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Short Communication

Galerkin finite element derivation for vibration of a thermopiezoelectric structure

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Abstract

A solution for the equations governing dynamic thermopiezoelectric behavior is assumed in the form of an exponential $e^{i(kz+m\theta+\omega t)}$. The resulting formulation is reduced to one-dimension and will have both real and imaginary parts. The Galerkin method and assumed shape functions are used to illustrate the formulation of the resulting eigenvalue problem.

1. Introduction

The vibration analysis of an infinite cylindrical piezoelectric body has been studied by numerous researchers of which Paul and Venkatesan [1] are representative of an analytical analysis. Their analysis has been verified by Buchanan and Peddieson [2] using a one-dimensional finite element analysis and later by Siao et al. [3] using a more definitive finite element formulation. The intent of this report is to extend the formulation presented in Ref. [3] to include thermal effects and to illustrate the formalism and organizational power of the Galerkin method for deriving finite element models. An analytical analysis has been given by Paul and Raman [4] for hollow pyroelectric circular cylinders.

2. Governing equations

The equations of linear thermopiezoelectricity were proposed in the classic paper by Mindlin [5]. The formulation used by Yang and Batra [6] will be followed in this communication. The governing equations are

$$\begin{gathered} T_{kj,j} = \rho \ddot{u}_k, \quad T_{kj} = T_{jk}, \quad \text{for } k \neq j, \quad D_{k,k} = 0, \quad q_{k,k} + T_o \dot{\eta} = 0, \\ T_{kj} = C_{kjrs} S_{rs} - e_{rkj} E_r - \beta_{kj} T, \quad S_{kj} = \tfrac{1}{2}(u_{k,j} + u_{j,k}), \\ D_k = e_{klr} S_{lr} + \varepsilon_{kl} E_l + P_k T, \quad E_k = -\phi_{,k}, \\ \eta = \beta_{kj} S_{kj} + P_k E_k + aT, \quad a = \tfrac{\rho C_v}{T_o}, \quad q_k = -k_{kj} T_{,j}. \end{gathered} \tag{1}$$

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The stress and strain tensors are defined as T_{kj} and S_{kj}, the remaining terms are defined as u_k the mechanical displacement, D_k the electric displacement, E_k the electric field, T the temperature change from a reference temperature T_o, q_k the heat flux, ϕ the electric potential, η the entropy, ρ the mass density and C_{kjrs}, e_{rkj}, β_{kj}, ε_{kj}, P_k are material constants with C_v the specific heat. The usual notation is employed, that a comma followed by a lower case letter indicates spatial derivative and a superimposed dot indicates time derivative.

The nine equations (1) are combined into three second-order equations that can then be written as finite element equations using the Galerkin method. The constitutive equations are substituted into the balance equations to yield [6]

$$C_{klrs}u_{r,sl} + e_{rkl}\varphi_{,rl} - \beta_{kl}T_{,l} = \rho\ddot{u}_k, \tag{2}$$

$$e_{klr}u_{r,lk} - \varepsilon_{kl}\varphi_{,lk} + P_kT_{,k} = 0, \tag{3}$$

$$\beta_{kl}\dot{u}_{k,l} - P_k\dot{\varphi}_{,k} + a\dot{T} - \frac{k_{kl}}{T_o}T_{,kl} = 0. \tag{4}$$

3. Finite element model

The derivation of the finite element equations using the Galerkin approximation is discussed in numerous textbooks, for instance Ref. [7], and would require assumed trial solutions in terms of shape functions that correspond to u_k, φ and T as follows:

$$u_i = [N_u]\{u\}, \quad \varphi = [N_\varphi]\{\varphi\}, \quad T = [N_T]\{T\}, \tag{5}$$

where $\{u\}$, $\{\phi\}$ and $\{T\}$ are unknown nodal point variables while $[N_u]$, $[N_\varphi]$ and $[N_T]$ are shape functions and may be different for each variable. In most applications it is convenient to assume all shape functions the same. The governing equations given by Eqs. (2)–(4) are valid for a Cartesian system, however, for the purpose of deriving a finite element model they are assumed to be symbolic. The final local finite element model can be cast in any convenient coordinate system. In this application an infinite cylinder will be modeled and that will require that the fundamental equations (1) be written in cylindrical (r, θ, z) coordinates.

