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On the dynamical axisymmetric stress field in a finite pre-stretched bilayered slab resting on a rigid foundation

S.D. Akbarov[a,b,*]

[a]*Faculty of Chemistry and Metallurgy, Department of Mathematical Engineering, Yildiz Technical University, Davutpasa Campus No. 127, 34210 Topkapi, Istanbul, Turkey*
[b]*Institute of Mathematics and Mechanics of National Academy of Sciences of Azerbaijan, 370141 Baku, Azerbaijan*

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Abstract

The dynamic axisymmetric stress field in an initially finite pre-stretched bilayered slab resting on a rigid foundation is studied within the framework of the piecewise homogeneous body model. The three-dimensional linearized theory of elastic waves in initially stressed bodies is used. It is assumed that a time-harmonic point-located normal force acts on the free face plane of the slab. The considered problem is solved by employing the Hankel integral transformation. The materials of the layers are assumed to be incompressible, and the elastic relations are given through the Treloar potential. The formulation and solution to the problem coincide with the corresponding ones of classical linear theory of elasticity for an incompressible body in the case where there is no initial stretching in the layers. Numerical results are presented and these results involve stresses acting on the interface planes. In particular, it is established that stresses on the interface planes decrease as the pre-stretching is increased.

1. Introduction

Many modern elastodynamic problems, especially elastic-wave propagation problems, are not solvable within the framework of the classical linear theory of elasticity. As a result, a general nonlinear theory of elastic waves with various simplifying modifications has been introduced since the mid-1900s and is still being developed. Comprehensive treatments of the subject can be found in Refs. [1–4]. A review of these treatments was presented in Ref. [5].

A class of interesting and urgent elastodynamic problems, which cannot be solved within the framework of classical linear theory of elastic waves, is that of elastodynamic problems for initially stressed bodies. Such problems have a wide range of applications in practice. For example, initial stresses occur in structural elements after manufacturing and assembly. Initial stresses are also presented in the Earth's crust due to the action of geostatic and geodynamic forces, in rocks and so on. Accordingly, a large number of theoretical and

*Corresponding author at: Faculty of Chemistry and Metallurgy, Department of Mathematical Engineering, Yildiz Technical University, Davutpasa Campus No. 127, 34010 Esenler, Istanbul, Turkey. Tel.: +90 212 449 1684; fax: +90 212 449 1574.
E-mail address: akbarov@yildiz.edu.tr.

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experimental investigations (for example Refs. [6–13] and many others) have been made in this field. A systematic analysis of results obtained before 1986 was made in monographs [14,15]. Subsequent research is reviewed in a paper [16]. Current investigations in this field are being pursued intensively; see for example Refs. [17–24].

Almost all of these investigations were made within the framework of the three-dimensional linearized theory of elastic waves in initially stressed bodies (TLTEWISB). Also, a substantial portion of the studies is based on wave propagation in layered composite materials with homogeneous initial stresses.

The study of the influence of the initial stresses on the dynamic stress-state in a homogeneous and layered medium is of great significance, in both theoretical and practical sense. Until now, there were a few studies in this field; see for example Refs. [25–29].

In the paper [25], the Lamb problem for a compressible half-plane with initial stresses was considered. In the papers [26,27] an attempt was made to study the time-harmonic two-dimensional Lamb problem for the half-plane covered with the pre-stretched layer. In Ref. [28] the investigations [25,26] have been developed for a strip load acting on the covering layer. The development of the studies [26,27] for the three-dimensional Lamb problem has been made in the paper [29].

In the foregoing investigations it is assumed that the region occupied by the body is semi-infinite. Therefore, the results obtained in Refs. [25–29] cannot be applied, for example, in the cases where the aforementioned dynamical stress field is studied for the layered material, which rests on the rigid foundation. Nor these results can be applied for structural elements whose basic material is covered with the layered ones. If the stiffness of the basic material (modulus of elasticity) is significantly greater than those of the covering layers, then the basic material can be modelled as a rigid foundation. It is well known that as a result of the covering procedure the residual (initial) stresses arise in the covering layers and it is almost inevitable to alert these stresses. Therefore, under studying the dynamical stress field in such structural members it is necessary to take the foregoing initial stresses into account.

Because of the above discussions in the present paper, the investigations carried out in Refs. [25–29] are developed for systems, which comprise bilayered infinite slab and rigid foundation. It is assumed that a time-harmonic point-located normal force acts on the free face plane of the upper layer of the slab and the axisymmetric stress state in this slab is studied. Furthermore, it is assumed that the layers of the slab are finite pre-strained (-stretched) radially. We suppose that the materials of the layers are incompressible neo-Hookean materials and the stress-strain relation for those are given through the Treloar potential. The investigations are carried out within the framework of the piecewise-homogeneous body model by the use of the TLTEWISB.

2. Formulation of the problem

We consider the bilayered slab resting on the rigid foundation (Fig. 1). Assume that in the natural state, the thicknesses of the upper and lower layers of the slab are h_1 and h_2, respectively. In the natural state, we determine the positions of the points of the layers by the Lagrangian coordinates in the Cartesian system of

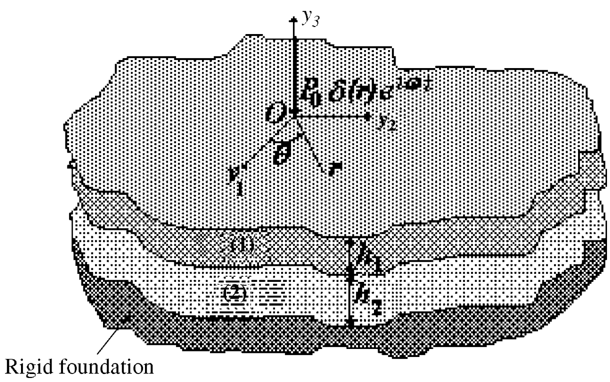

Fig. 1. The geometry of the bilayered slab resting on the rigid foundation.

coordinates $Oy_1y_2y_3$ as well as in the cylindrical system of coordinates $Or\theta y_3$. Assume that the layers of the slab have infinite length in the radial direction. We aim that the layers before the compounding with each other and with a rigid foundation be stretched separately along the radial direction and that in each of them, the homogeneous axisymmetric initial finite strain state appear. These initial strains are caused by the static forces acting in the radial direction at infinity. Note that the action of these forces continues during all further dynamical processes.

With the initial state of the layers of the slab we associate the Lagrangian cylindrical system of coordinates $O'r'\theta'y'_3$ and the Cartesian system of coordinates $O'y'_1y'_2y'_3$. Assume that the material of the layers is of incompressible neo-Hookean materials and the values related to the upper and lower layers of the slab are denoted by upper indices (1) and (2), respectively. Furthermore, we denote the values related to the initial state by upper index 0. Thus, according to the above-stated, the initial state in the layers can be determined as follows:

$$u_m^{(k),0} = \left(\lambda_m^{(k)} - 1\right)y_m, \quad \lambda_1^{(k)} = \lambda_2^{(k)} \neq \lambda_3^{(k)}, \quad \lambda_m^{(k)} = \text{const.},$$
$$\lambda_1^{(k)}\lambda_2^{(k)}\lambda_3^{(k)} = 1, \quad m = 1,2,3, \quad k = 1,2, \tag{1}$$

where $u_m^{(k),0}$ is a displacement and $\lambda_m^{(k)}$ is the elongation along the Oy_m-axis. We introduce the following notation:

$$\lambda_1^{(k)} = \lambda_2^{(k)} = \lambda^{(k)}, \quad \lambda_3^{(k)} = (\lambda^{(k)})^{-2}. \tag{2}$$

It follows from Eq. (1) that

$$y'_i = \lambda_i^{(k)}y_i, \quad r' = \lambda^{(k)}r, \quad h'_1 = (\lambda^{(1)})^{-2}h_1, \quad h'_2 = (\lambda^{(2)})^{-2}h_2. \tag{3}$$

Below the values related to the system of coordinates associated with initial state, i.e. with $O'y'_1y'_2y'_3$, will be denoted by upper prime.

Within the above-stated, let us investigate the stress state in the considered slab in the case where on the free face plane of the upper layer, the point-located normal time-harmonic force acts. We will make this investigation by the use of coordinates r' and y'_3 in the framework of the TLTEWISB.

In the construction of the field equations of the TLTEWISB, one considers two states of a deformable solid. The first is regarded as the initial or unperturbed state and the second is a perturbed state with respect to the unperturbed. By the "state of a deformable solid" both motion and equilibrium (as a particular case of motion) are meant. It is assumed that all values in a perturbed state can be represented as a sum of the values in the initial state and perturbations. The latter is also assumed to be small in comparison with the corresponding values in the initial state. It is also assumed that both initial (unperturbed) and perturbed states are described by the equations of nonlinear solid mechanics. Owing to the fact that perturbations are small, the relationships for the perturbed state in the vicinity of appropriate values for the unperturbed state are linearized and then, the relations for perturbed state are subtracted from them. The result is the equations of the TLTEWISB. The general problems of the TLTEWISB have been elaborated in many investigations such as Refs. [2,14,15,30,31] and others. In the present paper, we will follow the style and notation used in the monographs [14,15].

