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Journal of Sound and Vibration 294 (2006) 368–373
JOURNAL OF SOUND AND VIBRATION
www.elsevier.com/locate/jsvi

Short Communication

Dispersion of torsional waves in a thick-walled transversely isotropic circular cylinder of infinite length

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Received 28 July 2004; received in revised form 27 October 2005; accepted 27 October 2005
Available online 27 December 2005

Abstract

The dispersion properties of torsional waves of a thick-walled transversely isotropic circular cylinder of infinite length are investigated. The assumed waves are axisymmetric and propagate along the x-axis of the cylinder. The wall of the cylinder consists of a transversely isotropic material with the axis of isotropy parallel to the x-axis of the cylinder. The dispersion curves for both the phase and group velocities are determined and graphically illustrated. The amplitudes of the tangential displacements for some dispersion branches are shown. The purpose of the paper is to show the present technique is useful in solving some energy flow problems when torsional waves travel along arbitrarily laminated cylinders with various kinds of anisotropy. It seems that by controlling the anisotropy of a shell-like structure, the optimisation of amplitude and energy flow distributions in different parts of the structure can be achieved.

1. Introduction

Many scientists have dealt with propagation of elastic waves in isotropic and anisotropic hollow cylinders. A classification of the possible free wave motions in a three-layered composite thick-walled cylinder has been presented by Markuš and Mead [1]. Both the inner and outer layers were composed of transversely isotropic material. The middle layer was made of an isotropic rubbery material. To obtain correct and closed form solutions for the propagating waves, Bessel functions and special Frobenius series were adopted. Numerical results have been given in the form of dispersion curves.

Kudlička [2] considered an arbitrarily laminated orthotropic cylindrical thick-walled pipe of infinite length in order to obtain dispersion curves for an axisymmetric problem. A solution by means of finite exponential expansions was found. The first five dispersion curves and relative amplitudes of axial and radial displacements for axisymmetric elastic waves propagating along the x-axis of the boron–epoxy pipe were determined.

Kudlička [3] has also analysed an energy flow density problem, when axisymmetric elastic waves propagation was allowed along the axis of a thick-walled three-layered (transtropic–isotropic–transtropic) shell. The results for some wavenumbers and phase velocities belonging to the basic set of dispersion curves were compared with those obtained for an isotropic epoxy cylinder with the same geometry as the anisotropic one.

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0022-460X/$ - see front matter
doi:10.1016/j.jsv.2005.10.011

Brepta and Prokopec [4] have described the problem of the propagation of rotary and axially symmetric longitudinal, transverse (bending) and torsional waves in an isotropic cylindrical rod of infinite length. They have used Pochhammer's equation of motion in cylindrical coordinates introducing propagating wave-forms for a given circular frequency.

The wall of the shell to be considered in this paper consists of a transversely isotropic GFRP (glass fibres, resin polyester) layer, where the plane of symmetry is created by axial and tangential coordinates. The dispersion curves for both the phase and group velocities are determined and graphically illustrated. The amplitudes of the tangential displacements for some dispersion branches are also presented.

2. Dispersion and propagation of torsional waves

In the cylindrical coordinate system (Fig. 1), axial, tangential and radial coordinates are in the x, θ, r directions. For torsional waves, the displacements u_x and u_r are zero. The tangential displacements u_θ are not dependent on the angle θ, so $\partial u_\theta/\partial\theta = 0$. A harmonic torsional wave can be sought in the form:

$$u_\theta = V(r)\,\cos[K(x-ct)], \tag{1}$$

where $V(r)$ is the wave function depending upon the r coordinate only, $K = 2\pi/\lambda$ is the wavenumber, λ is the wavelength, c is the phase velocity of the wave, t is time.

The generalised Hooke's law equations (the constitutive equations) are given in matrix form:

$$\begin{bmatrix}\sigma_x\\ \sigma_\theta\\ \sigma_r\\ \tau_{\theta r}\\ \tau_{rx}\\ \tau_{x\theta}\end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0\\ C_{12} & C_{22} & C_{23} & 0 & 0 & 0\\ C_{13} & C_{23} & C_{33} & 0 & 0 & 0\\ 0 & 0 & 0 & C_{44} & 0 & 0\\ 0 & 0 & 0 & 0 & C_{55} & 0\\ 0 & 0 & 0 & 0 & 0 & C_{66}\end{bmatrix} \begin{bmatrix}\varepsilon_x\\ \varepsilon_\theta\\ \varepsilon_r\\ \gamma_{\theta r}\\ \gamma_{rx}\\ \gamma_{x\theta}\end{bmatrix}, \tag{2}$$