The finite element in Ref. [3] is based upon a variational formulation and the thermopiezoelectric finite element to be discussed will reduce to that given in Ref. [3] for piezoelectricity. Following Ref. [3], assume a solution for an infinite cylinder vibrating with harmonic motion as

$$u_i(r,\theta,z,t) = U_i(r)\,\mathrm{e}^{\mathrm{i}(kz+m\theta+\omega t)}, \quad \varphi(r,\theta,z,t) = \Psi(r)\,\mathrm{e}^{\mathrm{i}(kz+m\theta+\omega t)}, \quad T(r,\theta,z,t) = \chi(r)\,\mathrm{e}^{\mathrm{i}(kz+m\theta+\omega t)}. \tag{6}$$

In Eqs. (6) k is the axial wavenumber, m is the circumferential wavenumber and ω is the circular frequency. In Eq. (5) shape functions are spatial functions and the nodal variables are temporal functions and it follows that Eqs. (6) can be rewritten as follows (dropping the functional notation):

$$u_i = [N_u]\,\mathrm{e}^{\mathrm{i}(kz+m\theta)}\{U_i\}\,\mathrm{e}^{\mathrm{i}\omega t}, \quad \varphi = [N_\varphi]\,\mathrm{e}^{\mathrm{i}(kz+m\theta)}\{\Psi\}\,\mathrm{e}^{\mathrm{i}\omega t}, \quad T = [N_T]\,\mathrm{e}^{\mathrm{i}(kz+m\theta)}\{\chi\}\,\mathrm{e}^{\mathrm{i}\omega t}. \tag{7}$$

Eqs. (7) are to be substituted into Eqs. (2)–(4), multiplied by a suitable weight function and integrated over the corresponding volume.

The stiffness matrix for the local dynamic finite element can be written symbolically as

$$[K] = \begin{bmatrix} +[K_{uu}] & +[K_{u\phi}] & -[K_{uT}] \\ +[K_{u\phi}^{\mathrm{T}}] & -[K_{\phi\phi}] & +[K_{\phi T}] \\ -[\dot{K}_{Tu}] & +[\dot{K}_{T\phi}] & -[\dot{K}_T] - [K_{TT}] \end{bmatrix}, \tag{8}$$

where each sub-matrix corresponds to the position of terms in Eqs. (2)–(4) and a superscript T indicates the transpose. The mass matrix of Eqs. (2) will be dealt with later.

The sub-matrix corresponding to mechanical displacements can be written in terms of the so-called $[B]$ matrix and subsequently in terms of an operator matrix and the shape function matrix as

$$[K_{uu}] = \int_V [B_u]^{\mathrm{T}}[C][B_u]\,\mathrm{d}V = \int_V [N_u]^{\mathrm{T}}[L_u]^{\mathrm{T}}[C][L_u][N_u]\,\mathrm{d}V. \tag{9}$$

In three dimensions $[L_u]$ is written as the sum of three matrices that separate the coordinates and their format is dependent upon the coordinate system. In cylindrical coordinates $[L_u]$ could be written

$$[L_u] = [L_{ur}] + [L_{u\theta}] + [L_{uz}], \tag{10}$$

thereby treating each coordinate direction separately and allowing for a separation of real and imaginary parts. In other words, the classification of real terms and those multiplied by m, k, i or ω or combinations thereof. Dropping the integral sign, for the time being, the stiffness matrix of Eq. (9) can be expanded as

$$[K_{uu}] = [N_u]^{\mathrm{T}}\left[[L_{ur}]^{\mathrm{T}} + [L_{u\theta}]^{\mathrm{T}} + [L_{uz}]^{\mathrm{T}}\right][C][[L_{ur}] + [L_{u\theta}] + [L_{uz}]][N_u]. \tag{11}$$