Thus, according to Refs. [14,15], we write the basic relations of the TLTEWISB for the incompressible body under axisymmetrical state. These relations are satisfied within each layer because we use the piecewise-homogeneous body model.

The equations of motion are

$$\frac{\partial}{\partial r'}Q'^{(k)}_{rr} + \frac{\partial}{\partial y'_3}Q'^{(k)}_{r'3} + \frac{1}{r'}\left(Q'^{(k)}_{r'r'} - Q'^{(k)}_{\theta'\theta'}\right) = \rho'^{(k)}\frac{\partial^2}{\partial t^2}u'^{(k)}_{r'},$$

$$\frac{\partial}{\partial r'}Q'^{(k)}_{3r'} + \frac{\partial}{\partial y'_3}Q'^{(k)}_{33} + \frac{1}{r'}Q'^{(k)}_{3r'} = \rho'^{(k)}\frac{\partial^2}{\partial t^2}u'^{(k)}_{3}. \tag{4}$$

The mechanical relations are

$$Q'^{(k)}_{r'r'} = \chi'^{(k)}_{1111}\frac{\partial u'^{(k)}_{r'}}{\partial r'} + \chi'^{(k)}_{1122}\frac{u'^{(k)}_{r'}}{r'} + \chi'^{(k)}_{1133}\frac{\partial u'^{(k)}_{3}}{\partial y'_3} + p'^{(k)},$$

$$Q'^{(k)}_{\theta'\theta'} = \chi'^{(k)}_{2211}\frac{\partial u'^{(k)}_{r'}}{\partial r'} + \chi'^{(k)}_{2222}\frac{u'^{(k)}_{r'}}{r'} + \chi'^{(k)}_{2233}\frac{\partial u'^{(k)}_{3}}{\partial y'_3} + p'^{(k)},$$

$$Q'^{(k)}_{33} = \chi'^{(k)}_{3311}\frac{\partial u'^{(k)}_{r'}}{\partial r'} + \chi'^{(k)}_{3322}\frac{u'^{(k)}_{r'}}{r'} + \chi'^{(k)}_{3333}\frac{\partial u'^{(k)}_{3}}{\partial y'_3} + p'^{(k)},$$

$$Q'^{(k)}_{r'3} = \chi'^{(k)}_{1313}\frac{\partial u'^{(k)}_{r'}}{\partial y'_3} + \chi'^{(k)}_{1331}\frac{\partial u'^{(k)}_{3}}{\partial r'}, \quad Q'^{(k)}_{3r'} = \chi'^{(k)}_{3113}\frac{\partial u'^{(k)}_{r'}}{\partial y'_3} + \chi'^{(k)}_{3131}\frac{\partial u'^{(k)}_{3}}{\partial r'}. \tag{5}$$

In Eqs. (4) and (5) through $Q'^{(k)}_{r'r'}, \ldots, Q'^{(k)}_{3r'}$ the perturbations of the components of Kirchhoff stress tensor are determined. The notation $u'^{(k)}_{r'}, u'^{(k)}_{3}$ shows the perturbations of the components of the displacement vector, $p'^{(k)} = p'^{(k)}(r', y'_3, t)$ is an unknown function. The constants $\chi'^{(k)}_{1111}, \ldots, \chi'^{(k)}_{3333}$ in Eqs. (4), (5) are determined through the mechanical constants of the layers' materials and through the initial stress state, $\rho'^{(k)}$ is a density of the kth layer material. Note that for the considered initial strain state the expression of the constants $\chi'^{(k)}_{1111}, \ldots, \chi'^{(k)}_{3333}$ is given through the expression of those in the system of coordinates $Or\theta y_3$ (denoted by $\chi^{(k)}_{1111}, \ldots, \chi^{(k)}_{3333}, \rho^{(k)}$) by the following formulae:

$$\chi'^{(k)}_{1111} = (\lambda^{(k)})^2\chi^{(k)}_{1111}, \quad \chi'^{(k)}_{1122} = (\lambda^{(k)})^2\chi^{(k)}_{1122}, \quad \chi'^{(k)}_{1133} = (\lambda^{(k)})^{-1}\chi^{(k)}_{1133}, \quad \chi'^{(k)}_{2222} = (\lambda^{(k)})^2\chi^{(k)}_{2222},$$

$$\chi'^{(k)}_{1221} = (\lambda^{(k)})^2\chi^{(k)}_{1221}, \quad \chi'^{(k)}_{1313} = (\lambda^{(k)})^{-1}\chi^{(k)}_{1313}, \quad \chi'^{(k)}_{1331} = (\lambda^{(k)})^2\chi^{(k)}_{1331}, \quad \chi'^{(k)}_{3131} = \chi'^{(k)}_{1313},$$

$$\chi'^{(k)}_{2211} = \chi'^{(k)}_{1122}, \quad \chi'^{(k)}_{2233} = \chi'^{(k)}_{1133}, \quad \chi'^{(k)}_{3311} = \chi'^{(k)}_{1133} = \chi'^{(k)}_{3322} = \chi'^{(k)}_{2233},$$

$$\chi'^{(k)}_{3113} = (\lambda^{(k)})^2\chi^{(k)}_{3113}, \quad \chi'^{(k)}_{3333} = (\lambda^{(k)})^{-4}\chi^{(k)}_{3333}, \quad \rho'^{(k)} = \rho^{(k)}. \tag{6}$$

In the present investigation we assume that the elasticity relations of the layers' materials are given by neo-Hookean-type (Treloar) potential. This potential is given as follows:

$$\Phi = C_{10}(I_1 - 3), \quad I_1 = 3 + 2A_1, \;\; A_1 = \varepsilon_{rr} + \varepsilon_{\theta\theta} + \varepsilon_{33}, \tag{7}$$

where C_{10} is an elastic constant; A_1 is the first algebraic invariant of the Green's strain tensor, ε_{rr}, $\varepsilon_{\theta\theta}$ and ε_{33} are the components of this tensor. For the considered axisymmetric case the components of the Green's strain tensor are determined through the components of the displacement vector by the following expressions:

$$\varepsilon_{rr} = \frac{\partial u_r}{\partial r} + \frac{1}{2}\left(\frac{\partial u_r}{\partial r}\right)^2 + \frac{1}{2}\left(\frac{\partial u_3}{\partial r}\right)^2, \quad \varepsilon_{\theta\theta} = \frac{u_r}{r} + \frac{1}{2}\left(\frac{u_r}{r}\right)^2,$$

$$\varepsilon_{r3} = \frac{1}{2}\left(\frac{\partial u_3}{\partial r} + \frac{\partial u_r}{\partial y_3} + \frac{\partial u_r}{\partial r}\frac{\partial u_r}{\partial y_3} + \frac{\partial u_3}{\partial r}\frac{\partial u_3}{\partial y_3}\right), \quad \varepsilon_{33} = \frac{\partial u_3}{\partial y_3} + \frac{1}{2}\left(\frac{\partial u_3}{\partial r}\right)^2 + \frac{1}{2}\left(\frac{\partial u_3}{\partial y_3}\right)^2. \tag{8}$$

In this case the components S_{ij} of the Lagrange stress tensor are determined as follows:

$$S_{rr} = \frac{\partial \Phi}{\partial \varepsilon_{rr}} + pg^*_{rr}, \quad S_{\theta\theta} = \frac{\partial \Phi}{\partial \varepsilon_{\theta\theta}} + pg^*_{\theta\theta}, \quad S_{33} = \frac{\partial \Phi}{\partial \varepsilon_{33}} + pg^*_{33}, \quad S_{r3} = \frac{\partial \Phi}{\partial \varepsilon_{r3}}, \quad S_{r3} = S_{3r},$$

$$g^*_{11} = 1 + 2\frac{\partial u_r}{\partial r} + \left(\frac{\partial u_r}{\partial r}\right)^2 + \left(\frac{\partial u_3}{\partial r}\right)^2, \quad g^*_{33} = 1 + 2\frac{\partial u_3}{\partial y_3} + \left(\frac{\partial u_3}{\partial y_3}\right)^2 + \left(\frac{\partial u_r}{\partial y_3}\right)^2,$$

$$g^*_{\theta\theta} = 1 + \frac{2}{r}u_r + \frac{1}{r^2}(u_r)^2. \tag{9}$$

Note that the expressions (7)–(9) are written in the arbitrary system of cylindrical coordinate system, without any restriction related to the association of this system to the natural or the initial state of the considered bilayered slab.