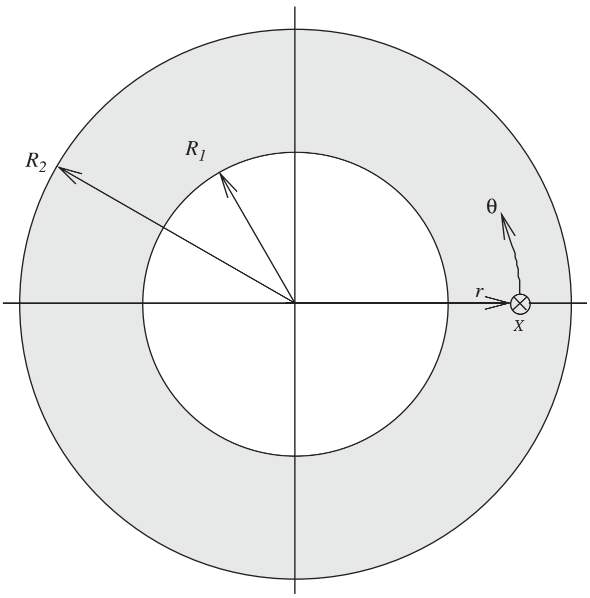

Fig. 1. The geometry and coordinate system.

where $\sigma_x, \sigma_\theta, \sigma_r, \tau_{\theta r}, \tau_{rx}, \tau_{x\theta}$ are the stresses, $\varepsilon_x, \varepsilon_\theta, \varepsilon_r, \gamma_{\theta r}, \gamma_{rx}, \gamma_{x\theta}$ the strains, and the elements C_{ij} of the matrix are the elastic moduli of the material of the cylinder. The strain–displacement relations are:

$$\varepsilon_x = \frac{\partial u_x}{\partial x} = 0, \quad \varepsilon_\theta = \frac{1}{r}\frac{\partial u_\theta}{\partial r} + \frac{u_r}{r} = 0, \quad \varepsilon_r = \frac{\partial u_r}{\partial r} = 0,$$
$$\gamma_{\theta r} = \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r} \neq 0, \quad \gamma_{rx} = \frac{\partial u_x}{\partial r} + \frac{\partial u_r}{\partial x} = 0, \quad \gamma_{x\theta} = \frac{\partial u_\theta}{\partial x} \neq 0. \tag{3}$$

The differential equations of wave motion for the displacements u_x and u_r are satisfied identically. The displacement u_θ is governed by the equation:

$$\frac{\partial \tau_{\theta r}}{\partial r} + \frac{\partial \tau_{x\theta}}{\partial x} + \frac{2\tau_{\theta r}}{r} = \rho \frac{\partial^2 u_\theta}{\partial t^2}, \tag{4}$$

where ρ is the density of the material.

Substituting Eqs. (1)–(3) into Eq. (4), one obtains for the function $V(r)$:

$$\frac{\mathrm{d}^2 V(r)}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{d}V(r)}{\mathrm{d}r} + \left(k'^2 - \frac{1}{r^2}\right) V(r) = 0, \tag{5}$$

where

$$k'^2 = \left(\frac{\rho c^2}{C_{44}} - \frac{C_{66}}{C_{44}}\right) K^2. \tag{6}$$

The general solution of Eq. (5) can be assumed in the form:

$$V(r) = C_1 J_1(k'r) + C_2 Y_1(k'r), \tag{7}$$

where functions $J_1(k'r)$ and $Y_1(k'r)$ are Bessel functions of the first and second kind with argument $k'r$; C_1 and C_2 are constants. The solution of Eq. (5) must satisfy the boundary conditions on the inner and outer surface of the cylinder. The conditions $\sigma_r = 0$ and $\tau_{rx} = 0$ are satisfied automatically in accordance with Eqs. (2) and (3). From the condition $\tau_{\theta r} = 0$ one gets the relation to be fulfilled on the boundaries in the following form:

$$r \frac{\partial}{\partial r}\left(\frac{u_\theta}{r}\right) = 0, \tag{8}$$

for $r = R_1$ and $r = R_2$, where R_1 and R_2 are the inner and outer radii, respectively.