The individual matrices of $[L_u]$ corresponding to Eq. (10) and cylindrical coordinates are

$$[L_{ur}] = \begin{bmatrix} \partial/\partial r & 0 & 0 \\ 1/r & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \partial/\partial r \\ 0 & \partial/\partial r - 1/r & 0 \end{bmatrix}, \quad [L_{u\theta}] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \partial/\partial\theta & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \partial/\partial\theta \\ 0 & 0 & 0 \\ \partial/\partial\theta & 0 & 0 \end{bmatrix}, \quad [L_{uz}] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \partial/\partial z \\ 0 & \partial/\partial z & 0 \\ \partial/\partial z & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{12}$$

Visualize substituting the operator matrices of Eqs. (12) into Eqs. (11). The terms in the stiffness matrix that would contain m, k and i are identified in Eq. (13):

$$\begin{aligned}[K_{uu}] = [N_u]^{\mathrm{T}} \Bigg[& [L_{ur}]^{\mathrm{T}}[C][L_{ur}] + [L_{ur}]^{\mathrm{T}}[C]\underbrace{[L_{u\theta}]}_{\mathrm{i}m} + [L_{ur}]^{\mathrm{T}}[C]\underbrace{[L_{uz}]}_{\mathrm{i}k} \\ & + \underbrace{[L_{u\theta}]}_{\mathrm{i}m}{}^{\mathrm{T}}[C][L_{ur}] + \underbrace{[L_{u\theta}]}_{\mathrm{i}m}{}^{\mathrm{T}}[C]\underbrace{[L_{u\theta}]}_{\mathrm{i}m} + \underbrace{[L_{u\theta}]}_{\mathrm{i}m}{}^{\mathrm{T}}[C]\underbrace{[L_{uz}]}_{\mathrm{i}k} \\ & + \underbrace{[L_{uz}]}_{\mathrm{i}k}{}^{\mathrm{T}}[C][L_{ur}] + \underbrace{[L_{uz}]}_{\mathrm{i}k}{}^{\mathrm{T}}[C]\underbrace{[L_{u\theta}]}_{\mathrm{i}m} + \underbrace{[L_{uz}]}_{\mathrm{i}k}{}^{\mathrm{T}}[C]\underbrace{[L_{uz}]}_{\mathrm{i}k} \Bigg][N_u]. \end{aligned} \tag{13}$$

The diagonal terms $[K_{\phi\phi}]$ and $[K_{TT}]$ are similar, only differing in the material matrix. The form of $[K_{\phi\phi}]$ is

$$[K_{\phi\phi}] = [N_\phi]^{\mathrm{T}}\left[[L_{\phi r}]^{\mathrm{T}} + [L_{\phi\theta}]^{\mathrm{T}} + [L_{\phi z}]^{\mathrm{T}}\right][\varepsilon][[L_{\phi r}] + [L_{\phi\theta}] + [L_{\phi z}]][N_\phi], \tag{14}$$

where the operator matrices become

$$[L_\varphi] = [L_{\varphi r}] + [L_{\varphi\theta}] + [L_{\varphi z}] = \begin{bmatrix} -\dfrac{\partial}{\partial r} \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ -\dfrac{1}{r}\dfrac{\partial}{\partial\theta} \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ -\dfrac{\partial}{\partial z} \end{bmatrix}. \tag{15}$$

An equation similar to Eq. (13) can be written for $[K_{\phi\phi}]$ that identifies the terms containing im, ik, i^2m^2, i^2k^2 and i^2mk.

The sub-matrix $[K_{u\phi}]$ is computed as

$$[K_{u\phi}] = \int_V [B_u]^{\mathrm{T}}[e][B_\phi]\,\mathrm{d}V = \int_V [N_u]^{\mathrm{T}}[L_u]^{\mathrm{T}}[e][L_\phi][N_\phi]\,\mathrm{d}V. \tag{16}$$

Again, omitting the integral sign and combining Eqs. (12) and (15) the form of $[K_{u\phi}]$ is

$$[K_{u\phi}] = \left[[N_u]^{\mathrm{T}}[[L_{ur}]^{\mathrm{T}} + [L_{u\theta}]^{\mathrm{T}} + [L_{uz}]^{\mathrm{T}}\right][e][[L_{\phi r}] + [L_{\phi\theta}] + [L_{\phi z}]][N_\phi] \tag{17}$$

and the terms containing im, ik, i^2m^2, i^2k^2 and i^2mk can be separated as in Eq. (13). The stiffness matrices that have been developed thus far can be used to formulate the analysis that was presented by Siao et al. [3]. The remaining stiffness matrices of Eq. (8) are necessary for the thermopiezoelectric vibration problem.