For the considered case the relations between the perturbation of the Kirchhoff stress tensor and the perturbation of the components of the Lagrange stress tensor can be written as follows:

$$Q'^{(k)}_{r'r'} = \lambda^{(k)} S^{(k)}_{r'r'} + S^{0(k)}_{rr} \frac{\partial u'^{(k)}_{r'}}{\partial r'}, \quad Q'^{(k)}_{\theta'\theta'} = \lambda^{(k)} S^{(k)}_{\theta'\theta'} + S^{0(k)}_{rr} \frac{u'^{(k)}_{r'}}{r'},$$

$$Q'^{(k)}_{33} = (\lambda^{(k)})^{-2} S^{(k)}_{33}, \quad Q'^{(k)}_{r'3} = \lambda^{(k)} S^{(k)}_{r'3} + S^{0(k)}_{rr} \frac{\partial u'^{(k)}_{3}}{\partial r'}, \quad Q'^{(k)}_{3r'} = (\lambda^{(k)})^{-2} S^{(k)}_{3r'}. \tag{10}$$

According to Refs. [14,15], by linearization of Eq. (9) and taking Eqs. (10), (1) and (2) into account, we obtain the following expressions for the constants $\chi^{(k)}_{1111}, \ldots, \chi^{(k)}_{3333}$ in Eq. (6):

$$\chi^{(k)}_{1111} = 2C^{(k)}_{10} (\lambda^{(k)})^{-2} ((\lambda^{(k)})^2 + (\lambda^{(k)})^{-4}),$$

$$\chi^{(k)}_{1122} = \chi^{(k)}_{1133} = \chi^{(k)}_{2233} = \chi^{(k)}_{3311} = \chi^{(k)}_{2211} = \chi^{(k)}_{3322} = 0, \quad \chi^{(k)}_{1331} = 2C^{(k)}_{10}, \quad \chi^{(k)}_{1221} = 2C^{(k)}_{10}, \quad \chi^{(k)}_{3333} = 4C^{(k)}_{10},$$

$$\chi^{(k)}_{1313} = 2C^{(k)}_{10} (\lambda^{(k)})^{-3}, \quad \chi^{(k)}_{3113} = 2C^{(k)}_{10}. \tag{11}$$

It should be noted that to the above-written equations the incompressibility conditions of the layers' materials must be added. These conditions for the considered case can be written as follows:

$$\frac{1}{\lambda^{(k)}} \left(\frac{\partial u'^{(k)}_{r'}}{\partial r'} + \frac{u'^{(k)}_{r'}}{r'} \right) + (\lambda^{(k)})^2 \frac{\partial u'^{(k)}_{3}}{\partial y'_3} = 0. \tag{12}$$

Thus, the stress state in the bilayered slab will be investigated by the use of Eqs. (4)–(12). In this case we will assume that the following boundary and contact conditions are satisfied.

$$Q'^{(1)}_{33}\Big|_{y'_3=0} = -P_0 \delta(r') e^{i\omega t} \frac{1}{(\lambda^{(1)})^2}, \quad Q'^{(1)}_{3r'}\Big|_{y'_3=0} = 0,$$

$$Q'^{(1)}_{33}\Big|_{y'_3=-h_1/(\lambda^{(1)})^2} = Q'^{(2)}_{33}\Big|_{y'_3=-h_1/(\lambda^{(1)})^2}, \quad Q'^{(1)}_{3r'}\Big|_{y'_3=-h_1/(\lambda^{(1)})^2} = Q'^{(2)}_{3r'}\Big|_{y'_3=-h_1/(\lambda^{(1)})^2},$$

$$u'^{(1)}_{r'}\Big|_{y'_3=-h_1/(\lambda^{(1)})^2} = u'^{(2)}_{r'}\Big|_{y'_3=-h_1/(\lambda^{(1)})^2}, \quad u'^{(1)}_{3}\Big|_{y'_3=-h_1/(\lambda^{(1)})^2} = u'^{(2)}_{3}\Big|_{y'_3=-h_1/(\lambda^{(1)})^2},$$

$$u'^{(2)}_{r'}\Big|_{y'_3=-h_1/(\lambda^{(1)})^2-h_2/(\lambda^{(2)})^2} = 0, \quad u'^{(2)}_{3}\Big|_{y'_3=-h_1/(\lambda^{(1)})^2-h_2/(\lambda^{(2)})^2} = 0, \tag{13}$$

where $\delta(r')$ is the Dirac delta function.

With the above-stated we exhaust the formulation of the problem and the consideration of the governing field equations. It should be noted that in the case where $\lambda^{(k)} = 1$, $k = 1, 2$ Eqs. (4)–(6), (10)–(12) and conditions (13) for kth layer transform to the corresponding ones of the classical linear theory of the elasticity for the incompressible body.

3. Solution procedure

Substituting Eq. (5) in Eq. (4), we obtain the following equation of motion in the displacement terms:

$$\chi_{1111}^{\prime(k)}\frac{\partial^2 u_{r'}^{\prime(k)}}{\partial r'^2}+\chi_{1122}^{\prime(k)}\frac{\partial}{\partial r'}\left(\frac{u_{r'}^{\prime(k)}}{r'}\right)+\left(\chi_{1133}^{\prime(k)}+\chi_{1331}^{\prime(k)}\right)\frac{\partial^2 u_3^{\prime(k)}}{\partial r'\partial y_3'}+\chi_{1313}^{\prime(k)}\frac{\partial^2 u_{r'}^{\prime(k)}}{\partial y_3'^2}$$
$$+\frac{1}{r'}\left(\chi_{1111}^{\prime(k)}-\chi_{2211}^{\prime(k)}\right)\frac{\partial u_{r'}^{\prime(k)}}{\partial r'}+\left(\chi_{1122}^{\prime(k)}-\chi_{2222}^{\prime(k)}\right)\frac{u_{r'}^{\prime(k)}}{r'^2}+\left(\chi_{1133}^{\prime(k)}-\chi_{2233}^{\prime(k)}\right)\frac{1}{r'}\frac{\partial u_3^{\prime(k)}}{\partial y_3'}$$
$$=\rho^{\prime(k)}\frac{\partial^2 u_{r'}^{\prime(k)}}{\partial t^2}-\frac{\partial p^{\prime(k)}}{\partial r'},$$

$$\chi_{3133}^{\prime(k)}\frac{\partial^2 u_{r'}^{\prime(k)}}{\partial r'\partial y_3'}+\chi_{3131}^{\prime(k)}\frac{\partial^2 u_3^{\prime(k)}}{\partial r'^2}+\frac{1}{r'}\chi_{3113}^{\prime(k)}\frac{\partial u_{r'}^{\prime(k)}}{\partial y_3'}+\frac{1}{r'}\chi_{3131}^{\prime(k)}\frac{\partial u_3^{\prime(k)}}{\partial r'}+\chi_{3311}^{\prime(k)}\frac{\partial^2 u_{r'}^{\prime(k)}}{\partial y_3'\partial r'}$$
$$+\chi_{3322}^{\prime(k)}\frac{1}{r'}\frac{\partial u_{r'}^{\prime(k)}}{\partial y_3'}+\chi_{3333}^{\prime(k)}\frac{\partial^2 u_3^{\prime(k)}}{\partial y_3'^2}=\rho^{\prime(k)}\frac{\partial^2 u_3^{\prime(k)}}{\partial t^2}-\frac{\partial p^{\prime(k)}}{\partial y_3'}. \tag{14}$$

Eqs. (12) and (14) compose the complete system with respect to the unknown functions $u_{r'}^{\prime(k)}$, $u_3^{\prime(k)}$ and $p^{\prime(k)}$. According to Refs. [14,15], we use the following representation for the displacement and unknown function $p^{\prime(k)}$:

$$u_{r'}^{\prime(k)}=-\frac{\partial^2}{\partial r'\partial y_3'}X^{\prime(k)},\quad u_3^{\prime(k)}=\Delta_1'X^{\prime(k)},$$

$$p^{\prime(k)}=\left[\left(\chi_{1111}^{\prime(k)}-\chi_{1133}^{\prime(k)}-\chi_{1313}^{\prime(k)}\right)\Delta_1'+\chi_{3113}^{\prime(k)}\frac{\partial^2}{\partial {y_3'}^2}-\rho^{\prime(k)}\frac{\partial^2}{\partial t^2}\right]\frac{\partial}{\partial y_3'}X^{\prime(k)}, \tag{15}$$

where

$$\Delta_1'=\frac{d^2}{dr'^2}+\frac{1}{r'}\frac{d}{dr'}. \tag{16}$$

The function $X^{\prime(k)}$ satisfies the following equation:

$$\left[\left(\Delta_1'+\left(\xi_2^{\prime(k)}\right)^2\frac{\partial^2}{\partial y_3'^2}\right)\left(\Delta_1'+\left(\xi_3^{\prime(k)}\right)^2\frac{\partial^2}{\partial y_3'^2}\right)-\frac{\rho^{\prime(k)}}{\chi_{1331}^{\prime(k)}}\left(\Delta_1'+\frac{\partial^2}{\partial y_3'^2}\right)\frac{\partial^2}{\partial t^2}\right]X^{\prime(k)}=0, \tag{17}$$

where for the considered case

$$\left(\xi_2^{\prime(k)}\right)^2=1,\quad \left(\xi_3^{\prime(k)}\right)^2=\left(\lambda^{(k)}\right)^{-6}. \tag{18}$$

Now we consider the solution to Eq. (17). Because the point load is harmonic in time, only the stationary case will be considered; all dependent variables become harmonic and can be represented as

$$\left\{Q_{r'r'}^{\prime(k)},\ldots,Q_{33}^{\prime(k)},u_{r'}^{\prime(k)},u_3^{\prime(k)},p^{\prime(k)},X^{\prime(k)}\right\}=\left\{\overline{Q}_{r'r'}^{\prime(k)},\ldots,\overline{Q}_{33}^{\prime(k)},\overline{u}_{r'}^{\prime(k)},\overline{u}_3^{\prime(k)},\overline{p}^{\prime(k)},\overline{X}^{\prime(k)}\right\}e^{i\omega t}, \tag{19}$$

where a superimposed dash denotes the amplitude of the relevant quantity. From here on we will omit this superimposed dash.