After substituting Eqs. (1) and (7) into Eq. (8) and executing several rearrangements, we obtain

$$\begin{aligned} k'R_1[J_0(k'R_1) + CY_0(k'R_1)] - 2[J_1(k'R_1) + CY_1(k'R_1)] = 0, \\ k'R_2[J_0(k'R_2) + CY_0(k'R_2)] - 2[J_1(k'R_2) + CY_1(k'R_2)] = 0, \end{aligned} \tag{9}$$

where $C = C_2/C_1$. From Eq. (9), the constant C is given by

$$C = -\frac{k'R_1 J_0(k'R_1) - 2J_1(k'R_1)}{k'R_1 Y_0(k'R_1) - 2Y_1(k'R_1)}. \tag{10}$$

Introducing C into the second Eq. of (9) we finally get:

$$k'R_2 J_0(k'R_2) - 2J_1(k'R_2) - \frac{k'R_1 J_0(k'R_1) - 2J_1(k'R_1)}{k'R_1 Y_0(k'R_1) - 2Y_1(k'R_1)} \times [k'R_2 Y_0(k'R_2) - 2Y_1(k'R_2)] = 0. \tag{11}$$

The correct values of k' are found by employing a root-finding procedure (using the software package Mathematica [5]). In this way, it is possible to gain as many solutions as are desired with the required accuracy.

If $c_2 = \sqrt{C_{44}/\rho}$ is the velocity of a simple shear wave with an amplitude in the plane (θ, r) or (r, x), and

$$K' = \frac{R_1}{\lambda}, \quad c' = \frac{c}{c_2}, \tag{12}$$

are the non-dimensional wavenumber and phase velocity of the torsional wave, then c' is found as a function of K' from Eq. (6):

$$c' = \sqrt{\left(\frac{k'R_1}{2\pi K'}\right)^2 + \frac{C_{66}}{C_{44}}}. \tag{13}$$

For any solution k', one dispersion branch is gained. For long waves $K' \to 0$ and then $c' \to \infty$. For very short waves $K' \to \infty$ and then $c' \to \sqrt{C_{66}/C_{44}}$.

A group velocity c_g satisfies the equation

$$c_g = c - \lambda \frac{\mathrm{d}c}{\mathrm{d}\lambda}. \tag{14}$$

Let us denote $c'_g = c_g/c_2$ which denotes the non-dimensional group velocity. Using Eqs. (12) and (14), we obtain

$$c'_g = c' - \frac{1}{c'}\left(\frac{k'R_1}{2\pi K'}\right)^2. \tag{15}$$

To obtain the exact expression for the function $V(r)$, we must first use Eq. (7) to get the constants C_1 and C_2. Let the function $V(r)$ be normalised in relation to the radius R_1 such that:

$$V(R_1) = C_1 J_1(k'R_1) + C_2 Y_1(k'R_1) = 1. \tag{16}$$

Since $C = C_2/C_1$ we get

$$C_1 = \frac{1}{J_1(k'R_1) + CY_1(k'R_1)}. \tag{17}$$

Following the above procedure, the normalised functions of $V(r)$ can be determined from Eq. (7) for arbitrary k' as a function of r.

3. Numerical results

The following data were used for the computational analysis: Ratio of outer radius to the inner radius of the shell was $R_2/R_1 = 2$. The mechanical properties of the chosen GFRP composite were: in-plane Young's moduli $E_x = E_\theta = 15.65\,\mathrm{GPa}$, transverse Young's modulus $E_r = 7.7\,\mathrm{GPa}$, shear modulus $G_{rx} = 5.0\,\mathrm{GPa}$, Poisson ratios $v_{x\theta} = 0.310$, $v_{rx} = 0.315$, $v_{\theta r} = 0.215$, density $\rho = 1576\,\mathrm{kg\,m^{-3}}$.

The mechanical properties for the transversely isotropic GFRP cylinder yield the following elastic matrix:

$$\mathbf{C} = \begin{bmatrix} 21 & 9 & 7 & 0 & 0 & 0 \\ 9 & 21 & 7 & 0 & 0 & 0 \\ 7 & 7 & 10 & 0 & 0 & 0 \\ 0 & 0 & 0 & 5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 6 \end{bmatrix} \mathrm{GPa}.$$

The first three solutions of Eq. (11) for the parameter k' are 3.407, 6.428 and 9.523. Using Eq. (13) the phase velocity c' was computed for different wavenumbers K' within the interval $(0; 3\rangle$. The three lowest dispersion curves are shown in Fig. 2. The non-dimensional velocities are plotted with thin unbroken lines and represent simple shear waves with amplitudes of motion in the plane of (θ, r) or (r, x) given by $c' = \sqrt{C_{66}/C_{44}} = \sqrt{6\,\mathrm{GPa}/5\,\mathrm{GPa}} = 1.2$. The dependence of phase velocities c' upon parameter k' and wavenumbers K' are clearly illustrated. The same features for group velocities c'_g (three increasing dotted curves) are worth noticing.