As mentioned previously the sub-matrix $[K_{TT}]$ is identical with $[K_{\phi\phi}]$ with the permittivity matrix $[\varepsilon]$ replaced with the thermal conductivity $[k]$. The sub-matrix $[K_{uT}]$ of Eq. (8) is

$$[K_{uT}] = \int_V [B_u]^{\mathrm{T}}[\beta][N_T]\,\mathrm{d}V = \int_V [N_u]^{\mathrm{T}}[L_u]^{\mathrm{T}}[\beta][N_T]\,\mathrm{d}V. \tag{18}$$

Use Eq. (12) to arrive at the following representation, again, dropping the integral sign:

$$[K_{uT}] = [N_u]^{\mathrm{T}}\left[[L_{ur}]^{\mathrm{T}} + [L_{u\theta}]^{\mathrm{T}} + [L_{uz}]^{\mathrm{T}}\right][\beta][N_\phi] \tag{19}$$

or

$$[K_{uT}] = [N_u]^{\mathrm{T}}\left[[L_{ur}]^{\mathrm{T}}[\beta][N_T] + \underbrace{[L_{u\theta}]}_{\mathrm{i}m}{}^{\mathrm{T}}[\beta][N_T] + \underbrace{[L_{uz}]}_{\mathrm{i}k}{}^{\mathrm{T}}[\beta][N_T]\right]. \tag{20}$$

It follows that $[K_{uT}]$ has only real terms and terms containing im and ik. The sub-matrix $[K_{\phi T}]$ is similar to $[K_{uT}]$ and becomes

$$[K_{\phi T}] = [N_\phi]^{\mathrm{T}}\left[[L_{ur}]^{\mathrm{T}}[P][N_T] + \underbrace{[L_{u\theta}]}_{\mathrm{i}m}{}^{\mathrm{T}}[P][N_T] + \underbrace{[L_{uz}]}_{\mathrm{i}k}{}^{\mathrm{T}}[P][N_T]\right]. \tag{21}$$

The remaining sub-matrices contain time derivatives and referring to Eq. (7) can be formulated as

$$[\dot{K}_{Tu}] = [N_T]^{\mathrm{T}}[\beta]\left[\underbrace{[L_{ur}]}_{\mathrm{i}\omega} + \underbrace{[L_{u\theta}]}_{\mathrm{i}m\mathrm{i}\omega} + \underbrace{[L_{uz}]}_{\mathrm{i}k\mathrm{i}\omega}\right][N_u], \tag{22}$$

$$[\dot{K}_{T\phi}] = [N_T]^{\mathrm{T}}[P]\left[\underbrace{[L_{\phi r}]}_{\mathrm{i}\omega} + \underbrace{[L_{\phi\theta}]}_{\mathrm{i}m\mathrm{i}\omega} + \underbrace{[L_{\phi z}]}_{\mathrm{i}k\mathrm{i}\omega}\right][N_\phi], \tag{23}$$

where each contribution to the total stiffness matrix contains an imaginary part iω and real parts $\mathrm{i}^2 m\omega$ and $\mathrm{i}^2 k\omega$. Eqs. (22) and (23) are the transpose of Eqs. (20) and (21) with the exception that they are multiplied by specific terms containing ω. The last sub-matrix of Eq. (8) to be evaluated is

$$[\dot{K}_T] = [N_T]^{\mathrm{T}}[a][N_T](\mathrm{i}\omega). \tag{24}$$

The right-hand side of Eq. (2) will be made up of the mass matrix and shape functions. Substituting Eq. (7) will lead to an expression of the form

$$-\omega^2\int_V [N_u]^{\mathrm{T}}[\rho][N_u]\,\mathrm{d}V, \quad \text{where } [M] = \int_V [N_u]^{\mathrm{T}}[\rho][N_u]\,\mathrm{d}V \tag{25}$$

is the mass matrix.

