are therefore left with the equation

$$U(\eta) = A \sin \alpha\eta. \tag{26}$$

For the case of free boundaries a condition of zero shear stress must be imposed along the edge, $\eta = 1$. We therefore write

$$A\alpha \cos \alpha\eta|_{\eta=1} = 0. \tag{27}$$

A non-zero solution for the quantity A is possible only if we impose the condition

$$\alpha = \frac{\pi}{2}, \frac{3\pi}{2}, \ldots, (2n-1)\frac{\pi}{2}, \text{ etc.}$$

Using our expression for α we obtain

$$\lambda^2 = (2n-1)\frac{\pi}{2}[a_{66}]^{1/2}/\phi. \tag{28}$$

Exact eigenvalues (λ^2) for this one-dimensional mode family are therefore provided explicitly by Eq. (28), and associated mode shapes are provided by Eq. (26) where any arbitrary value may be assigned to the constant A.

For the case of clamped edge conditions a condition of zero displacement parallel to the edge, $\eta = 1$, must be imposed.

Returning to Eq. (26) we see that the quantity α must take on values of

$$\alpha = \pi, 2\pi, \ldots, n\pi, \quad n = 1, 2, \ldots \text{etc.}$$

This leads to the expression

$$\lambda^2 = n\pi[a_{66}]^{1/2}/\phi. \tag{29}$$

Here, exact eigenvalues, λ^2, are obtained from Eq. (29) with mode shapes provided by Eq. (26).

An analysis for antisymmetric and symmetric–antisymmetric modes of this set is provided in Appendix A.

2.4. Analysis of plates with type SS2 boundary conditions

The analysis to follow differs only slightly from that already described in detail for plates with SS1 type simple supports. Accordingly, in what is to follow, only the differences in analytical procedure required to take care of this alternative type of simple edge support will be elaborated upon.

2.4.1. Fully symmetric modes

Here we refer to Voight type solutions utilized in Ref. [4] when analyzing fully symmetric in-plane modes of the completely free plate. Focusing on individual terms of the series employed we write

$$V(\xi, \eta) = V_m(\eta) \sin \frac{(2m-1)\pi\xi}{2} \tag{30}$$

and

$$U(\xi, \eta) = U_m(\eta) \cos \frac{(2m-1)\pi\xi}{2}, \tag{31}$$

where $m = 1, 2, \ldots$ etc.

It is easily shown that the SS2 boundary conditions are satisfied along the edge, $\xi = 1$, as well as the conditions of symmetry required along the η-axis, by the above trigonometric functions.

Substituting the above expressions for displacements U and V into the governing differential equations we again obtain Eqs. (5) and (6), with the same constants a_{m1}, b_{m1}, etc. Here the quantity EMP equals $(2m-1)\pi/2$. Again, manipulating Eqs. (5) and (6) we obtain Eq. (7) with the same associated constants b and c. Solutions are as given by Eqs. (8)–(10). It will be apparent that all terms for $V_m(\eta)$ which are antisymmetric with respect to the ξ-axis must be deleted. Eigenvalues and mode shapes for the fully symmetric modes under investigation here are obtained by following procedures identical to those discussed earlier.

Case 1: Solution 1 applicable.

$$V_m(\eta) = B_m \cosh \beta_m\eta + D_m \cos \gamma_m\eta \tag{32}$$

and

$$U_m(\eta) = B_m\alpha_{2m} \sinh \beta_m\eta + D_m\alpha_{4m} \sin \gamma_m\eta, \tag{33}$$

where $\alpha_{2m} = \beta_m(a_{m1}a_{m2}\beta_m^2 + a_{m1}c_{m2} - b_{m1}b_{m2})/c_{m1}b_{m2}, \alpha_{4m} = \gamma_m(a_{m1}a_{m2}\gamma_m^2 - a_{m1}c_{m2} - b_{m1}b_{m2})/c_{m1}b_{m2}$.

(1) Plate with free edges.

$$V_m(\eta) = B_m[\cosh \beta_m\eta + \theta_{1m} \cos \gamma_m\eta] \tag{34}$$

and

$$U_m(\eta) = B_m[\alpha_{2m} \sinh \beta_m\eta + \theta_{1m}\alpha_{4m} \sin \gamma_m\eta], \tag{35}$$

with

$$\theta_{1m} = -\frac{[\alpha_{2m}\beta_m + \phi\text{EMP}] \cosh \beta_m}{[\alpha_{4m}\gamma_m + \phi\text{EMP}] \cos \gamma_m}$$

and

$$\theta_{11m} = \left[\frac{\beta_m}{\phi} - v\text{EMP}\alpha_{2m}\right] \sinh \beta_m - \theta_{1m}\left[\frac{\gamma_m}{\phi} + v\text{EMP}\alpha_{4m}\right] \sin \gamma_m.$$

(2) Plate with clamped edges.
Displacements are given by Eqs. (34) and (35):

$$\theta_{1m} = -\frac{\alpha_{2m} \sinh \beta_m}{\alpha_{4m} \sin \gamma_m}$$

and

$$\theta_{11m} = \cosh \beta_m + \theta_{1m} \cos \gamma_m.$$

Case 2: Solution 2 applicable.

$$V_m(\eta) = B_m \cos \beta_m\eta + D_m \cos \gamma_m\eta \tag{36}$$

and

$$U_m(\eta) = B_m\alpha_{2m} \sin \beta_m\eta + D_m\alpha_{4m} \sin \gamma_m\eta, \tag{37}$$

where $\alpha_{2m} = \beta_m(a_{m1}a_{m2}\beta_m^2 - a_{m1}c_{m2} + b_{m1}b_{m2})/c_{m1}b_{m2}, \alpha_{4m} = \gamma_m(a_{m1}a_{m2}\gamma_m^2 - a_{m1}c_{m2} + b_{m1}b_{m2})/c_{m1}b_{m2}$.
(1) Plate with free edges.

$$V_m(\eta) = B_m[\cos \beta_m\eta + \theta_{1m} \cos \gamma_m\eta] \tag{38}$$

and

$$U_m(\eta) = B_m[\alpha_{2m} \sin \beta_m\eta + \theta_{1m}\alpha_{4m} \sin \gamma_m\eta], \tag{39}$$

with

$$\theta_{1m} = -\frac{[\alpha_{2m}\beta_m + \phi\text{EMP}] \cos \beta_m}{[\alpha_{4m}\gamma_m + \phi\text{EMP}] \cos \gamma_m}$$

and

$$\theta_{11m} = -\left[\frac{\beta_m}{\phi} + v\text{EMP}\alpha_{2m}\right] \sin \beta_m + \theta_{1m}\left[\frac{\gamma_m}{\phi} + v\text{EMP}\alpha_{4m}\right] \sin \gamma_m.$$

(1) Plate with clamped edges.
Displacements are given by Eqs. (38) and (39):

$$\theta_{1m} = -\frac{\alpha_{2m} \sin \beta_m}{\alpha_{4m} \sin \gamma_m}$$

and

$$\theta_{11m} = \cos \beta_m + \theta_{1m} \cos \gamma_m.$$

Case 3: Solution 3 applicable.

$$V_m(\eta) = B_m \cosh \beta_m\eta + D_m \cosh \gamma_m\eta \tag{40}$$

and

$$U_m(\eta) = B_m\alpha_{2m} \sinh \beta_m\eta + D_m\alpha_{4m} \sinh \gamma_m\eta, \tag{41}$$

where $\alpha_{2m} = \beta_m(a_{m1}a_{m2}\beta_m^2 + a_{m1}c_{m2} - b_{m1}b_{m2})/c_{m1}b_{m2}, \alpha_{4m} = \gamma_m(a_{m1}a_{m2}\gamma_m^2 + a_{m1}c_{m2} - b_{m1}b_{m2})/c_{m1}b_{m2}$.
(1) Plate with free edges

$$V_m(\eta) = B_m[\cosh \beta_m\eta + \theta_{1m} \cosh \gamma_m\eta] \tag{42}$$

and

$$U_m(\eta) = B_m[\alpha_{2m} \sinh \beta_m\eta + \theta_{1m}\alpha_{4m} \sinh \gamma_m\eta], \tag{43}$$

with

$$\theta_{1m} = -\frac{[\alpha_{2m}\beta_m + \phi \mathrm{EMP}]\cosh \beta_m}{[\alpha_{4m}\gamma_m + \phi \mathrm{EMP}]\cosh \gamma_m}$$

and

$$\theta_{11m} = \left[\frac{\beta_m}{\phi} - v\mathrm{EMP}\alpha_{2m}\right]\sinh \beta_m + \theta_{1m}\left[\frac{\gamma_m}{\phi} - v\mathrm{EMP}\alpha_{4m}\right]\sinh \gamma_m.$$

(2) Plate with clamped edges.

Displacements are given by Eqs. (42) and (43):

$$\theta_{1m} = -\frac{\alpha_{2m}\sinh \beta_m}{\alpha_{4m}\sinh \gamma_m}$$

and

$$\theta_{11m} = \cosh \beta_m + \theta_{1m}\cosh \gamma_m.$$

An analysis for antisymmetric and symmetric–antisymmetric modes of this set is provided in Appendix B.

3. Presentation of computed results

The reader can, of course, easily compute exact eigenvalues and mode shapes for any plate in-plane vibration problem among the families of problems introduced here. In the case of problems with one-dimensional mode shapes, exact solutions are provided in the text with no computation required. Otherwise it is only necessary to search for values of the parameter, λ^2, which cause the pertinent value of the quantity, θ_{11m}, as provided herein to take on a zero value.

Nevertheless, a set of tables of limited scope, based on computed exact eigenvalues, are provided for the benefit of the reader.

3.1. Listing of computed eigenvalues

Data related to the four distinct plate plate-boundary configurations considered here are to be found in Tables 1–4. Following conventional notation practices each plate configuration is designated by a set of four symbols in ordered sequence. The first symbol indicates the boundary condition enforced along the left edge of the full plate (Fig. 1). Subsequent symbols indicate boundary conditions enforced along the remaining edges, in order, as we move counter clockwise around the plate. For example, the designation, SS1-F-SS1-F indicates a plate with two opposite edges given SS1 type edge support, the other two edges being free. The designation SS1-C-SS1-C differs from that immediately above only in that free edges are replaced by edges with clamped support. Of course, there will be two other mode families where the SS1 edge conditions will be replaced by edge conditions of the type SS2. The nature of these two types of edge support were explained in detail earlier.