If Eq. (19) is employed in Eqs. (14)–(17), by replacing the operator $\partial^2/\partial t^2$ with $-\omega^2$ we obtain the same equations and conditions for the amplitude of the quantities sought. Consequently, introducing the dimensionless coordinates $r'\to r'/h_1, y_3'\to y_3'/h_1$ and the dimensionless frequency

$$\Omega^2=\frac{(\omega h_1)^2\rho^{\prime(2)}}{2C_{10}^{(2)}}, \tag{20}$$

we obtain the following equation for the potential $X'^{(k)}$ from Eqs. (17) and (18).

$$\left[\left(\Delta_1' + \frac{\partial^2}{\partial y_3'^2}\right)\left(\Delta_1' + (\lambda^{(k)})^{-6}\frac{\partial^2}{\partial y_3'^2}\right) - \frac{\Omega^2}{(\lambda^{(k)})^{(2)}}\left(\Delta_1' + \frac{\partial^2}{\partial y_3'^2}\right)\frac{C_{10}^{(2)}\rho'^{(k)}}{C_{10}^{(k)}\rho'^{(2)}}\right]X'^{(k)} = 0. \tag{21}$$

For the solution to Eq. (21) we use the Hankel integral representation for the function $X'^{(k)}$:

$$X'^{(k)} = \int_0^\infty F_1^{(k)} e^{\gamma^{(k)} y_3'} J_0(sr) s \, ds, \tag{22}$$

where $J_0(sr)$ is the Bessel function of zeroth order.

Substituting Eq. (22) into Eq. (21) we obtain the following algebraic equation for $\gamma^{(k)}$:

$$A^{(k)}(\gamma^{(k)})^4 + B^{(k)}(\gamma^{(k)})^2 + C^{(k)} = 0, \tag{23}$$

where

$$A^{(k)} = (\lambda^{(k)})^{-6}, \quad B^{(k)} = \frac{1}{(\lambda^{(k)})^2}\frac{C_{10}^{(2)}}{C_{10}^{(k)}}\frac{\rho'^{(k)}}{\rho'^{(2)}}\Omega^2 - (1 + (\lambda^{(k)})^{-6})s^2,$$

$$C^{(k)} = s^4 - s^2 \frac{C_{10}^{(2)}}{C_{10}^{(k)}}\frac{\rho'^{(k)}}{\rho'^{(2)}}\frac{\Omega^2}{(\lambda^{(k)})^2}. \tag{24}$$

We obtain from Eq. (23) that

$$(\gamma^{(k)})^2 = \frac{-B^{(k)} \pm \sqrt{(B^{(k)})^2 - 4A^{(k)}C^{(k)}}}{2A^{(k)}}. \tag{25}$$

By using the expressions (18) by direct verification and transformation, it is proven that

$$(\gamma_1^{(k)})^2 = \frac{-B^{(k)} + \sqrt{(B^{(k)})^2 - 4A^{(k)}C^{(k)}}}{2A^{(k)}} = s^2 > 0,$$

$$(\gamma_2^{(k)})^2 = \frac{-B^{(k)} - \sqrt{(B^{(k)})^2 - 4A^{(k)}C^{(k)}}}{2A^{(k)}} = s^2(\lambda^{(k)})^6\left(1 - \frac{\Omega_k^2}{s^2}\right), \tag{26}$$

where

$$\Omega_k^2 = \frac{C_{10}^{(2)}}{C_{10}^{(k)}}\frac{\rho'^{(k)}}{\rho'^{(2)}}\frac{\Omega^2}{(\lambda^{(k)})^2}. \tag{27}$$

According to the expression of $(\gamma_2^{(k)})^2$ in Eq. (26), the following two cases may occur:

Case 1:

$$(\gamma_2^{(k)})^2 > 0 \quad \text{or} \quad \Omega_k^2 > s^2. \tag{28}$$

Case 2:

$$(\gamma_2^{(k)})^2 < 0 \quad \text{or} \quad \Omega_k^2 < s^2. \tag{29}$$

In Case 1, the solution to Eq. (21) is determined as

$$X'^{(k)} = \int_0^\infty \left[F_1^{(k)}(s)e^{sy_3'} + F_2^{(k)}(s)e^{-sy_3'} + F_3^{(k)}(s)e^{\gamma_2^{(k)}y_3'} + F_4^{(k)}(s)e^{-\gamma_2^{(k)}y_3'}\right], \tag{30}$$

where

$$\gamma_2^{(k)} = s(\lambda^{(k)})^3\sqrt{1 - \frac{\Omega_k^2}{s^2}}. \tag{31}$$

In Case 2, this solution is determined as

$$X'^{(k)} = \int_0^\infty \left[F_1^{(k)}(s)e^{sy'_3} + F_2^{(k)}(s)e^{-sy'_3} + F_3^{(k)}(s)e^{i\gamma_2^{(k)}y'_3} + F_4^{(k)}(s)e^{-i\gamma_2^{(k)}y'_3}\right], \tag{32}$$

where

$$\gamma_2^{(k)} = s(\lambda^{(k)})^3\sqrt{\frac{\Omega_k^2}{s^2} - 1}. \tag{33}$$

Thus, we get the following expressions for stresses and displacements from Eqs. (5), (7), (15), (30) and (32).

Case 1:

$$\begin{aligned}
u_{r'}'^{(k)} &= \int_0^\infty [F_1^{(k)}(s)se^{sy'_3} - F_2^{(k)}(s)se^{-sy'_3} + \gamma_2^{(k)}(F_3^{(k)}e^{\gamma_2^{(k)}s} - F_4^{(k)}e^{-\gamma_2^{(k)}s})]J_1(sr')s^2\,ds,\\
u_3'^{(k)} &= -\int_0^\infty s^2[F_1^{(k)}(s)e^{sy'_3} + F_2^{(k)}(s)se^{-sy'_3} + F_3^{(k)}e^{\gamma_2^{(k)}s} + F_4^{(k)}e^{-\gamma_2^{(k)}s}]J_0(sr')s\,ds,\\
Q_{3r}'^{(k)} &= C_{10}^{(k)}\int_0^\infty \left[\frac{2s^2}{(\lambda^{(k)})^4}(F_1^{(k)}(s)e^{sy'_3} + F_2^{(k)}(s)e^{-sy'_3})\right.\\
&\quad \left.+\left(s^2 + s^2(\lambda^{(k)})^2\left(1 - \frac{\Omega_k^2}{s^2}\right)\right)\frac{1}{(\lambda^{(k)})^4}(F_3^{(k)}(s)e^{\gamma_2^{(k)}y'_3} + F_4^{(k)}(s)e^{-\gamma_2^{(k)}y'_3})\right]s^2J_1(sr')\,ds,\\
Q_{33}'^{(k)} &= C_{10}^{(k)}\int_0^\infty \left[\left(\Omega_k^2 - s^2(\lambda^{(k)})^2 - \frac{s^2}{(\lambda^{(k)})^4}\right)s(F_1^{(k)}(s)e^{sy'_3} - F_2^{(k)}(s)e^{-sy'_3})\right.\\
&\quad \left.+\gamma_2^{(k)}\left(\Omega_k^2(1 - (\lambda^{(k)})^2) - \frac{s^2}{(\lambda^{(k)})^4}\right)(F_3^{(k)}(s)e^{\gamma_2^{(k)}y'_3} - F_4^{(k)}(s)e^{-\gamma_2^{(k)}y'_3})\right]J_0(sr')s\,ds, \ldots .
\end{aligned} \tag{34}$$