4. Eigenvalue problem

The total stiffness matrix $[K]$ of Eq. (8) can be written as the sum of nine individual matrices. Following Ref. [3] visualize Eqs. (13), (14), (17), (20), (21) and (23) grouped as in Eq. (8) and written as

$$[K] = [K_1] + [K_2]\mathrm{i}m + [K_3]\mathrm{i}k + [K_4]\mathrm{i}^2m^2 + [K_5]\mathrm{i}^2mk + [K_6]\mathrm{i}^2k^2 + [K_7]\mathrm{i}\omega + [K_8]\mathrm{i}^2m\omega + [K_9]\mathrm{i}^2k\omega.$$

It follows that $[K]$ can be combined into two matrices, one real and one imaginary:

$$[K_R] = [K_1] + [K_4]\mathrm{i}^2m^2 + [K_5]\mathrm{i}^2mk + [K_6]\mathrm{i}^2k^2 + [K_8]\mathrm{i}^2m\omega + [K_9]\mathrm{i}^2k\omega, \tag{26}$$

$$[K_I] = [K_2]m + [K_3]k + [K_3]\omega. \tag{27}$$

The stiffness matrix becomes

$$[K] = [K_R] + \mathrm{i}[K_I]. \tag{28}$$

The finite element model for Eqs. (2)–(4) becomes

$$[K_R]\{U\} + [K_I]\mathrm{i}\{U\} = \omega^2[M]\{U\}, \tag{29}$$

where $\{U\}$ is a vector that represents the nodal variables of Eq. (7) and the mass density matrix of Eq. (25) can be written as

$$[\rho] = \begin{bmatrix} \rho & 0 & 0 & 0 & 0 \\ 0 & \rho & 0 & 0 & 0 \\ 0 & 0 & \rho & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \tag{30}$$

The eigenvalue problem defined by Eq. (29) requires that m be defined as an integer. A real value can be assigned to k and ω is the eigenvalue or real values can be assigned to ω and k is the eigenvalue. In either case the result differs from Ref. [3] since either assumption gives a quadratic eigenvalue problem. The analysis methods discussed in Ref. [3] are pertinent for continuing toward an eigenvalue solution.

5. Material matrices

A brief discussion of material matrices is in order. Fundamental concepts are given by Nye [8]. A paper by Tauchert and Ashida [9] gives an analytical static piezothermoelectric analysis using cadmium selenide as a representative material. Cadmium selenide is of material class 6 mm and referring to Nye [8] the three-dimensional material matrices are

$$[C] = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ C_{12} & C_{11} & C_{13} & 0 & 0 & 0 \\ C_{13} & C_{13} & C_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{66} \end{bmatrix}, \quad [e] = \begin{bmatrix} 0 & 0 & e_{31} \\ 0 & 0 & e_{31} \\ 0 & 0 & e_{33} \\ 0 & e_{15} & 0 \\ e_{15} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad [\beta] = \begin{bmatrix} \beta_{11} \\ \beta_{11} \\ \beta_{33} \\ 0 \\ 0 \\ 0 \end{bmatrix},$$
$$[\varepsilon] = \begin{bmatrix} \varepsilon_{11} & 0 & 0 \\ 0 & \varepsilon_{11} & 0 \\ 0 & 0 & \varepsilon_{33} \end{bmatrix}, \quad [P] = \begin{bmatrix} 0 \\ 0 \\ P_{33} \end{bmatrix}, \quad a = \rho C_v / T_o. \tag{31}$$

6. Conclusions

A derivation of the finite element equations for vibration of a thermopiezoelectric solid has been presented using the Galerkin method. A complete harmonic solution is assumed and the equations are specialized for the case of an infinite cylinder. The derivation that follows separates the resulting finite element formulation into real and imaginary parts for a final eigenvalue analysis. Typical material parameters are discussed.

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