Examining the eigenvalue listings of Tables 1–4, it will be seen that for each distinct type of mode, a three by three array of eigenvalues is provided for a range of plate aspect ratios. The value of 'm', moving across the top of the array, indicates the number of half or quarter waves in displacement, as appropriate, as we move along the ξ-axis. The parameter n increases from 1 to 3 as we move down the array and indicates the order of the mode, first, second, etc. This does not include the one-dimensional modes where they exist. A pair of asterisks adjacent to a mode family heading indicates that one-dimensional modes also exist for that family and that exact eigenvalues and mode shapes for these modes are to be found in the text.

It will be noted that tabulated eigenvalues are given to four significant digits. Also, it will be noted that for plates with aspect ratios less than one, tabulated eigenvalues are non-dimensionalized with respect to b, the shorter of the two edge lengths of the quarter plate (Fig. 2). This is in keeping with practices followed earlier in plate lateral free vibration analysis.

Table 1
Eigenvalues, λ^2, for the SS1-F-SS1-F rectangular plate. Eigenvalues, λ_b^2, stored when inverse of aspect ratio, ϕ^{I}, listed

*Fully symmetric modes***

	m											
	$\phi = 1.0$			$\phi = 1.25$			$\phi = 1.5$			$\phi = 2.0$		
n	1	2	3	1	2	3	1	2	3	1	2	3
1	1.624	3.391	5.107	1.659	3.401	5.108	1.679	3.404	5.109	1.695	3.406	5.109
2	3.449	4.037	5.737	2.359	3.893	5.666	2.190	3.825	5.632	2.019	3.768	5.604
3	4.306	6.313	6.642	3.214	4.741	6.242	3.021	4.428	6.024	3.157	4.104	5.812

	$\phi^{\text{I}} = 1.25$			$\phi^{\text{I}} = 1.5$			$\phi^{\text{I}} = 2.0$		
n	1	2	3	1	2	3	1	2	3
1	1.259	2.694	4.080	1.013	2.223	3.391	0.7038	1.624	2.518
2	2.338	3.427	4.697	2.098	3.058	4.037	1.749	3.449	3.285
3	3.065	4.699	5.789	2.892	4.202	6.313	2.776	4.306	4.531

Fully antisymmetric modes

	m											
	$\phi = 1.0$			$\phi = 1.25$			$\phi = 1.5$			$\phi = 2.0$		
n	1	2	3	1	2	3	1	2	3	1	2	3
1	1.682	2.607	4.263	1.156	2.573	4.258	1.046	2.561	4.257	0.9331	2.555	4.257
2	2.517	4.705	5.338	1.597	3.550	5.063	1.569	3.310	4.919	1.530	3.063	4.785
3	3.801	5.186	7.718	2.196	4.531	5.931	1.983	4.160	5.546	1.713	3.620	5.144

	$\phi^{\text{I}} = 1.25$			$\phi^{\text{I}} = 1.5$			$\phi^{\text{I}} = 2.0$		
n	1	2	3	1	2	3	1	2	3
1	1.137	2.146	3.423	0.9717	1.866	2.874	0.7402	1.568	2.221
2	1.533	3.506	4.609	1.497	3.059	4.160	1.493	2.353	3.590
3	2.341	3.862	5.882	2.229	3.425	5.174	2.098	2.974	3.994

Modes symmetric about the ξ-axis and anti-symmetric about the η-axis

	m											
	$\phi = 1.0$			$\phi = 1.25$			$\phi = 1.5$			$\phi = 2.0$		
n	1	2	3	1	2	3	1	2	3	1	2	3
1	0.7038	2.518	4.252	0.7485	2.539	4.256	0.7779	2.548	4.257	0.8121	2.553	4.257
2	1.749	3.285	4.867	1.611	3.072	4.767	1.502	2.963	4.721	1.725	2.869	4.683
3	2.776	4.531	5.926	2.283	4.073	5.463	1.994	3.724	5.202	2.153	3.321	4.942

	$\phi^{\text{I}} = 1.25$			$\phi^{\text{I}} = 1.5$			$\phi^{\text{I}} = 2.0$		
n	1	2	3	1	2	3	1	2	3
1	0.5199	1.985	3.391	0.4000	1.624	2.811	0.2570	1.167	2.074
2	1.532	2.884	4.037	1.390	3.449	3.524	1.222	2.253	2.949
3	2.749	3.904	6.313	2.745	4.306	4.801	2.751	2.991	4.018

Table 1 (*continued*)

Modes symmetric about the η-axis and anti-symmetric about the ξ-axis

	m											
	$\phi = 1.0$			$\phi = 1.25$			$\phi = 1.5$			$\phi = 2.0$		
n	1	2	3	1	2	3	1	2	3	1	2	3
1	1.682	2.607	4.263	1.156	2.573	4.258	1.046	2.561	4.257	0.9331	2.555	4.257
2	2.517	5.186	5.338	1.597	3.550	5.063	1.569	3.310	4.919	1.530	3.063	4.785
3	3.801	4.705	7.718	2.196	4.531	5.931	1.983	4.160	5.546	1.713	3.620	5.144

	$\phi^{\mid} = 1.25$			$\phi^{\mid} = 1.5$			$\phi^{\mid} = 2.0$		
n	1	2	3	1	2	3	1	2	3
1	1.137	2.146	3.423	0.9717	1.866	2.874	0.7402	1.568	2.221
2	1.533	3.506	4.609	1.497	3.059	4.160	1.493	2.352	3.590
3	2.341	3.862	5.882	2.229	3.425	5.174	2.098	2.974	3.994

Table 2
Eigenvalues, λ^2, for the SS1-C-SS1-C rectangular plate. Eigenvalues, λ_b^2, stored when inverse of aspect ratio, $\phi^{\mid}$, listed

*Fully symmetric modes***

	m											
	$\phi = 1.0$			$\phi = 1.25$			$\phi = 1.5$			$\phi = 2.0$		
n	1	2	3	1	2	3	1	2	3	1	2	3
1	2.205	3.865	5.667	2.070	3.808	5.633	1.999	3.778	5.614	1.933	3.750	5.597
2	3.422	4.825	6.338	3.051	4.444	6.062	2.769	4.225	5.912	2.413	4.002	5.762
3	4.241	6.104	7.450	3.735	5.424	6.822	3.459	4.967	6.457	3.052	4.453	6.077

	$\phi^{\mid} = 1.25$			$\phi^{\mid} = 1.5$			$\phi^{\mid} = 2.0$		
n	1	2	3	1	2	3	1	2	3
1	1.935	3.169	4.580	1.783	2.722	3.865	1.638	2.205	2.999
2	3.054	4.285	5.404	2.796	3.930	4.825	2.479	3.422	4.153
3	4.007	5.397	6.649	3.900	4.837	6.104	3.800	4.241	5.188

Fully antisymmetric modes

	m											
	$\phi = 1.0$			$\phi = 1.25$			$\phi = 1.5$			$\phi = 2.0$		
n	1	2	3	1	2	3	1	2	3	1	2	3
1	1.747	3.513	5.079	1.640	3.256	4.918	1.561	3.110	4.832	1.405	2.965	4.748
2	2.814	4.682	6.137	2.310	4.238	5.633	2.007	3.864	5.340	1.725	3.427	5.039
3	3.570	5.116	7.432	3.000	4.841	6.607	2.626	4.680	6.073	2.164	4.047	5.484

	$\phi^{\mid} = 1.25$			$\phi^{\mid} = 1.5$			$\phi^{\mid} = 2.0$		
n	1	2	3	1	2	3	1	2	3
1	1.510	3.097	4.265	1.362	2.811	3.755	1.193	2.342	3.166
2	2.783	3.942	5.464	2.773	3.450	4.993	2.771	3.010	4.072
3	3.440	4.627	6.399	3.361	4.327	5.492	3.274	3.939	4.701

Table 2 (*continued*)

Modes symmetric about the ξ-axis and anti-symmetric about the η-axis

	m											
	$\phi = 1.0$			$\phi = 1.25$			$\phi = 1.5$			$\phi = 2.0$		
n	1	2	3	1	2	3	1	2	3	1	2	3
1	1.638	2.999	4.760	1.393	2.916	4.716	1.250	2.874	4.694	1.103	2.834	4.673
2	2.479	4.153	5.555	2.172	3.722	5.232	1.972	3.455	5.052	1.711	3.169	5.245
3	3.810	5.188	6.780	3.099	4.727	6.099	2.642	4.301	5.685	2.121	3.725	5.245

	$\phi^{|} = 1.25$			$\phi^{|} = 1.5$			$\phi^{|} = 2.0$		
n	1	2	3	1	2	3	1	2	3
1	1.581	2.508	3.865	1.558	2.205	3.283	1.545	1.875	2.588
2	2.301	3.741	4.825	2.191	3.422	4.373	2.066	2.958	3.814
3	3.773	4.573	6.104	3.754	4.241	5.531	3.737	3.963	4.669

Modes symmetric about the η-axis and anti-symmetric about the ξ-axis

	m											
	$\phi = 1.0$			$\phi = 1.25$			$\phi = 1.5$			$\phi = 2.0$		
n	1	2	3	1	2	3	1	2	3	1	2	3
1	1.747	3.513	5.079	1.640	3.256	4.918	1.561	3.110	4.832	1.405	2.965	4.748
2	2.814	4.682	6.137	2.310	4.238	5.633	2.007	3.864	5.341	1.725	3.427	5.039
3	3.570	5.118	7.432	3.000	4.841	6.607	2.626	4.680	6.073	2.164	4.047	5.484

	$\phi^{|} = 1.25$			$\phi^{|} = 1.5$			$\phi^{|} = 2.0$		
n	1	2	3	1	2	3	1	2	3
1	1.510	3.097	4.265	1.362	2.811	3.755	1.193	2.342	3.166
2	2.783	3.942	5.464	2.773	3.450	4.993	2.771	3.010	4.072
3	3.440	4.627	6.399	3.361	4.327	5.492	3.274	3.939	4.701

Table 3
Eigenvalues, λ^2, for the SS2-F-SS2-F rectangular plate. Eigenvalues, λ_b^2, stored when inverse of aspect ratio, $\phi^{|}$, listed