Case 2:

$$\begin{aligned}
u_{r'}'^{(k)} &= \int_0^\infty [F_1^{(k)}(s)se^{sy'_3} - F_2^{(k)}(s)se^{-sy'_3}\\
&\quad + \gamma_2^{(k)}(-F_3^{(k)}\sin(\gamma_2^{(k)}s) + F_4^{(k)}\cos(\gamma_2^{(k)}s))]J_1(sr')s^2\,ds,\\
u_3'^{(k)} &= -\int_0^\infty s^2[F_1^{(k)}(s)e^{sy'_3} + F_2^{(k)}(s)se^{-sy'_3}\\
&\quad + F_3^{(k)}\cos(\gamma_2^{(k)}y'_3) + F_4^{(k)}\sin(\gamma_2^{(k)}y'_3)]J_0(sr')s\,ds,\\
Q_{3r}'^{(k)} &= C_{10}^{(k)}\int_0^\infty \left[\frac{2s^2}{(\lambda^{(k)})^4}(F_1^{(k)}(s)e^{sy'_3} + F_2^{(k)}(s)e^{-sy'_3})\right.\\
&\quad \left.+\left(s^2 - s^2(\lambda^{(k)})^2\left(\frac{\Omega_k^2}{s^2} - 1\right)\right)\frac{1}{(\lambda^{(k)})^4}(F_3^{(k)}(s)\cos(\gamma_2^{(k)}y'_3) + F_4^{(k)}(s)\sin(\gamma_2^{(k)}y'_3))\right]s^2J_1(sr')\,ds,\\
Q_{33}'^{(k)} &= C_{10}^{(k)}\int_0^\infty \left[\left(\Omega_k^2 - s^2(\lambda^{(k)})^2 - \frac{s^2}{(\lambda^{(k)})^4}\right)s(F_1^{(k)}(s)e^{sy'_3} - F_2^{(k)}(s)e^{-sy'_3})\right.\\
&\quad \left.+\gamma_2^{(k)}\left(\Omega_k^2(1 - (\lambda^{(k)})^2) - \frac{2s^2}{(\lambda^{(k)})^4}\right)(-F_3^{(k)}(s)\sin(\gamma_2^{(k)}y'_3) + F_4^{(k)}(s)\cos(\gamma_2^{(k)}y'_3))\right]J_0(sr')s\,ds, \ldots .
\end{aligned} \tag{35}$$

To find the unknowns $F_1^{(k)}(s), \ldots, F_4^{(k)}(s)$ we use the boundary and contact conditions (13). For this purpose we determine the Hankel transformation of the right-hand side of the first condition in Eq. (13). Using the equality $P_0\delta(r') = \lim_{r'\to 0}(P_0/\pi r'^2)$ we obtain $P_0/2\pi$ for Hankel transformation of $P_0\delta(r')$ from

$$\lim_{\varepsilon\to 0}\int_0^\varepsilon \frac{P_0}{\pi\varepsilon^2}r'J_0(sr')\,dr'.$$

Thus, we derive from the conditions (13) the following equations for the above-listed unknowns:

$$\sum_{j=1}^{4} F_j^{(1)}(s)\alpha_{ij}^{(1)}(s) = P_0/(2\pi(\lambda^{(1)})^2)\delta_i^1, \quad i = 1, 2,$$

$$\sum_{j=1}^{4}[F_j^{(1)}(s)\alpha_{ij}^{(1)}(s) + F_j^{(2)}(s)\alpha_{ij}^{(2)}(s)] = 0, \quad i = 3, 4, 5, 6;$$

$$\sum_{j=1}^{4} F_j^{(2)}(s)\alpha_{ij}^{(2)}(s) = 0, \quad i = 7, 8. \tag{36}$$

The coefficients of the unknowns in Eq. (36) are determined through the coefficients of those in expressions (34) and (35). Thus, we determine the unknowns $F_1^{(k)}(s), \ldots, F_4^{(k)}(s)$ from Eq. (36) and the stresses and displacements are determined from expressions (34) and (35).

Now we consider the calculation of the integrals in Eqs. (34) and (35). As it follows from the expressions in Eqs. (33) and (34), these integrals are the improved ones. Therefore, under calculation procedure, they are replaced by the corresponding definite integrals, i.e. we use the following relation:

$$\int_0^{\infty} (.)\,\mathrm{d}s \approx \int_0^{S_*} (.)\,\mathrm{d}s. \tag{37}$$

The values of S_* in Eq. (37) are determined from the convergence criterion of the improved integrals.

Consider the determination of the singular points of the integrated expressions in Eqs. (34), (35). Since the solution to Eq. (36) can be expressed as

$$(F_1^{(1)}(s), \ldots, F_4^{(2)}(s)) = \frac{1}{\det \|\alpha_{ij}(s)\|}(\det \|\beta_{ij}^{F_1^{(1)}(s)}\|, \ldots, \det \|\beta_{ij}^{F_4^{(2)}(s)}\|), \tag{38}$$

these singular points coincide with the roots of the equation

$$\det \|\alpha_{ij}(s)\| = 0, \quad i; j = 1, 2, \ldots, 8 \tag{39}$$

in s, where

$$\alpha_{ij}(s) = \alpha_{ij}^{(1)}(s) \quad \text{for } i = 1, 2, \ldots, 6,\ j = 1, 2, 3, 4,$$
$$\alpha_{ij}(s) = \alpha_{ij}^{(2)}(s) \quad \text{for } i = 3, \ldots, 8,\ j = 5, \ldots, 8. \tag{40}$$

Note that the expressions for $\|\beta_{ij}^{F_1^{(1)}(s)}\|, \ldots, \|\beta_{ij}^{F_4^{(2)}(s)}\|$ are obtained from $\|\alpha_{ij}(s)\|$ by replacing the corresponding column with the right-hand side of Eq. (36).

A numerical analysis shows that the order of the roots of Eq. (39) is one. Therefore, the order of all singular points of the integrated expressions is also one. Taking this situation into account in the solution to Eq. (39) with respect to s we employ the well-known bisection method.

Let us denote the roots of Eq. (39) as

$$s_1 < s_2 < \cdots < s_k < \cdots < s_N. \tag{41}$$

The number N in Eq. (41) depends on the values of dimensionless frequency Ω and the mechanical and geometrical parameters. After determining the roots (41), the interval $[0, S_*]$ in Eq. (37) is partitioned as follows:

$$\int_0^{S_*} (.)\,\mathrm{d}s = \int_0^{s_1-\varepsilon} (.)\,\mathrm{d}s + \int_{s_1+\varepsilon}^{s_2-\varepsilon} (.)\,\mathrm{d}s + \cdots + \int_{s_k+\varepsilon}^{s_{k+1}-\varepsilon} (.)\,\mathrm{d}s + \cdots + \int_{s_N+\varepsilon}^{S_*} (.)\,\mathrm{d}s. \tag{42}$$

Consequently, the calculation of the integral (37) is performed in the Cauchy's principal value sense. Here, ε is a very small value determined numerically from the convergence requirement of the integral (42). Each interval $[s_k + \varepsilon, s_{k+1} - \varepsilon]$ is further divided into a certain number of shorter intervals, which are used in Gauss integration algorithm. In this integration procedure the values of the integrated expressions, i.e. the values of unknowns $F_1^{(1)}(s),\ldots,F_4^{(2)}(s)$ in the Gauss integration points are determined through Eqs. (36). All these procedures are performed automatically in PC by the use of corresponding programs constructed by the author.

Thus, now we consider some numerical results obtained within the framework of the above-discussed solution procedure and related the influence of the pre-stretching of the layers on the distribution of the normal stresses acting on the interface planes.

4. Numerical results and discussions

For testing the validity of the algorithm and programmes we consider the case where the slab consists of a single layer. Analyse the distribution of the stress Q'_{33} on the plane between the rigid foundation and the slab. We examine the influence of the Ω (20) on this distribution. According to the mechanical consideration, under the absence of the initial stretching of the slab, the values of Q'_{33} must approach the values obtained for the corresponding static problem studied in Ref. [32] as $\Omega\to 0$. Note that in Ref. [32], the expression for the stress $Q'_{33}=\sigma_{33}$ was obtained in the integral form within the framework of the classical linear theory of elasticity and it was assumed that the slab material is compressible.

Now, consider the comparison of the present results with corresponding ones obtained by the use of the integral expression given for σ_{33} in Ref. [32]. In the latter case we assume that $\nu=0.499$ where ν is a Poisson's ratio of the slab material. Fig. 2 shows the graphs of the dependencies between $Q'_{33}h_1^2/P_0$ and r'/h_1 (h_1 is a slab thickness) for various Ω. It follows from these graphs that the values of $Q'_{33}h_1^2/P_0$ obtained for the dynamical problem approach the corresponding ones obtained for the static problem as $\Omega\to 0$. This situation holds for the correctness of the algorithm and programmes used.

We consider the influence of the initial pre-stretching of the single-layer slab on the dependencies between $Q'_{33}h_1^2/P_0$ (at $r'/h_1=0$) and Ω. The graphs of these dependencies are given in Fig. 3. The graphs show that for the considered range of the change of the Ω, i.e. for

$$0<\Omega\leqslant 2.5, \tag{43}$$

the absolute values of $Q'_{33}h_1^2/P_0$ increase monotonically with Ω. As a result of the pre-stretching of the slab the values of the $Q'_{33}h_1^2/P_0$ decrease monotonically with λ. The explanation of these results will be considered below.