Normalised values of the tangential displacements $V(r)$ for the first three wave-forms are shown in Fig. 3. The solid, dashed and dash–dotted lines show the amplitudes of tangential displacements for the iterated values of k'. The integration constants C_1 and C_2 for corresponding k', calculated from Eq. (17),

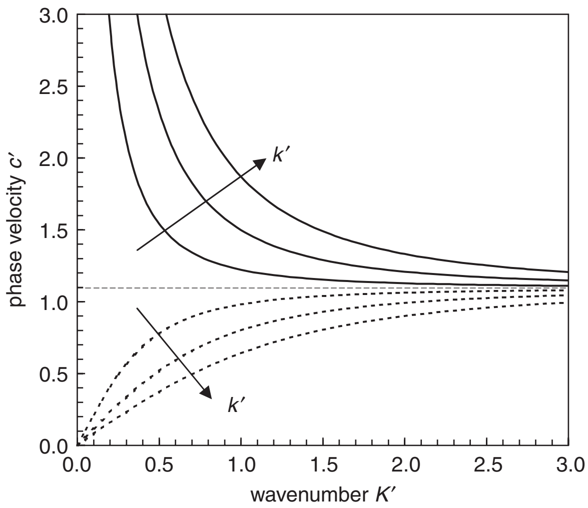

Fig. 2. First three dispersion curves for phase and group velocities. Solid lines (from bottom up): Phase velocities for $k' = 3.407$, 6.428, 9.523. Dotted lines (from top down): Group velocities for the same k'. Thin line: The velocity of a transverse wave.

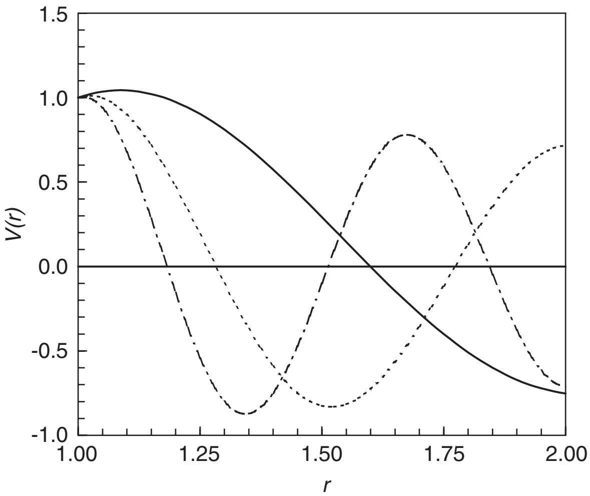

Fig. 3. Normalised amplitudes of tangential displacements for torsional waves. Solid line: $k' = 3.407$; dashed line: $k' = 6.428$; dash–dotted line: $k' = 9.523$.

Table 1
Constants of Eq. (7)

1	2	3	4	5
N	k'	c'	C_1	C_2
1	3.407	1.222	−0.048	2.510
2	6.428	1.499	−1.116	−3.054
3	9.523	1.870	1.842	3.447

are presented in Table 1. As can be seen in Fig. 3, vibration nodes $[V(r) = 0]$ are found at the following radii: 1.60 (first branch), 1.27 and 1.77 (second branch), 1.17, 1.52 and 1.85 (third branch) of the dispersion curve.

4. Conclusions

Some important conclusions can be drawn from the study:

1. The analysis has shown under what conditions pure torsional waves can propagate in a thick shell-like structure with defined transverse isotropy. Typical dispersion effects take place similar to those in a cylinder made from isotropic material.
2. The dispersion of torsional waves is strongly dependent on shear elastic constants, i.e. C_{66}/C_{44} (see Eq. (13)). For very large values of wavenumbers (K'), phase velocities reach asymptotic values, approaching the value $\sqrt{C_{66}/C_{44}}$. This feature suggests the possibility of tailoring structures when energy flow or vibration control problems are involved.
3. Further, we conclude that the phase velocity of long waves increases indefinitely as the long wave values of K' approach zero. On the other hand, the group velocity always remains finite, approaching zero as $K' \to 0$ and approaching an asymptotic value of $\sqrt{C_{66}/C_{44}}$ as $K' \to \infty$.
4. Finally the study has proved that for a wide range of engineering materials such as reinforced composites, wood, molybdenum, barites, etc. (all of which possess special planes of elastic symmetry) a powerful computational procedure exists for analysis of their dynamical processes.

Acknowledgements

The author gratefully acknowledges the Grant Agency of the Slovak Academy of Sciences for supporting this work under Grant No. 2/3205/23.

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