*Fully symmetric modes***

	m											
	$\phi = 1.0$			$\phi = 1.25$			$\phi = 1.5$			$\phi = 2.0$		
n	1	2	3	1	2	3	1	2	3	1	2	3
1	0.7038	2.518	4.252	0.7485	2.539	4.256	0.7779	2.548	4.257	0.8121	2.553	4.257
2	1.749	3.285	4.867	1.611	3.072	4.767	1.502	2.963	4.721	1.724	2.869	4.683
3	2.776	4.531	5.926	2.283	4.073	5.463	1.994	3.724	5.202	2.153	3.321	4.942

	$\phi^{|} = 1.25$			$\phi^{|} = 1.5$			$\phi^{|} = 2.0$		
n	1	2	3	1	2	3	1	2	3
1	0.5199	1.985	3.391	0.4000	1.624	2.811	0.2570	1.167	2.074
2	1.532	2.884	4.037	1.390	3.449	3.524	1.222	2.253	2.949
3	2.749	3.904	6.313	2.745	4.306	4.801	2.751	2.991	4.018

Table 3 (*continued*)

Fully antisymmetric modes

	m											
	$\phi = 1.0$			$\phi = 1.25$			$\phi = 1.5$			$\phi = 2.0$		
n	1	2	3	1	2	3	1	2	3	1	2	3
1	1.866	3.423	5.111	1.776	3.410	5.109	1.738	3.407	5.109	1.712	3.406	5.109
2	3.059	4.609	6.125	2.872	4.270	5.903	3.137	4.084	5.789	2.304	3.904	5.685
3	3.425	5.882	7.241	3.195	6.174	6.655	3.457	4.827	6.319	2.941	4.349	5.981

	$\phi^{\mid} = 1.25$			$\phi^{\mid} = 1.5$			$\phi^{\mid} = 2.0$		
n	1	2	3	1	2	3	1	2	3
1	1.622	2.767	4.094	1.485	2.347	3.423	1.682	1.866	2.607
2	2.497	4.073	5.187	2.113	3.716	4.609	2.517	3.060	4.705
3	3.064	4.990	6.440	2.822	4.225	5.882	3.801	3.425	5.186

*Modes symmetric about the ξ-axis and anti-symmetric about the η-axis***

	m											
	$\phi = 1.0$			$\phi = 1.25$			$\phi = 1.5$			$\phi = 2.0$		
n	1	2	3	1	2	3	1	2	3	1	2	3
1	3.582	7.025	9.935	3.582	6.767	9.754	3.582	6.623	9.655	3.512	6.477	9.555
2	4.443	7.164	10.75	4.023	7.164	10.68	3.776	7.164	10.31	3.582	7.025	9.935
3	7.025	8.886	11.33	5.928	8.046	10.75	5.236	7.551	10.75	4.443	7.164	10.54

	$\phi^{\mid} = 1.25$			$\phi^{\mid} = 1.5$			$\phi^{\mid} = 2.0$		
n	1	2	3	1	2	3	1	2	3
1	2.866	5.731	8.168	2.388	4.776	7.025	1.791	3.582	5.373
2	4.023	5.928	8.597	3.776	5.236	7.164	3.512	4.443	5.664
3	6.767	8.046	9.815	6.623	7.551	8.886	6.477	7.025	7.854

Modes symmetric about the η-axis and anti-symmetric about the ξ-axis

	m											
	$\phi = 1.0$			$\phi = 1.25$			$\phi = 1.5$			$\phi = 2.0$		
n	1	2	3	1	2	3	1	2	3	1	2	3
1	1.682	2.607	4.263	1.156	2.573	4.258	1.046	2.561	4.257	0.9331	2.555	4.257
2	2.517	5.186	5.338	1.597	3.550	5.063	1.569	3.310	4.919	1.530	3.063	4.785
3	3.801	4.705	7.718	2.196	4.531	5.931	1.983	4.160	5.546	1.713	3.620	5.144

	$\phi^{\mid} = 1.25$			$\phi^{\mid} = 1.5$			$\phi^{\mid} = 2.0$		
n	1	2	3	1	2	3	1	2	3
1	1.137	2.146	3.423	0.9717	1.866	2.874	0.7402	1.568	2.221
2	1.533	3.506	4.609	1.497	3.059	4.160	1.493	2.352	3.590
3	2.341	3.862	5.882	2.229	3.425	5.174	2.098	2.974	3.994

Table 4
Eigenvalues, λ^2, for the SS2-C-SS2-C rectangular plate. Eigenvalues, λ_b^2, stored when inverse of aspect ratio, $\phi^|$, listed

Fully symmetric modes

	m											
	$\phi = 1.0$			$\phi = 1.25$			$\phi = 1.5$			$\phi = 2.0$		
n	1	2	3	1	2	3	1	2	3	1	2	3
1	1.638	2.999	4.760	1.393	2.916	4.716	1.250	2.874	4.694	1.103	2.834	4.672
2	2.479	4.153	5.555	2.172	3.722	5.232	1.972	3.455	5.052	1.711	3.169	4.872
3	3.809	5.188	6.780	3.099	4.727	6.099	2.643	4.301	5.685	2.121	3.725	5.245

	$\phi^\| = 1.25$			$\phi^\| = 1.5$			$\phi^\| = 2.0$		
n	1	2	3	1	2	3	1	2	3
1	1.581	2.508	3.865	1.558	2.205	3.283	1.545	1.875	2.588
2	2.301	3.741	4.825	2.191	3.422	4.373	2.066	2.958	3.814
3	3.773	4.573	6.104	3.754	4.241	5.531	3.737	3.963	4.669

*Fully antisymmetric modes***

	m											
	$\phi = 1.0$			$\phi = 1.25$			$\phi = 1.5$			$\phi = 2.0$		
n	1	2	3	1	2	3	1	2	3	1	2	3
1	2.811	4.265	5.931	2.532	4.063	5.798	2.342	3.954	5.728	2.132	3.847	5.660
2	3.450	5.464	6.855	3.257	4.910	6.409	3.121	4.570	6.157	2.732	4.207	5.903
3	4.327	6.399	8.095	3.761	5.945	7.288	3.411	5.397	6.801	3.199	4.733	6.284

	$\phi^\| = 1.25$			$\phi^\| = 1.5$			$\phi^\| = 2.0$		
n	1	2	3	1	2	3	1	2	3
1	2449	3.657	4.912	2.150	3.280	4.265	1.747	2.811	3.513
2	3.076	4.879	6.000	2.923	4.286	5.464	2.814	3.450	4.682
3	4.017	5.331	7.284	3.812	4.828	6.399	3.570	4.327	5.116

Modes symmetric about the ξ-axis and anti-symmetric about the η-axis

	m											
	$\phi = 1.0$			$\phi = 1.25$			$\phi = 1.5$			$\phi = 2.0$		
n	1	2	3	1	2	3	1	2	3	1	2	3
1	2.205	3.865	5.667	2.070	3.808	5.638	1.999	3.778	5.614	1.933	3.751	5.597
2	3.422	4.825	6.338	3.051	4.444	6.062	2.769	4.225	5.912	2.413	4.002	5.762
3	4.241	6.104	7.450	3.735	5.424	6.822	3.459	4.967	6.457	3.052	4.453	6.077

	$\phi^\| = 1.25$			$\phi^\| = 1.5$			$\phi^\| = 2.0$		
n	1	2	3	1	2	3	1	2	3
1	1.935	3.169	4.580	1.783	2.722	3.865	1.638	2.205	2.999
2	3.054	4.285	5.404	2.796	3.930	4.825	2.479	3.422	4.153
3	4.000	5.397	6.649	3.900	4.837	6.104	3.809	4.241	5.188

Table 4 (*continued*)

Modes symmetric about the η-axis and anti-symmetric about the ξ-axis

	m											
	$\phi = 1.0$			$\phi = 1.25$			$\phi = 1.5$			$\phi = 2.0$		
n	1	2	3	1	2	3	1	2	3	1	2	3
1	2.811	4.625	5.931	2.532	4.063	5.798	2.342	3.954	5.729	2.132	3.847	5.660
2	3.450	5.464	6.855	3.257	4.910	6.409	3.122	4.570	6.157	2.732	4.207	5.903
3	4.327	6.399	8.095	3.761	5.945	7.288	3.411	5.397	6.801	3.200	4.733	6.284

	$\phi^{|} = 1.25$			$\phi^{|} = 1.5$			$\phi^{|} = 2.0$		
n	1	2	3	1	2	3	1	2	3
1	2.450	3.657	4.912	2.150	3.280	4.265	1.747	2.811	3.513
2	3.076	4.879	6.000	2.923	4.286	5.464	2.814	3.450	4.682
3	4.017	5.331	7.284	3.812	4.828	6.399	3.570	4.327	5.116

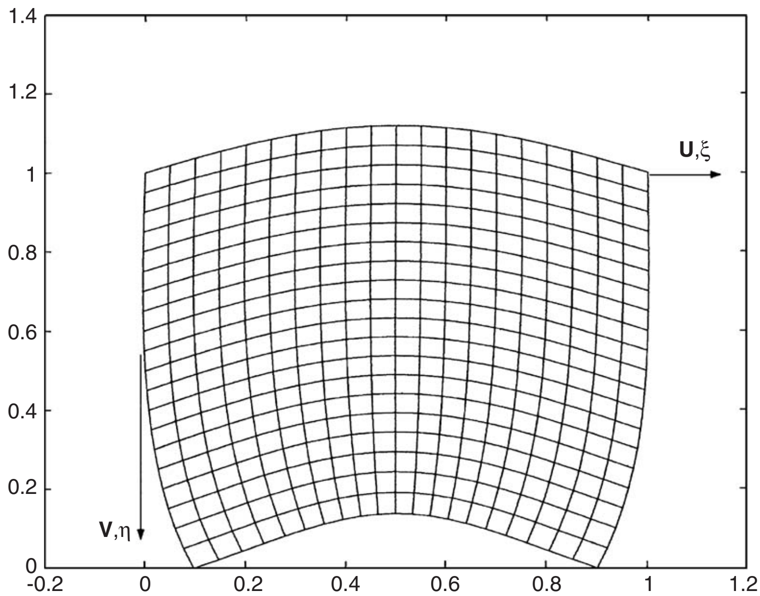

Fig. 3. Vibratory displacement pattern for first fully symmetric mode of square plate with designation SS1-F-SS1-F.

3.2. Mode shape studies

Mode shapes associated with any eigenvalue are easily generated. The practice has been to plot the shape of the quarter plate only. Numerous such mode shapes have been generated, however, only a limited number are presented here for the purposes of discussion.