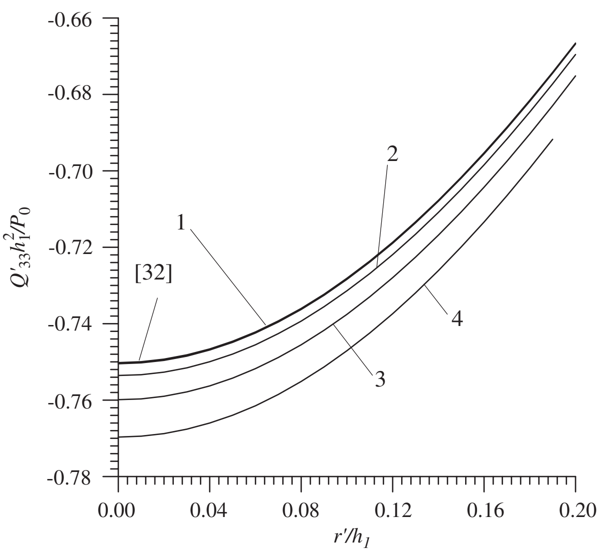

Fig. 2. The comparison of the results obtained for single-layer slab with corresponding ones given in Ref. [32] as $\Omega\to 0$. Lines 1, 2, 3 and 4 correspond to the values of $\Omega=0.00$–0.1, 0.2, 0.5 and 0.7, respectively.

We turn to the consideration of the stress distribution in the bilayered slab resting on the rigid foundation. Introduce the notation

$$e = \frac{C_{10}^{(1)}}{C_{10}^{(2)}}, \quad H = \frac{h_2}{h_1}, \quad q_{33}^{(1)} = \left(\frac{Q_{33}^{\prime(1)} h_1^2}{P_0} \right) \Bigg|_{y_3' = -h_1/(\lambda^{(1)})^2},$$

$$q_{33}^{(2)} = \left(\frac{Q_{33}^{\prime(2)} h_1^2}{P_0} \right) \Bigg|_{y_3' = -h_1/(\lambda^{(1)})^2 - h_2/(\lambda^{(2)})^2}. \tag{44}$$

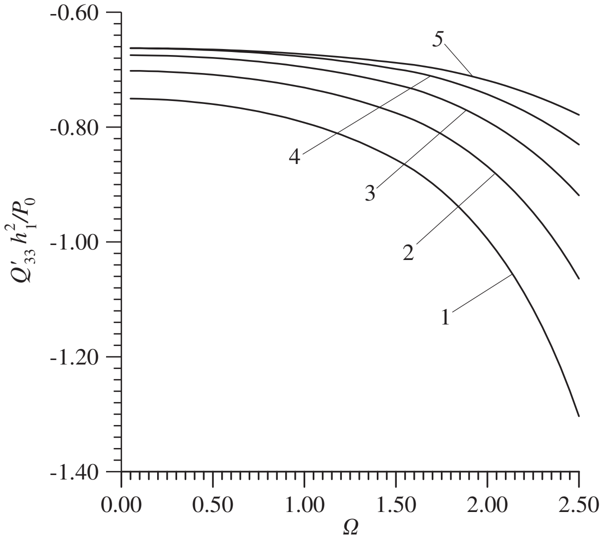

Fig. 3. The influence of the pre-stretching of the single-layer slab to the dependencies between stress $Q'_{33}h_1^2/P_0$ (at $r'/h_1 = 0.0$) and dimensionless frequency Ω (20). Lines 1, 2, 3, 4 and 5 correspond to the values of $\lambda = 1.0$, 1.05, 1.10, 1.15 and 1.20, respectively.

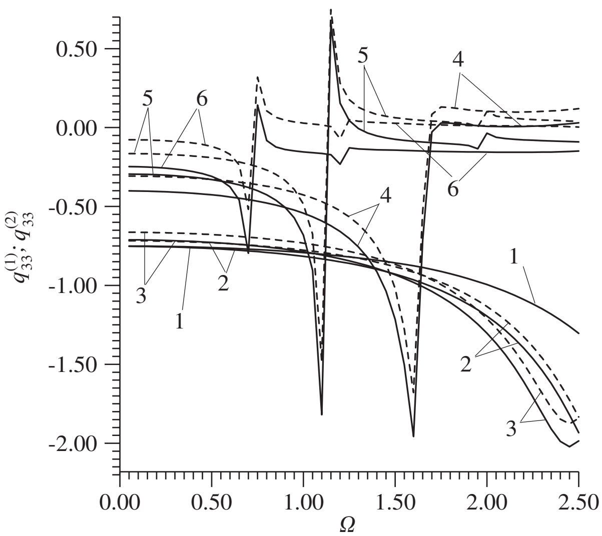

Fig. 4. The influence of the parameter $H = h_2/h_1$ on the dependencies among the stresses $q_{33}^{(1)}$ (solid lines), $q_{33}^{(2)}$ (dashed lines) (44) and frequency Ω (20) for the case where the initial pre-stretching is absent, i.e. $\lambda^{(1)} = \lambda^{(2)} = 1.0$ and $e = 5.0$, $r'/h_1 = 0.0$. Lines 1, 2, 3, 4, 5 and 6 correspond to the cases where $H = 0.00$, 0.05, 0.1, 0.5, 1.0 and 2.0, respectively.

Assume that the values of Ω change in the interval (43). We consider the case where $C_{10}^{(1)}/\rho'^{(1)} = C_{10}^{(2)}/\rho'^{(2)}$ and analyse the influence of H on the character of the dependencies among $q_{33}^{(1)}$, $q_{33}^{(2)}$ and Ω for the case where $\lambda^{(1)} = \lambda^{(2)} = 1.0$, i.e. for the case where the initial stretching is absent. The graphs of these dependencies are given in Fig. 4 for $e = 5$. The values of $q_{33}^{(1)}$, $q_{33}^{(2)}$ are calculated at the point $r'/h_1 = 0$. Note that in Fig. 4 and in the following figures which will be considered, unless otherwise specified, the solid (dashed) lines show the values of $q_{33}^{(1)}$ ($q_{33}^{(2)}$).

It follows from the analyses of the graphs that under certain values of Ω, the absolute values of $q_{33}^{(1)}$ and $q_{33}^{(2)}$ reach the extrema. We will call these values of Ω as its "resonance" values. Moreover, these graphs show that the "resonance" values of Ω decrease with H. We attempt to explain the described character of the considered dependencies.

According to Refs. [33–35], the behaviour of the half-space or half-plane under forced vibrations is similar to that of the system, which comprises a mass, a parallel connected spring and a dashpot. The similar