In Fig. 3 the computed quarter-plate first mode displacement pattern is presented for a square plate with the designation SS1-F-SS1-F. It is evident that displacement parallel to the axes of the quarter plate equals zero. This required condition will be observed in all plate displacement patterns for fully symmetric mode shapes presented here. It is also evident in Fig. 3 that there is zero displacement parallel to the edge, $\xi = 1$ (Fig. 2). This is a requirement of simply supported edges with the designation SS1.

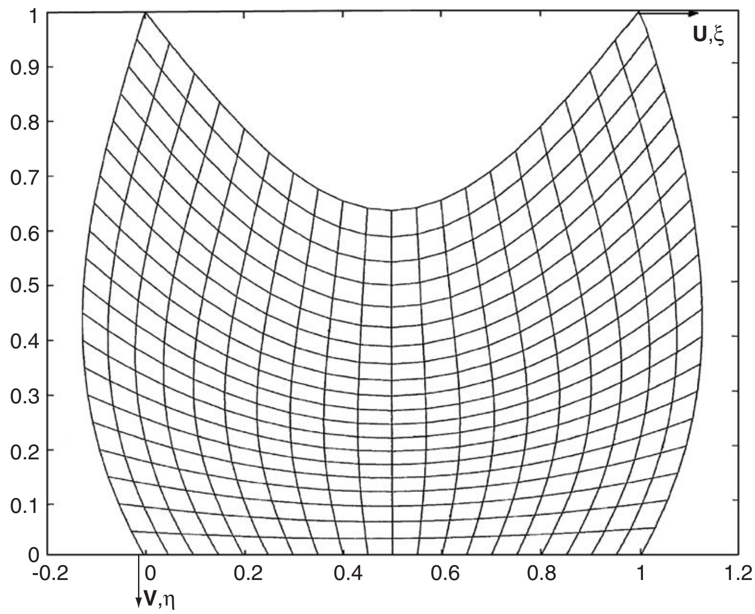

Fig. 4. Vibratory displacement pattern for first fully symmetric mode of square plate with designation SS1-C-SS1-C.

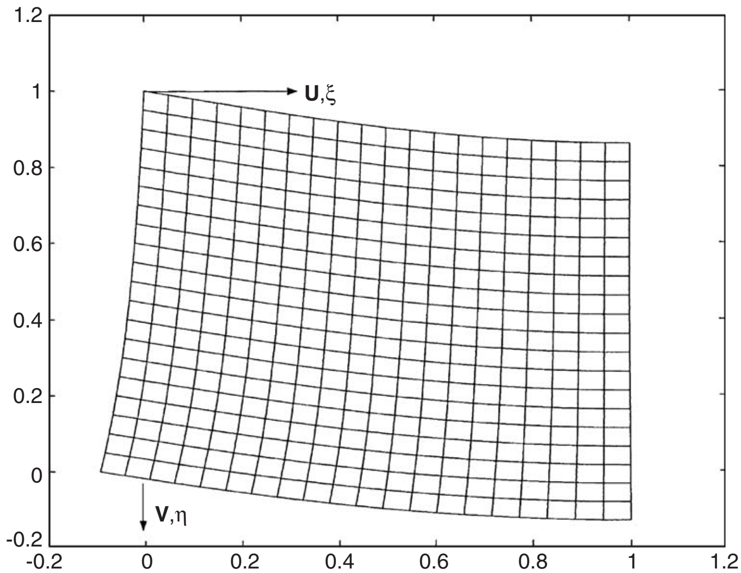

Fig. 5. Vibratory displacement pattern for first fully symmetric mode of square plate with designation SS2-F-SS2-F.

Turning to Fig. 4 related to the SS1-C-SS1-C plate we find that edge conditions discussed above in connection with Fig. 3 are also satisfied. Here, however, the clamped condition along the edge, $\eta = 1$, is highly evident.

Boundary conditions related to the mode shape of Fig. 5 (the SS2-F-SS2-F plate) differ from those of the plate of Fig. 3 only in that now class SS2 conditions are imposed along the edge, $\xi = 1$. This is evidenced by the fact that displacement normal to the edge, $\xi = 1$, is equal to zero.

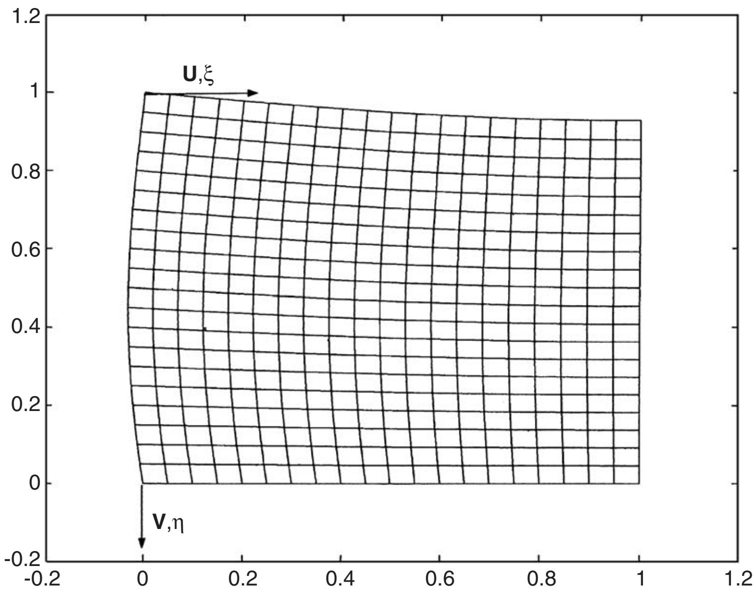

Fig. 6. Vibratory displacement pattern for first fully symmetric mode of square plate with designation SS2-C-SS2-C.

Finally, boundary conditions related to the mode shape of Fig. 6 (the SS2-C-SS2-C plate) differ from those of the plate of Fig. 4 only in connection with the class SS2 conditions imposed along the edge, $\xi = 1$. It is seen that displacement normal to this edge is again zero.

4. Discussion and conclusions

Exact solutions have been obtained in an orderly fashion for the free in-plane vibration eigenvalues and mode shapes of two families of rectangular plates, each with a pair of opposite edges simply supported. The two distinct classes of simple support are clearly defined. It is pointed out that they have a counterpart in the well known simple support conditions utilized in rectangular plate free lateral vibration analysis. The major difference is that in in-plane vibration there exists two distinct edge condition formulations which are considered to act as simple support.

It is well known that in the study of rectangular plate free lateral vibration exact solutions can be obtained for a vast array of problems provided one pair of opposite edges are given what is referred to as simple support. It is shown here that a vast array of exact solutions can also be obtained for rectangular plate in-plane free vibration provided one pair of opposite edges are given what is referred to here as simple edge support. However, here there are two distinct simple support edge condition formulations, each of which leads to a vast array of exact solutions.

The present study has been limited to plates, where non-simply supported edges are each free or are given clamped edge support. It will be obvious to the reader that exact solutions can be obtained for many other combinations of support enforced along these latter edges. Furthermore, plates with combinations of the two classes of simple support discussed here can be analyzed. This represents future work for investigators.

To the author's knowledge the present study represents the first thorough and orderly attempt to classify these simple support boundary conditions. It has been shown that exact solutions are obtained for the in-plane vibration of rectangular plates with either of these classes of simple support acting along a pair of opposite edges. The work presented here is expected to provide further insight into the overall subject of free in-plane vibration of rectangular plates.

Appendix A. Plates with type SS1 boundary conditions

A.1. Fully antisymmetric modes

Again we turn to Ref. [5] and focus on the series utilized in this earlier paper to analyze fully antisymmetric modes. Taking a single term from this series we have

$$V(\xi,\eta) = V_m(\eta)\cos\frac{(2m-1)\pi\xi}{2} \tag{A.1}$$

and

$$U_m(\xi,\eta) = U_m(\eta)\sin\frac{(2m-1)\pi\xi}{2}, \tag{A.2}$$

where m takes on the values $1, 2, 3$, etc.

It is easily verified that the above trigonometric functions satisfy exactly all the conditions at their extremities as required by the problem presently under investigation.

Substituting the above expressions in the governing differential equations we again arrive at Eqs. (5) and (6) of the main text. Now the quantity $\text{EMP} = (2m-1)\pi/2$. The coefficients appearing in these equations are unchanged with the exception of b_{m2} which must now be replaced by its negative.

A differential equation governing the quantity, $V_m(\eta)$, identical in form to Eq. (7), is now obtained with the same expressions for the quantities b and c. The same three possible forms of solution for $V_m(\eta)$ as given by Eqs. (8)–(10) are therefore applicable. Of course, this time terms symmetric about the ξ-axis must be deleted. Quantities $U_m(\eta)$ are obtained in the manner described earlier. This leads to the following results.

Case 1: Solution 1 applicable.

$$V_m(\eta) = A_m \sinh \beta_m\eta + C_m \sin \gamma_m\eta \tag{A.3}$$

and

$$U_m(\eta) = A_m\alpha_{1m}\cosh\beta_m\eta + C_m\alpha_{3m}\cos\gamma_m\eta, \tag{A.4}$$

where $\alpha_{1m} = \beta_m[a_{m1}a_{m2}\beta_m^2 + a_{m1}c_{m2} - b_{m1}b_{m2}]/c_{m1}b_{m2}$ and $\alpha_{3m} = -\gamma_m[a_{m1}a_{m2}\gamma_m^2 + a_{m1}c_{m2} - b_{m1}b_{m2}]/c_{m1}b_{m2}$.

(1) Plate with free edges.

$$V_m(\eta) = A_m[\sinh \beta_m\eta + \theta_{1m}\sin \gamma_m\eta] \tag{A.5}$$

and

$$U_m(\eta) = A_m[\alpha_{1m} \cosh\beta_m\eta + \theta_{1m}\alpha_{3m}\cos\gamma_m\eta], \tag{A.6}$$

with

$$\theta_{1m} = \frac{[\alpha_{1m}\beta_m - \phi\text{EMP}]\sinh\beta_m}{[\alpha_{3m}\gamma_m + \phi\text{EMP}]\sin\gamma_m}$$

and

$$\theta_{11m} = \left[\frac{\beta_m}{\phi} + v\text{EMP}\alpha_{1m}\right]\cosh\beta_m + \theta_{1m}\left[\frac{\gamma_m}{\phi} + v\text{EMP}\alpha_{3m}\right]\cos\gamma_m.$$

(2) Plate with clamped edges.

Displacements are given by Eqs. (A.5) and (A.6):

$$\theta_{1m} = -\frac{\alpha_{1m}\cosh\beta_m}{\alpha_{3m}\cos\gamma_m}$$

and $\theta_{11m} = \sinh\beta_m + \theta_{1m}\sin\gamma_m$.

Case 2: Solution 2 applicable.

$$V_m(\eta) = A_m\sin\beta_m\eta + C_m\sin\gamma_m\eta \tag{A.7}$$

and

$$U_m(\eta) = A_m\alpha_{1m}\cos\beta_m\eta + C_m\alpha_{3m}\cos\gamma_m\eta, \tag{A.8}$$

where $\alpha_{1m} = -\beta_m[a_{m1}a_{m2}\beta_m^2 - a_{m1}c_{m2} + b_{m1}b_{m2}]/c_{m1}b_{m2}$ and

$$\alpha_{3m} = -\gamma_m[a_{m1}a_{m2}\gamma_m^2 - a_{m1}c_{m2} + b_{m1}b_{m2}]/c_{m1}b_{m2}.$$

(1) Plate with free edges.

$$V_m(\eta) = A_m[\sin\beta_m\eta + \theta_{1m}\sin\gamma_m\eta] \tag{A.9}$$

and

$$U_m(\eta) = A_m[\alpha_{1m}\cos\beta_m\eta + \theta_{1m}\alpha_{3m}\cos\gamma_m\eta], \tag{A.10}$$

with

$$\theta_{1m} = -\frac{[\alpha_{1m}\beta_m + \phi\mathrm{EMP}]\sin\beta_m}{[\alpha_{3m}\gamma_m + \phi\mathrm{EMP}]\sin\gamma_m}$$

and

$$\theta_{11m} = \left[\frac{\beta_m}{\phi} + v\mathrm{EMP}\alpha_{1m}\right]\cos\beta_m + \theta_{1m}\left[\frac{\gamma_m}{\phi} + v\mathrm{EMP}\alpha_{3m}\right]\cos\gamma_m.$$

(2) Plate with clamped edges.

Displacements are given by Eqs. (A.9) and (A.10):

$$\theta_{1m} = -\frac{\alpha_{1m}\cos\beta_m}{\alpha_{3m}\cos\gamma_m}$$

and

$$\theta_{11m} = \sin\beta_m + \theta_{1m}\sin\gamma_m.$$

Case 3: Solution 3 applicable.

$$V_m(\eta) = A_m\sinh\beta_m\eta + C_m\sinh\gamma_m\eta \tag{A.11}$$

and

$$U_m(\eta) = A_m\alpha_{1m}\cosh\beta_m\eta + C_m\alpha_{3m}\cosh\gamma_m\eta, \tag{A.12}$$

where $\alpha_{1m} = \beta_m[a_{m1}a_{m2}\beta_m^2 + a_{m1}c_{m2} - b_{m1}b_{m2}]/c_{m1}b_{m2}$ and $\alpha_{3m} = \gamma_m[a_{m1}a_{m2}\gamma_m^2 + a_{m1}c_{m2} - b_{m1}b_{m2}]/c_{m1}b_{m2}$.

(1) Plate with free edges.

$$V_m(\eta) = A_m[\sinh\beta_m\eta + \theta_{1m}\sinh\gamma_m\eta] \tag{A.13}$$

and

$$U_m(\eta) = A_m[\alpha_{1m}\cosh\beta_m\eta + \theta_{1m}\alpha_{3m}\cosh\gamma_m\eta], \tag{A.14}$$

with

$$\theta_{1m} = -\frac{[\alpha_{1m}\beta_m - \phi\mathrm{EMP}]\sinh\beta_m}{[\alpha_{3m}\gamma_m + \phi\mathrm{EMP}]\sinh\gamma_m}$$

and

$$\theta_{11m} = \left[\frac{\beta_m}{\phi} + v\mathrm{EMP}\alpha_{1m}\right]\cosh\beta_m + \theta_{1m}\left[\frac{\gamma_m}{\phi} + v\mathrm{EMP}\alpha_{3m}\right]\cosh\gamma_m.$$

(2) Plate with clamped edges.

Displacements are given by Eqs. (A.13) and (A.14):

$$\theta_{1m} = -\alpha_{1m}\cosh\beta_m/(\alpha_{3m}\cosh\gamma_m)$$

and

$$\theta_{11m} = \sinh \beta_m + \theta_{1m} \sinh \gamma_m.$$

A.2. Modes symmetric about ξ-axis and antisymmetric about η-axis

It will be obvious that analysis of modes of this family differs from that of the antisymmetric–antisymmetric mode analysis just described, only in that now expressions for the displacement $V_m(\eta)$ must be symmetric with respect to the ξ-axis. Solutions obtained are as follows.

Case 1: Solution 1 applicable.

$$V_m(\eta) = B_m \cosh \beta_m \eta + D_m \cos \gamma_m \eta \tag{A.15}$$

and

$$U_m(\eta) = B_m \alpha_{2m} \sinh \beta_m \eta + C_m \alpha_{4m} \sin \gamma_m \eta, \tag{A.16}$$

where $\alpha_{2m} = \beta_m[a_{m1} a_{m2} \beta_m^2 + a_{m1} c_{m2} - b_{m1} b_{m2}]/c_{m1} b_{m2}$ and $\alpha_{4m} = \gamma_m[a_{m1} a_{m2} \gamma_m^2 - a_{m1} c_{m2} + b_{m1} b_{m2}]/c_{m1} b_{m2}$.

(1) Plate with free edges.

$$V_m(\eta) = B_m[\cosh \beta_m \eta + \theta_{1m} \cos \gamma_m \eta] \tag{A.17}$$

and

$$U_m(\eta) = B_m[\alpha_{2m} \sinh \beta_m \eta + \theta_{1m} \alpha_{4m} \sin \gamma_m \eta], \tag{A.18}$$

with

$$\theta_{1m} = -\frac{[\alpha_{2m} \beta_m - \phi \mathrm{EMP}] \cosh \beta_m}{[\alpha_{4m} \gamma_m - \phi \mathrm{EMP}] \cos \gamma_m}$$

and

$$\theta_{11m} = \left[\frac{\beta_m}{\phi} + v\mathrm{EMP}\alpha_{2m}\right] \sinh \beta_m - \theta_{1m} \left[\frac{\gamma_m}{\phi} - v\mathrm{EMP}\alpha_{4m}\right] \sin \gamma_m.$$

(2) Plate with clamped edges.

Displacements are given by Eqs. (A.17) and (A.18):

$$\theta_{1m} = -\frac{\alpha_{2m} \sinh \beta_m}{\alpha_{4m} \sin \gamma_m}$$

and $\theta_{11m} = \cosh \beta_m + \theta_{1m} \cos \gamma_m$.

Case 2: Solution 2 applicable.

$$V_m(\eta) = B_m \cos \beta_m \eta + D_m \cos \gamma_m \eta \tag{A.19}$$

and

$$U_m(\eta) = B_m \alpha_{2m} \sin \beta_m \eta + C_m \alpha_{4m} \sin \gamma_m \eta, \tag{A.20}$$

where $\alpha_{2m} = \beta_m[a_{m1} a_{m2} \beta_m^2 - a_{m1} c_{m2} + b_{m1} b_{m2}]/c_{m1} b_{m2}$ and

$$\alpha_{4m} = \gamma_m[a_{m1} a_{m2} \gamma_m^2 - a_{m1} c_{m2} + b_{m1} b_{m2}]/c_{m1} b_{m2}.$$

(1) Plate with free edges.

$$V_m(\eta) = B_m[\cos \beta_m \eta + \theta_{1m} \cos \gamma_m \eta] \tag{A.21}$$

and

$$U_m(\eta) = B_m[\alpha_{2m} \sin \beta_m \eta + \theta_{1m} \alpha_{4m} \sin \gamma_m \eta], \tag{A.22}$$

with

$$\theta_{1m} = -\frac{[\alpha_{2m} \beta_m - \phi \mathrm{EMP}] \cos \beta_m}{[\alpha_{4m} \gamma_m - \phi \mathrm{EMP}] \cos \gamma_m}$$

and

$$\theta_{11m} = -\left[\frac{\beta_m}{\phi} - v\text{EMP}\alpha_{2m}\right]\sin\beta_m - \theta_{1m}\left[\frac{\gamma_m}{\phi} - v\text{EMP}\alpha_{4m}\right]\sin\gamma_m.$$

(2) Plate with clamped edges.

Displacements are given by Eqs. (A.21) and (A.22):

$$\theta_{1m} = -\frac{\alpha_{2m}\sin\beta_m}{\alpha_{4m}\sin\gamma_m}$$

and $\theta_{11m} = \cos\beta_m + \theta_{1m}\cos\gamma_m$.

Case 3: Solution 3 applicable.

$$V_m(\eta) = B_m\cosh\beta_m\eta + D_m\cosh\gamma_m\eta \tag{A.23}$$

and

$$U_m(\eta) = B_m\alpha_{2m}\sinh\beta_m\eta + D_m\alpha_{4m}\sinh\gamma_m\eta, \tag{A.24}$$

where $\alpha_{2m} = \beta_m[a_{m1}a_{m2}\beta_m^2 + a_{m1}c_{m2} - b_{m1}b_{m2}]/c_{m1}b_{m2}$ and $\alpha_{4m} = \gamma_m[a_{m1}a_{m2}\gamma_m^2 + a_{m1}c_{m2} - b_{m1}b_{m2}]/c_{m1}b_{m2}$.

(1) Plate with free edges.

$$V_m(\eta) = B_m[\cosh\beta_m\eta + \theta_{1m}\cosh\gamma_m\eta] \tag{A.25}$$

and

$$U_m(\eta) = B_m[\alpha_{2m}\sinh\beta_m\eta + \theta_{1m}\alpha_{4m}\sinh\gamma_m\eta], \tag{A.26}$$

with

$$\theta_{1m} = -\frac{[\alpha_{2m}\beta_m - \phi\text{EMP}]\cosh\beta_m}{[\alpha_{4m}\gamma_m - \phi\text{EMP}]\cosh\gamma_m}$$

and

$$\theta_{11m} = \left[\frac{\beta_m}{\phi} + v\text{EMP}\alpha_{2m}\right]\sinh\beta_m + \theta_{1m}\left[\frac{\gamma_m}{\phi} + v\text{EMP}\alpha_{4m}\right]\sinh\gamma_m.$$

(2) Plate with clamped edges.