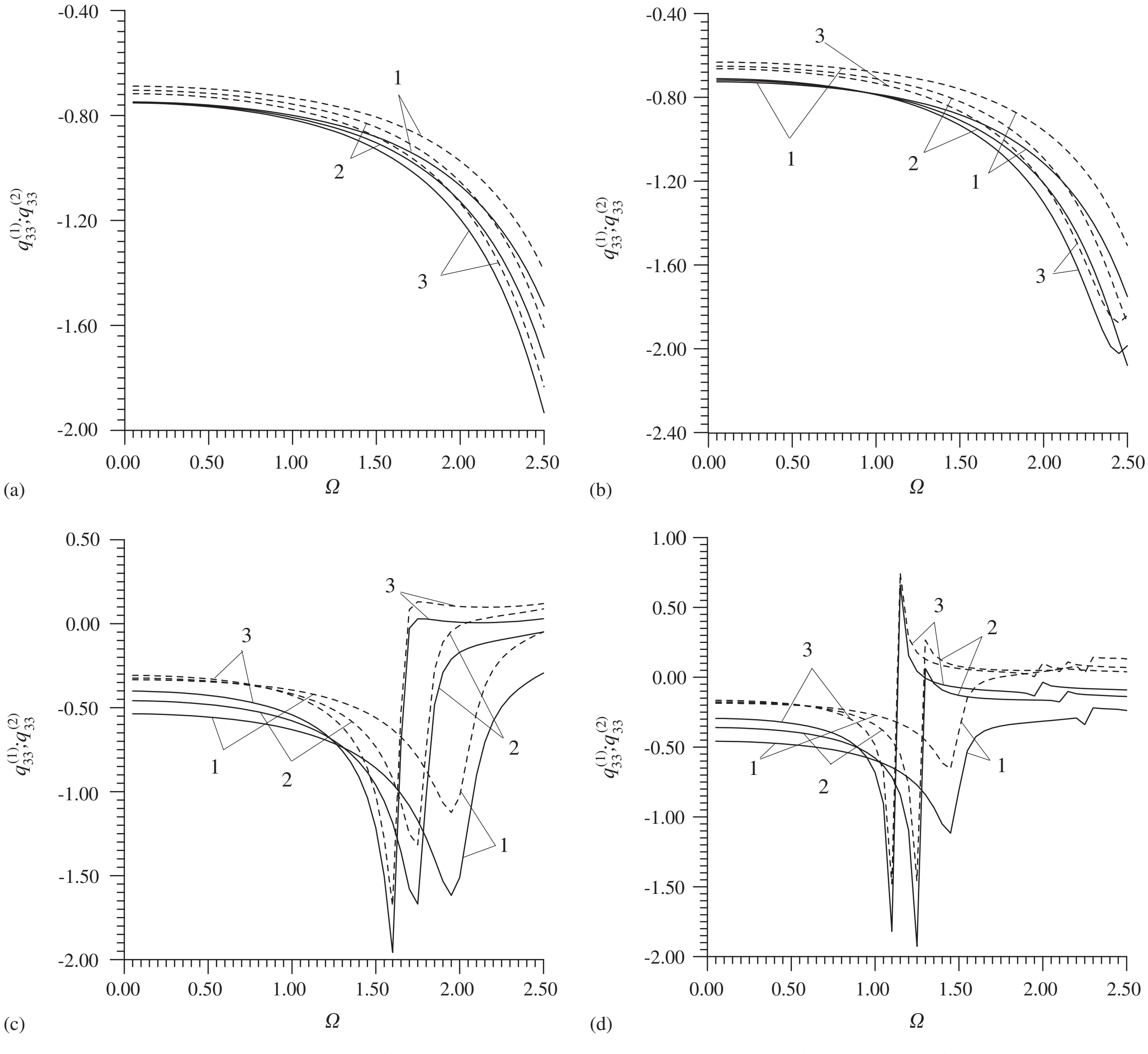

Fig. 5. The influence of the parameter $e = C_{10}^{(1)}/C_{10}^{(2)}$ on the dependencies among the stresses $q_{33}^{(1)}$ (solid lines), $q_{33}^{(2)}$ (dashed lines) (44) and frequency Ω (20) for the case where the initial pre-stretching is absent, i.e. $\lambda^{(1)} = \lambda^{(2)} = 1.0$, and $r'/h_1 = 0.0$. Graphs 1, 2 and 3 correspond to the values $e = 1.5$, 3.0 and 5.0, respectively: (a) $H = 0.05$; (b) $H = 0.10$; (c) $H = 0.5$ and (d) $H = 1.0$.

behaviour is also observed under dynamical (vibrating) contact problems [36]. The numerical results given in Fig. 4 and the other ones, which are not illustrated here, show that the behaviour of the bilayered slab on the rigid foundation under forced vibrations is also similar to the behaviour of the above-mentioned system comprising mass, spring and dashpot. Consequently, the occurrence of the "resonance" values of Ω follows from the nature of the considered mechanical object.

According to the well-known mechanical consideration, the stiffness of the investigated system must decrease with H in the considered cases (i.e. for the cases where $e = C_{10}^{(1)}/C_{10}^{(2)} > 1$). Therefore, the "resonance" values of Ω decrease, which follows from the observation of the graphs given in Fig. 4, with H. It should be noted that there are also the "resonance" values of Ω for all values of H, for example, for $H = 0.05$, 0.10 and for the system consisting of a single-layer slab and rigid foundation (i.e. for $H = 0.00$). However, these "resonance" values of Ω for these cases are out of the interval (43), i.e. are greater than $\Omega = 2.5$.

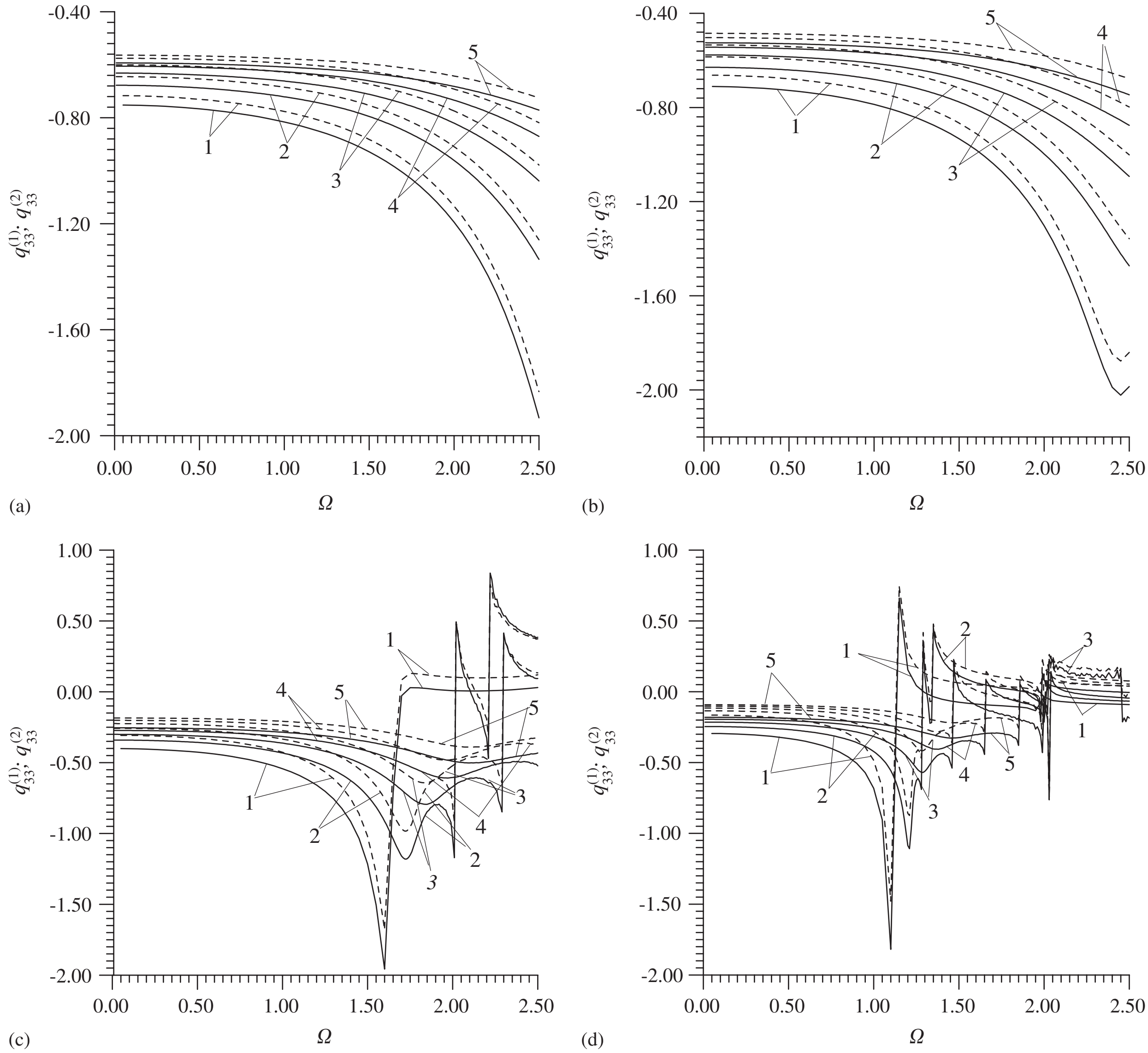

Fig. 6. The influence of the pre-stretching of the slab on the dependencies among the stresses $q_{33}^{(1)}$ (solid lines), $q_{33}^{(2)}$ (dashed lines) (44) and frequency Ω (20) obtained for the case where $e = 5.0$, $\lambda^{(2)} = 1.00$, $r'/h_1 = 0.0$. Graphs 1, 2, 3, 4 and 5 correspond to the cases where $\lambda^{(1)} = 1.00$, 1.05, 1.10, 1.15 and 1.20, respectively: (a) $H = 0.05$; (b) $H = 0.10$; (c) $H = 0.50$ and (d) $H = 1.00$.

Consider the influence of e on the dependencies among $q_{33}^{(1)}$, $q_{33}^{(2)}$ and Ω. The graphs of these dependencies are given in Fig. 5 for various e. In this figure, the graphs separated by letters (a), (b), (c) and (d) are obtained for $H = 0.05$, 0.10, 0.50 and 1.00, respectively. It follows from the graphs that under $H = 0.05$, 0.10 for the considered values of e, the "resonance" values of Ω are not observed in the interval (43). But under $H = 0.50$, 1.00, 2.00, the "resonance" values of Ω decrease with e. This situation agrees with the well-known mechanical and engineering considerations.

Now we analyse the influence of the pre-stretching of the layers on the above-discussed dependencies among $q_{33}^{(1)}$, $q_{33}^{(2)}$ and Ω. Note that various numerical results which are not given here show that the influence of the pre-stretching of the lower layer of the slab on the considered dependencies is insignificant. Therefore, here we will consider only the case where the pre-stretching exists only in the upper layer of the slab, i.e. $\lambda^{(2)} = 1.0$, $\lambda^{(1)} > 1.0$. The graphs of the dependencies are given in Fig. 6 for the case where $e = 5.0$ with various values of $\lambda^{(1)}$. In this figure the graphs separated by letters (a), (b), (c) and (d) are constructed for $H = 0.05$, 0.10, 0.50 and 1.00, respectively.

It follows from the results that the values of $q_{33}^{(1)}$, $q_{33}^{(2)}$ decrease with $\lambda^{(1)}$. Moreover, it follows from these results that the "resonance" values of Ω increase with the pre-stretching of the upper layer of the slab. At the same time, the pre-stretching of the layer causes the character of the dependencies to change in a vicinity of the "resonance" values of Ω and the extremum values of $q_{33}^{(1)}$, $q_{33}^{(2)}$ to decrease. The described type of influences of $\lambda^{(1)}$ on the considered dependencies are explained by the increase in the stiffness of the upper layer of the slab in the radial direction caused with the pre-stretching of that.