Displacements are given by Eqs. (A.25) and (A.26):

$$\theta_{1m} = -\frac{\alpha_{2m}\sinh\beta_m}{\alpha_{4m}\sinh\gamma_m}$$

and $\theta_{11m} = \cosh\beta_m + \theta_{1m}\cosh\gamma_m$.

A.3. Modes symmetric about the η-axis and antisymmetric about the ξ-axis

Analysis of modes of this family differs from that of fully symmetric modes discussed earlier only in that expressions for the displacement $V_m(\eta)$ must now be antisymmetric with respect to the ξ-axis. Solutions are as follows. For $m \geqslant 1$:

Case 1: Solution 1 applicable.

$$V_m(\eta) = A_m\sinh\beta_m\eta + C_m\sin\gamma_m\eta \tag{A.27}$$

and

$$U_m(\eta) = A_m\alpha_{1m}\cosh\beta_m\eta + C_m\alpha_{3m}\cos\gamma_m\eta, \tag{A.28}$$

where $\alpha_{1m} = \beta_m(a_{m1}a_{m2}\beta_m^2 + a_{m1}c_{m2} - b_{m1}b_{m2})/c_{m1}b_{m2}$ and $\alpha_{3m} = -\gamma_m(a_{m1}a_{m2}\gamma_m^2 + a_{m1}c_{m2} - b_{m1}b_{m2})/c_{m1}b_{m2}$.

(1) Plate with free edges.

$$V_m(\eta) = A_m[\sinh\beta_m\eta + \theta_{1m}\sin\gamma_m\eta] \tag{A.29}$$

and

$$U_m(\eta) = A_m[\alpha_{1m} \cosh \beta_m \eta + \theta_{1m} \alpha_{3m} \cos \gamma_m \eta], \tag{A.30}$$

with

$$\theta_{1m} = \frac{[\alpha_{1m}\beta_m + \phi\mathrm{EMP}] \sinh \beta_m}{[\alpha_{3m}\gamma_m - \phi\mathrm{EMP}] \sin \gamma_m}$$

and

$$\theta_{11m} = \left[\frac{\beta_m}{\phi} - v\mathrm{EMP}\alpha_{1m}\right] \cosh \beta_m + \theta_{1m}\left[\frac{\gamma_m}{\phi} - v\mathrm{EMP}\alpha_{3m}\right] \cos \gamma_m.$$

(2) Plate with clamped edges.
Displacements are given by Eqs. (A.29) and (A.30):

$$\theta_{1m} = -\frac{\alpha_{1m} \cosh \beta_m}{\alpha_{3m} \cos \gamma_m}$$

and $\theta_{11m} = \sinh \beta_m + \theta_{1m} \sin \gamma_m$.
Case 2: Solution 2 applicable.

$$V_m(\eta) = A_m \sin \beta_m \eta + c_m \sin \gamma_m \eta \tag{A.31}$$

and

$$U_m(\eta) = A_m \alpha_{1m} \cos \beta_m \eta + C_m \alpha_{3m} \cos \gamma_m \eta, \tag{A.32}$$

where $\alpha_{1m} = -\beta_m(a_{m1}a_{m2}\beta_m^2 + a_{m1}c_{m2} - b_{m1}b_{m2})/c_{m1}b_{m2}$ and $\alpha_{3m} = -\gamma_m(a_{m1}a_{m2}\gamma_m^2 + a_{m1}c_{m2} - b_{m1}b_{m2})/c_{m1}b_{m2}$.
(1) Plate with free edges.

$$V_m(\eta) = A_m[\sin \beta_m \eta + \theta_{1m} \sin \gamma_m \eta] \tag{A.33}$$

and

$$U_m(\eta) = A_m[\alpha_{1m} \cos \beta_m \eta + \theta_{1m} \alpha_{3m} \cos \gamma_m \eta], \tag{A.34}$$

with

$$\theta_{1m} = -\frac{[\alpha_{1m}\beta_m - \phi\mathrm{EMP}] \sin \beta_m}{[\alpha_{3m}\gamma_m - \phi\mathrm{EMP}] \sin \gamma_m}$$

and

$$\theta_{11m} = \left[\frac{\beta_m}{\phi} + v\mathrm{EMP}\alpha_{1m}\right] \cos \beta_m + \theta_{1m}\left[\frac{\gamma_m}{\phi} - v\mathrm{EMP}\alpha_{3m}\right] \cos \gamma_m.$$

(2) Plate with clamped edges.
Displacements are given by Eqs. (A.33) and (A.34):

$$\theta_{1m} = -\frac{\alpha_{1m} \cos \beta_m}{\alpha_{3m} \cos \gamma_m}$$

and $\theta_{11m} = \sin \beta_m + \theta_{1m} \sin \gamma_m$.
Case 3: Solution 3 applicable.

$$V_m(\eta) = A_m \sinh \beta_m \eta + C_m \sinh \gamma_m \eta \tag{A.35}$$

and

$$U_m(\eta) = A_m \alpha_{1m} \cosh \beta_m \eta + C_m \alpha_{3m} \cosh \gamma_m \eta, \tag{A.36}$$

where $\alpha_{1m} = -\beta_m(a_{m1}a_{m2}\beta_m^2 + a_{m1}c_{m2} - b_{m1}b_{m2})/c_{m1}b_{m2}$ and $\alpha_{3m} = \gamma_m(a_{m1}a_{m2}\gamma_m^2 + a_{m1}c_{m2} - b_{m1}b_{m2})/c_{m1}b_{m2}$.
(1) Plate with free edges.

$$V_m(\eta) = A_m[\sinh \beta_m \eta + \theta_{1m} \sinh \gamma_m \eta] \tag{A.37}$$

and

$$U_m(\eta) = A_m[\alpha_{1m} \cosh \beta_m \eta + \theta_{1m} \alpha_{3m} \cosh \gamma_m \eta], \tag{A.38}$$

with

$$\theta_{1m} = -\frac{[\alpha_{1m}\beta_m + \phi \text{EMP}] \sinh \beta_m}{[\alpha_{3m}\gamma_m + \phi \text{EMP}] \sinh \gamma_m}$$

and

$$\theta_{11m} = \left[\frac{\beta_m}{\phi} - v\text{EMP}\alpha_{1m}\right] \cosh \beta_m + \theta_{1m} \left[\frac{\gamma_m}{\phi} - v\text{EMP}\alpha_{3m}\right] \cosh \gamma_m.$$

(2) Plate with clamped edges.

Displacements are given by Eqs. (A.37) and (A.38):

$$\theta_{1m} = -\frac{\alpha_{1m} \cosh \beta_m}{\alpha_{3m} \cosh \gamma_m}$$

and $\theta_{11m} = \sinh \beta_m + \theta_{1m} \sinh \gamma_m$.

Next, the one-dimensional mode ($m = 1$). Again, the solution for displacement $U(\eta)$ is as given by Eq. (25). In view of the antisymmetry of displacement with respect to the ξ-axis, as discussed earlier, we have

$$U(\eta) = B \cos \alpha\eta. \tag{A.39}$$

Enforcing the condition of shear stress along the edge, $\eta = 1$, we obtain

$$B \sin \alpha\eta|_{\eta=1} = 0 \tag{A.40}$$

and hence, $\alpha = \pi, 2\pi, \ldots, n\pi,\ n = 1, 2, \ldots$, etc., with associated eigenvalues given by

$$\lambda^2 = n\pi[a_{66}]^{1/2}/\phi. \tag{A.41}$$

Exact solutions for the one-dimensional mode shapes and associated eigenvalues of the plate with free edges is therefore provided by Eqs. (A.39) and (A.41).

For the plate with clamped edges a condition of zero displacement along the edge, $\eta = 1$, leads to the equation

$$B \cos \alpha\eta|_{\eta=1} = 0. \tag{A.42}$$

We therefore have $\alpha = \pi/2, 3\pi/2, \ldots, (2n-1)\pi/2,\ n = 1, 2, \ldots$, etc., and hence,

$$\lambda^2 = (2n-1)\pi/2[a_{66}]^{1/2}/\phi. \tag{A.43}$$

Eqs. (A.39) and (A.43) provide mode shape and eigenvalues for the plate with clamped edges.

Appendix B. Plates with type SS2 boundary conditions

B.1. Fully antisymmetric modes

Again we turn to Ref. [4] and focus on the series utilized in this earlier paper to analyze fully antisymmetric modes. Taking a single term from this series we have

$$U(\xi, \eta) = U_m(\eta) \sin m\pi\xi \tag{B.1}$$

and

$$V(\xi, \eta) = V_m(\eta) \cos m\pi\xi, \tag{B.2}$$

with $m = 0, 1, 2$, etc.

We begin by considering terms with $m \geqslant 1$. It is easily verified that the above trigonometric functions satisfy all of the conditions at their extremities as required by the problem under investigation.

Substituting the above equations we again arrive at Eqs. (5) and (6). Now the quantity EMP equals $m\pi$. Coefficients appearing in these equations are unchanged with the exception of b_{m1} and b_{m2} which must be replaced by their negatives.

A differential equation governing the quantity $V_m(\eta)$, identical to Eq. (7), is now obtained with the same expressions for quantities b and c. The same three possible forms of solution for $V_m(\eta)$ as given by Eqs. (8)–(10) are applicable. This time terms symmetric about the ξ-axis must be deleted and quantities $U_m(\eta)$ are obtained as described earlier. This leads to the following results.

Case 1: Solution 1 applicable.

$$V_m(\eta) = A_m \sinh \beta_m \eta + C_m \sin \gamma_m \eta \tag{B.3}$$

and

$$U_m(\eta) = A_m \alpha_{1m} \cosh \beta_m \eta + C_m \alpha_{3m} \cos \gamma_m \eta, \tag{B.4}$$

where $\alpha_{1m} = \beta_m(a_{m1}a_{m2}\beta_m^2 + a_{m1}c_{m2} - b_{m1}b_{m2})/c_{m1}b_{m2}$ and $\alpha_{3m} = -\gamma_m(a_{m1}a_{m2}\gamma_m^2 - a_{m1}c_{m2} + b_{m1}b_{m2})/c_{m1}b_{m2}$.

(1) Plate with free edges.

$$V_m(\eta) = A_m[\sinh \beta_m \eta + \theta_{1m} \sin \gamma_m \eta] \tag{B.5}$$

and

$$U_m(\eta) = A_m[\alpha_{1m} \cosh \beta_m \eta + \theta_{1m} \alpha_{3m} \cos \gamma_m \eta], \tag{B.6}$$

with

$$\theta_{1m} = -\frac{[\alpha_{1m}\beta_m - \phi \text{EMP}] \sinh \beta_m}{[\alpha_{3m}\gamma_m + \phi \text{EMP}] \sin \gamma_m}$$

and

$$\theta_{11m} = \left[\frac{\beta_m}{\phi} + v\text{EMP}\alpha_{1m}\right] \cosh \beta_m + \theta_{1m}\left[\frac{\gamma_m}{\phi} + v\text{EMP}\alpha_{3m}\right] \cos \gamma_m.$$

(2) Plate with clamped edges.

Displacements are given by Eqs. (B.5) and (B.6):

$$\theta_{1m} = -\frac{\alpha_{1m} \cosh \beta_m}{\alpha_{3m} \cos \gamma_m}$$

and $\theta_{11m} = \sinh \beta_m + \theta_{1m} \sin \gamma_m$.

Case 2: Solution 2 applicable.

$$V_m(\eta) = A_m \sin \beta_m \eta + C_m \sin \gamma_m \eta \tag{B.7}$$

and

$$U_m(\eta) = A_m \alpha_{1m} \cos \beta_m \eta + C_m \alpha_{3m} \cos \gamma_m \eta, \tag{B.8}$$

where $\alpha_{1m} = -\beta_m(a_{m1}a_{m2}\beta_m^2 - a_{m1}c_{m2} + b_{m1}b_{m2})/c_{m1}b_{m2}$ and $\alpha_{3m} = -\gamma_m(a_{m1}a_{m2}\gamma_m^2 - a_{m1}c_{m2} + b_{m1}b_{m2})/c_{m1}b_{m2}$.

(1) Plate with free edges.

$$V_m(\eta) = A_m[\sin \beta_m \eta + \theta_{1m} \sin \gamma_m \eta] \tag{B.9}$$

and

$$U_m(\eta) = A_m[\alpha_{1m} \cos \beta_m \eta + \theta_{1m} \alpha_{3m} \cos \gamma_m \eta], \tag{B.10}$$

with

$$\theta_{1m} = -\frac{[\alpha_{1m}\beta_m + \phi \text{EMP}] \sin \beta_m}{[\alpha_{3m}\gamma_m + \phi \text{EMP}] \sin \gamma_m}$$

and

$$\theta_{11m} = \left[\frac{\beta_m}{\phi} + v\text{EMP}\alpha_{1m}\right]\cos\beta_m + \theta_{1m}\left[\frac{\gamma_m}{\phi} + v\text{EMP}\alpha_{3m}\right]\cos\gamma_m.$$

(2) Plate with clamped edges.
Displacements are given by Eqs. (B.9) and (B.10):

$$\theta_{1m} = -\frac{\alpha_{1m}\cos\beta_m}{\alpha_{3m}\cos\gamma_m}$$

and $\theta_{11m} = \sin\beta_m + \theta_{1m}\sin\gamma_m$.
Case 3: Solution 3 applicable

$$V_m(\eta) = A_m\sinh\beta_m\eta + C_m\sinh\gamma_m\eta \tag{B.11}$$

and

$$U_m(\eta) = A_m\alpha_{1m}\cosh\beta_m\eta + C_m\alpha_{3m}\cosh\gamma_m\eta, \tag{B.12}$$

where $\alpha_{1m} = \beta_m(a_{m1}a_{m2}\beta_m^2 + a_{m1}c_{m2} - b_{m1}b_{m2})/c_{m1}b_{m2}$ and $\alpha_{3m} = \gamma_m(a_{m1}a_{m2}\gamma_m^2 + a_{m1}c_{m2} - b_{m1}b_{m2})/c_{m1}b_{m2}$.
(1) Plate with free edges.

$$V_m(\eta) = A_m[\sinh\beta_m\eta + \theta_{1m}\sinh\gamma_m\eta] \tag{B.13}$$

and

$$U_m(\eta) = A_m[\alpha_{1m}\cosh\beta_m\eta + \theta_{1m}\alpha_{3m}\cosh\gamma_m\eta], \tag{B.14}$$

with

$$\theta_{1m} = -\frac{[\alpha_{1m}\beta_m - \phi\text{EMP}]\sinh\beta_m}{[\alpha_{3m}\gamma_m - \phi\text{EMP}]\sinh\gamma_m}$$

and

$$\theta_{11m} = \left[\frac{\beta_m}{\phi} + v\text{EMP}\alpha_{1m}\right]\cosh\beta_m + \theta_{1m}\left[\frac{\gamma_m}{\phi} + v\text{EMP}\alpha_{3m}\right]\cosh\gamma_m.$$

(2) Plate with clamped edges.
Displacements are given by Eqs. (B.13) and (B.14):

$$\theta_{1m} = -\frac{\alpha_{1m}\cosh\beta_m}{\alpha_{3m}\cosh\gamma_m}$$

and $\theta_{11m} = \sinh\beta_m + \theta_{1m}\sinh\gamma_m$.

Next, the one-dimensional case ($m = 0$). Returning to Eqs. (B.1) and (B.2) it is seen that for this mode only the quantity $V_m(\eta)$ will be non-zero. The governing differential equation becomes

$$V_m(\eta) + \alpha^2 V_m(\eta) = 0, \tag{B.15}$$

where $\alpha^2 = \lambda^4\phi^2/a_{11}$.

Deleting the term of the solution (Eq. (25)) antisymmetric about the ξ-axis from the solution to Eq. (B.15) we are left with

$$V_m(\eta) = A\sin\alpha\eta. \tag{B.16}$$

Enforcing the condition of zero stress normal to the edge, $\eta = 1$, we obtain the requirement

$$A\cos\alpha\eta|_{\eta=1} = 0, \tag{B.17}$$

from which we obtain $\alpha = \pi/2, 3\pi/2, \ldots, (2n-1)\pi/2$, $n = 1, 2, 3$, etc., hence the eigenvalues are

$$\lambda^2 = (2n-1)\frac{\pi}{2}[a_{11}]^{1/2}/\phi. \tag{B.18}$$

Exact mode shapes and eigenvalues for this family of modes are therefore available from Eqs. (B.16) and (B.18).

For the case of clamped edges, enforcing a condition of zero displacement normal to the edge, $\eta = 1$, we obtain

$$A \sin \alpha\eta|_{\eta=1} = 0 \tag{B.19}$$

and therefore, $\alpha = \pi, 2\pi, 3\pi, \ldots n\pi$, $n = 1, 2, 3$, etc. Eigenvalues are expressed as

$$\lambda^2 = n\pi[a_{11}]^{1/2}/\phi. \tag{B.20}$$

B.2. Modes symmetric about the ξ-axis and antisymmetric about the η-axis

Analysis of this mode family differs from that of antisymmetric–antisymmetric modes just described only in that now displacement $V_m(\eta)$ must be symmetric with respect to the ξ-axis. Solutions obtained are as follows.

First, terms for which $m \geqslant 1$ are considered.

Case 1: Solution 1 applicable.

$$V_m(\eta) = B_m \cosh \beta_m\eta + D_m \cos \gamma_m\eta \tag{B.21}$$

and

$$U_m(\eta) = B_m\alpha_{2m} \sinh \beta_m\eta + D_m\alpha_{4m} \sin \gamma_m\eta, \tag{B.22}$$

where $\alpha_{2m} = \beta_m(a_{m1}a_{m2}\beta_m^2 + a_{m1}c_{m2} - b_{m1}b_{m2})/c_{m1}b_{m2}$ and $\alpha_{4m} = \gamma_m(a_{m1}a_{m2}\gamma_m^2 - a_{m1}c_{m2} + b_{m1}b_{m2})/c_{m1}b_{m2}$.

(1) Plate with free edges.

$$V_m(\eta) = B_m[\cosh \beta_m\eta + \theta_{1m} \cos \gamma_m\eta] \tag{B.23}$$

and

$$U_m(\eta) = B_m[\alpha_{2m} \sinh \beta_m\eta + \theta_{1m}\alpha_{4m} \sin \gamma_m\eta], \tag{B.24}$$

with

$$\theta_{1m} = -\frac{[\alpha_{2m}\beta_m + \phi\text{EMP}] \cosh \beta_m}{[\alpha_{4m}\gamma_m + \phi\text{EMP}] \cos \gamma_m}$$

and

$$\theta_{11m} = \left[\frac{\beta_m}{\phi} + v\text{EMP}\alpha_{2m}\right] \sinh \beta_m - \theta_{1m}\left[\frac{\gamma_m}{\phi} + v\text{EMP}\alpha_{4m}\right] \sin \gamma_m.$$

(2) Plate with clamped edges.

Displacements are given by Eqs. (B.23) and (B.24):

$$\theta_{1m} = -\frac{\alpha_{2m} \sinh \beta_m}{\alpha_{4m} \sin \gamma_m}$$

and $\theta_{11m} = \cosh \beta_m + \theta_{1m} \cos \gamma_m$.

Case 2: Solution 2 applicable.

$$V_m(\eta) = B_m \cos \beta_m\eta + D_m \cos \gamma_m\eta \tag{B.25}$$

and

$$U_m(\eta) = B_m\alpha_{2m} \sin \beta_m\eta + D_m\alpha_{4m} \sin \gamma_m\eta, \tag{B.26}$$

where $\alpha_{2m} = \beta_m(a_{m1}a_{m2}\beta_m^2 - a_{m1}c_{m2} + b_{m1}b_{m2})/c_{m1}b_{m2}$, $\alpha_{4m} = \gamma_m(a_{m1}a_{m2}\gamma_m^2 - a_{m1}c_{m2} + b_{m1}b_{m2})/c_{m1}b_{m2}$.