So far we have considered the values of $q_{33}^{(1)}$, $q_{33}^{(2)}$ at the point $r'/h_1 = 0$. These values are the extrema of $q_{33}^{(1)}$, $q_{33}^{(2)}$ with respect to r'/h_1. This conclusion is also proven by the graphs given in Fig. 7 which show the distribution of $q_{33}^{(1)}$, $q_{33}^{(2)}$ with respect to r'/h_1 for $H = 0.50$ (Fig. 7(a)) and 1.00 (Fig. 7(b)), respectively. These graphs are constructed in the case where $\Omega = 1.0$, $e = 5$. Note that the above-discussed results on the dependencies among the $q_{33}^{(1)}$, $q_{33}^{(2)}$ (at $r'/h_1 = 0$) and Ω hold also in a qualitative sense for each point of the interface plane $y_3 = -h_1$. At the same time, these results are obtained in the case where $C_{10}^{(1)}/C_{10}^{(2)} = \rho'^{(2)}/\rho'^{(1)}$. Numerical analyses show that the results considered above hold also in a qualitative sense for the case where $C_{10}^{(1)}/C_{10}^{(2)} \neq \rho'^{(2)}/\rho'^{(1)} = 1$. However, the "resonance" values of Ω determined for the latter case are greater than the corresponding ones determined for the case where $C_{10}^{(1)}/C_{10}^{(2)} = \rho'^{(2)}/\rho'^{(1)}$.This result is testified by the

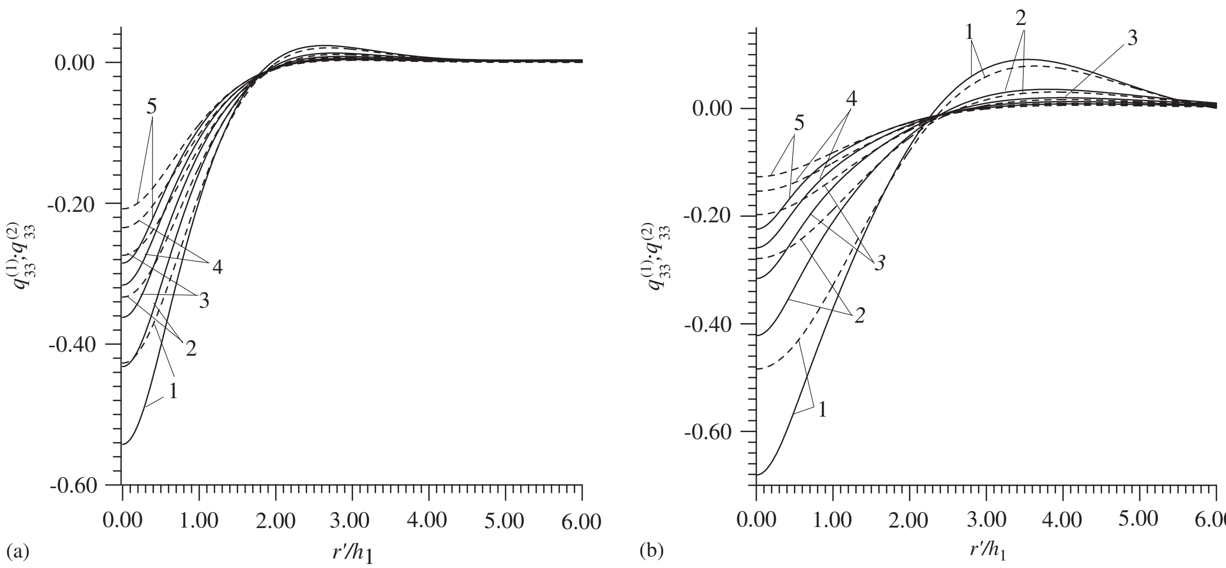

Fig. 7. The influence of the pre-stretching of the upper layer of the slab on the distribution of the stresses $q_{33}^{(1)}$ (solid lines), $q_{33}^{(2)}$ (dashed lines) (44) with respect to r'/h_1 in the case where $e = 5.0$, $\Omega = 1.0$, $\lambda^{(2)} = 1.00$. Lines 1, 2, 3, 4 and 5 correspond to the cases where $\lambda^{(1)} = 1.0$, 1.05, 1.10, 1.15 and 1.20, respectively: (a) $H = 0.5$ and (b) $H = 1.00$.

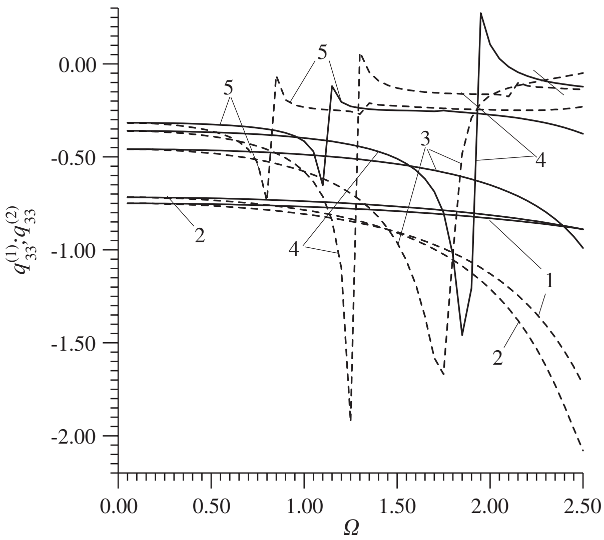

Fig. 8. The graphs of the dependencies between the stress $q_{33}^{(1)}$ (44) and the frequency Ω (20) obtained in the cases where $C_{10}^{(1)}/C_{10}^{(2)} = \rho'^{(1)}/\rho'^{(2)}$. (dashed lines) and $C_{10}^{(1)}/C_{10}^{(2)} \neq \rho'^{(1)}/\rho'^{(2)} = 1$ (solid lines) for $e = 3.0$, $r'/h_1 = 0.0$, $\lambda^{(1)} = \lambda^{(2)} = 1.0$. Graphs 1, 2, 3, 4 and 5 correspond to the cases where $H = 0.05$, 0.1, 0.5, 1.0 and 2.0, respectively.

graphs given in Fig. 8 which show the dependencies between $q_{33}^{(1)}$ and Ω for $e = 3.0$, $\lambda^{(1)} = \lambda^{(2)} = 1.0$, $r'/h_1' = 0$. Here, the solid (dashed) lines correspond to the case where $C_{10}^{(1)}/C_{10}^{(2)} \neq \rho'^{(2)}/\rho'^{(1)} = 1$ ($C_{10}^{(1)}/C_{10}^{(2)} = \rho'^{(2)}/\rho'^{(1)}$). It follows from these graphs that the extremum values of $q_{33}^{(1)}$ arising at the "resonance" values for Ω for the case $C_{10}^{(1)}/C_{10}^{(2)} \neq \rho'^{(2)}/\rho'^{(1)} = 1$ are less than the corresponding ones arising under $C_{10}^{(1)}/C_{10}^{(2)} = \rho'^{(2)}/\rho'^{(1)}$.

5. Conclusions

In this paper, the dynamical axisymmetric stress field in the initially finite pre-stretched bilayered slab resting on the rigid foundation is studied within the framework of the piecewise homogeneous bodies model with the use of the three-dimensional linearized theory of elastic waves in initially stressed bodies. It is assumed that a time-harmonic point-located normal force acts on the free face plane of the slab. The model problem is solved by employing the Hankel integral transformation. The materials of the layers are assumed to be incompressible neo-Hookean materials and the elastic relations of those are given through the Treloar potential. The formulation and solution to the problem coincide with the corresponding ones of the classical linear theory of elasticity for an incompressible body in the case where the initial stretching is absent in the layers. The algorithm for obtaining numerical results is developed. According to these numerical results, the dependencies among the normal stresses, which act on the interface planes, and frequency of the external force are analysed. It is assumed that $C_{10}^{(1)} > C_{10}^{(2)}$, where $C_{10}^{(k)}$ ($k = 1, 2$) is a material constant which enter the Treloar potential of the kth layer.

The numerical results indicate the following conclusions:

- the mechanical behaviour of the forced vibration of the bilayered slab resting on the rigid foundation is similar to that of the system comprising a mass, a spring and a dashpot;
- the "resonance" values of the frequency of the external force decrease with $C_{10}^{(1)}/C_{10}^{(2)}$ and with h_2/h_1, where h_1 (h_2) is a thickness of the upper (lower) layer of the slab;
- the normal stresses on the interface planes decrease as the pre-stretching of the layers is increased;
- the "resonance" values of the frequency increase with pre-stretching;

- the foregoing influence of the pre-stretching of the layers on the stress distribution and on the "resonance" values of the frequency is significant in the quantitative sense and must be taken into account in the regarding cases.

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