(1) Plate with free edges.

$$V_m(\eta) = B_m[\cos \beta_m\eta + \theta_{1m} \cos \gamma_m\eta] \tag{B.27}$$

and

$$U_m(\eta) = B_m[\alpha_{2m} \sin \beta_m\eta + \theta_{1m}\alpha_{4m} \sin \gamma_m\eta], \tag{B.28}$$

with

$$\theta_{1m} = -\frac{[\alpha_{2m}\beta_m + \phi \mathrm{EMP}]\cos\beta_m}{[\alpha_{4m}\gamma_m + \phi \mathrm{EMP}]\cos\gamma_m}$$

and

$$\theta_{11m} = -\left[\frac{\beta_m}{\phi} + v\mathrm{EMP}\alpha_{2m}\right]\sin\beta_m - \theta_{1m}\left[\frac{\gamma_m}{\phi} + v\mathrm{EMP}\alpha_{4m}\right]\sin\gamma_m.$$

(2) Plate with clamped edges.
Displacements are given by Eqs. (B.27) and (B.28):

$$\theta_{1m} = -\frac{\alpha_{2m}\sin\beta_m}{\alpha_{4m}\sin\gamma_m}$$

and $\theta_{11m} = \cos\beta_m + \theta_{1m}\cos\gamma_m$.
Case 3: Solution 3 applicable.

$$V_m(\eta) = B_m\cosh\beta_m\eta + D_m\cosh\gamma_m\eta \tag{B.29}$$

and

$$U_m(\eta) = B_m\alpha_{2m}\sinh\beta_m\eta + D_m\alpha_{4m}\sinh\gamma_m\eta, \tag{B.30}$$

where $\alpha_{2m} = \beta_m(a_{m1}a_{m2}\beta_m^2 + a_{m1}c_{m2} - b_{m1}b_{m2})/c_{m1}b_{m2}$ and $\alpha_{4m} = \gamma_m(a_{m1}a_{m2}\gamma_m^2 + a_{m1}c_{m2} - b_{m1}b_{m2})/c_{m1}b_{m2}$.
(1) Plate with free edges.

$$V_m(\eta) = B_m[\cosh\beta_m\eta + \theta_{1m}\cosh\gamma_m\eta] \tag{B.31}$$

and

$$U_m(\eta) = B_m[\alpha_{2m}\sinh\beta_m\eta + \theta_{1m}\alpha_{4m}\sinh\gamma_m\eta], \tag{B.32}$$

with

$$\theta_{1m} = -\frac{[\alpha_{2m}\beta_m + \phi \mathrm{EMP}]\cosh\beta_m}{[\alpha_{4m}\gamma_m + \phi \mathrm{EMP}]\cosh\gamma_m}$$

and

$$\theta_{11m} = \left[\frac{\beta_m}{\phi} - v\mathrm{EMP}\alpha_{2m}\right]\sinh\beta_m + \theta_{1m}\left[\frac{\gamma_m}{\phi} - v\mathrm{EMP}\alpha_{4m}\right]\sinh\gamma_m.$$

(2) Plate with clamped edges.
Displacements are given by Eqs. (B.31) and (B.32):

$$\theta_{1m} = -\frac{\alpha_{2m}\sinh\beta_m}{\alpha_{4m}\sinh\gamma_m}$$

and $\theta_{11m} = \cosh\beta_m + \theta_{1m}\cosh\gamma_m$.

Next, the one-dimensional problem ($m = 0$). Returning to Eq. (B.15) and deleting the term antisymmetric about the ξ-axis we obtain

$$V(\eta) = B\cos\alpha\eta. \tag{B.33}$$

Enforcing the condition of zero normal stress along the edge, $\eta = 1$, we obtain

$$B\sin\alpha\eta|_{\eta=1} = 0, \tag{B.34}$$

from which we obtain, $\alpha = \pi, 2\pi, 3\pi \ldots n\pi$ etc., $n = 1, 2, 3 \ldots$. Eigenvalues are therefore given by the expression

$$\lambda^2 = n\pi[a_{11}]^{1/2}/\phi. \tag{B.35}$$

Exact mode shapes and eigenvalues for one-dimensional modes of the plate with free edges are therefore given by Eqs. (B.33) and (B.35).

Returning to Eq. (B.33) and enforcing the condition of zero displacement normal to the edge, $\eta = 1$, we obtain the equation

$$B \cos \alpha\eta|_{\eta=1} = 0, \tag{B.36}$$

from which we obtain $\alpha = \pi/2, 3\pi/2, \ldots (2n-1)\pi/2$, etc., $n = 1, 2, 3$. Eigenvalues for this mode family therefore become

$$\lambda^2 = (2n-1)\frac{\pi}{2}[a_{11}]^{1/2}/\phi. \tag{B.37}$$

Exact mode shapes and eigenvalues for one-dimensional modes of the plate with clamped edges are therefore given by Eqs. (B.33) and (B.37).

B.3. Modes symmetric about the η-axis and antisymmetric about the ξ-axis

Analysis of modes of this family differs from that of fully symmetric modes discussed earlier only in that expressions for the displacement $V(\eta)$ must be antisymmetric with respect to the ξ-axis. Solutions are as follows.

Case 1: Solution 1 applicable.

$$V_m(\eta) = A_m \sinh \beta_m\eta + C_m \sin \gamma_m\eta \tag{B.38}$$

and

$$U_m(\eta) = A_m\alpha_{1m} \cosh \beta_m\eta + C_m\alpha_{3m} \cos \gamma_m\eta, \tag{B.39}$$

where $\alpha_{1m} = \beta_m(a_{m1}a_{m2}\beta_m^2 + a_{m1}c_{m2} - b_{m1}b_{m2})/c_{m1}b_{m2}$ and $\alpha_{3m} = -\gamma_m(a_{m1}a_{m2}\gamma_m^2 - a_{m1}c_{m2} + b_{m1}b_{m2})/c_{m1}b_{m2}$.

(1) Plate with free edges.

$$V_m(\eta) = A_m[\sinh \beta_m\eta + \theta_{1m} \sin \gamma_m\eta] \tag{B.40}$$

and

$$U_m(\eta) = A_m[\alpha_{1m} \cosh \beta_m\eta + \theta_{1m}\alpha_{3m} \cos \gamma_m\eta], \tag{B.41}$$

with

$$\theta_{1m} = \frac{[\alpha_{1m}\beta_m + \phi\mathrm{EMP}] \sinh \beta_m}{[\alpha_{3m}\gamma_m - \phi\mathrm{EMP}] \sin \gamma_m}$$

and

$$\theta_{11m} = \left[\frac{\beta_m}{\phi} - v\mathrm{EMP}\alpha_{1m}\right] \cosh \beta_m + \theta_{1m}\left[\frac{\gamma_m}{\phi} - v\mathrm{EMP}\alpha_{3m}\right] \cos \gamma_m.$$

(2) Plate with clamped edges.

Displacements are given by Eqs. (B.40) and (B.41):

$$\theta_{1m} = -\frac{\alpha_{1m} \cosh \beta_m}{\alpha_{3m} \cos \gamma_m}$$

and $\theta_{11m} = \sinh \beta_m + \theta_{1m} \sin \gamma_m$.

Case 2: Solution 2 applicable.

$$V_m(\eta) = A_m \sin \beta_m\eta + C_m \sin \gamma_m\eta \tag{B.42}$$

and

$$U_m(\eta) = A_m\alpha_{1m} \cos \beta_m\eta + C_m\alpha_{3m} \cos \gamma_m\eta, \tag{B.43}$$

where $\alpha_{1m} = -\beta_m(a_{m1}a_{m2}\beta_m^2 - a_{m1}c_{m2} + b_{m1}b_{m2})/c_{m1}b_{m2}$ and $\alpha_{3m} = -\gamma_m(a_{m1}a_{m2}\gamma_m^2 - a_{m1}c_{m2} + b_{m1}b_{m2})/c_{m1}b_{m2}$.

(1) Plate with free edges.

$$V_m(\eta) = A_m[\sin \beta_m\eta + \theta_{1m} \sin \gamma_m\eta] \tag{B.44}$$

and

$$U_m(\eta) = A_m[\alpha_{1m} \cos \beta_m \eta + \theta_{1m} \alpha_{3m} \cos \gamma_m \eta], \tag{B.45}$$

with

$$\theta_{1m} = -\frac{[\alpha_{1m}\beta_m - \phi \text{EMP}] \sin \beta_m}{[\alpha_{3m}\gamma_m - \phi \text{EMP}] \sin \gamma_m}$$

and

$$\theta_{11m} = \left[\frac{\beta_m}{\phi} - v\text{EMP}\alpha_{1m}\right] \cos \beta_m + \theta_{1m}\left[\frac{\gamma_m}{\phi} - v\text{EMP}\alpha_{3m}\right] \cos \gamma_m.$$

(2) Plate with clamped edges.

Displacements are given by Eqs. (B.44) and (B.45):

$$\theta_{1m} = -\frac{\alpha_{1m} \cos \beta_m}{\alpha_{3m} \cos \gamma_m}$$

and $\theta_{11m} = \sin \beta_m + \theta_{1m} \sin \gamma_m$.

Case 3: Solution 3 applicable.

$$V_m(\eta) = A_m \sinh \beta_m \eta + C_m \sinh \gamma_m \eta \tag{B.46}$$

and

$$U_m(\eta) = A_m \alpha_{1m} \cosh \beta_m \eta + C_m \alpha_{3m} \cosh \gamma_m \eta, \tag{B.47}$$

where $\alpha_{1m} = \beta_m(a_{m1}a_{m2}\beta_m^2 + a_{m1}c_{m2} - b_{m1}b_{m2})/c_{m1}b_{m2}$ and $\alpha_{3m} = \gamma_m(a_{m1}a_{m2}\gamma_m^2 + a_{m1}c_{m2} - b_{m1}b_{m2})/c_{m1}b_{m2}$.

(1) Plate with free edges.

$$V_m(\eta) = A_m[\sinh \beta_m \eta + \theta_{1m} \sinh \gamma_m \eta] \tag{B.48}$$

and

$$U_m(\eta) = A_m[\alpha_{1m} \cosh \beta_m \eta + \theta_{1m} \alpha_{3m} \cosh \gamma_m \eta], \tag{B.49}$$

with

$$\theta_{1m} = -\frac{[\alpha_{1m}\beta_m + \phi \text{EMP}] \sinh \beta_m}{[\alpha_{3m}\gamma_m + \phi \text{EMP}] \sinh \gamma_m}$$

and

$$\theta_{11m} = \left[\frac{\beta_m}{\phi} - v\text{EMP}\alpha_{1m}\right] \cosh \beta_m + \theta_{1m}\left[\frac{\gamma_m}{\phi} - v\text{EMP}\alpha_{3m}\right] \cosh \gamma_m.$$

(2) Plate with clamped edges.

Displacements are given by Eqs. (B.48) and (B.49):

$$\theta_{1m} = -\frac{\alpha_{1m} \cosh \beta_m}{\alpha_{3m} \cosh \gamma_m}$$

and $\theta_{11m} = \sinh \beta_m + \theta_{1m} \sinh \gamma_m$.

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