Available online at www.sciencedirect.com
SCIENCE DIRECT®
Journal of Sound and Vibration 294 (2006) 388–396
JOURNAL OF SOUND AND VIBRATION
www.elsevier.com/locate/jsvi

Short Communication

On the eigencharacteristics of an axially vibrating viscoelastic rod carrying a tip mass and its representation by a single degree-of-freedom system

M. Gürgöze*, S. Zeren

Faculty of Mechanical Engineering, Technical University of Istanbul, 34439 Gümüşsuyu, Istanbul, Turkey

Received 19 April 2005; received in revised form 13 September 2005; accepted 10 November 2005
Available online 24 January 2006

Abstract

The present note deals with the derivation of the characteristic equation of an axially vibrating viscoelastic rod (Kelvin–Voigt model), carrying a tip mass. Further, it is attempted to represent this continuous system by an ''equivalent'' spring-damper-mass system. Then, the ''first'' eigenvalues of these systems are calculated and tabulated for a wide range of the nondimensional mass parameter.

1. Introduction

Recently, in the context of a project study, the need arose to obtain the eigenvalues of an axially vibrating viscoelastic rod (Kelvin–Voigt model) carrying a tip mass. It has been necessary to establish and solve the characteristic equation of the system under consideration, after a futile search for it and its solution in the literature, even hoping to find a table of eigenvalues for particular ranges of dimensionless values of the mass and the damping parameters.

After having derived and solved the characteristic equation, it is attempted to represent the system mentioned above by an equivalent spring-damper-mass system. The present note presents some results of these efforts.

Although it is acknowledged that the contribution of this study does not solve a very complex problem, it is nevertheless thought that the characteristic equation established and the numerical results collected in tables can be helpful to design engineers working in this field.

2. Theory

The mechanical system under consideration is shown in Fig. 1. It consists of an axially vibrating viscoelastic rod carrying a tip mass M. It is assumed that its viscoelastic properties fit the Kelvin–Voigt model. The axial

*Corresponding author. Fax: +90 212 245 07 95.
E-mail address: gurgozem@itu.edu.tr (M. Gürgöze).

0022-460X/$ - see front matter
doi:10.1016/j.jsv.2005.11.006

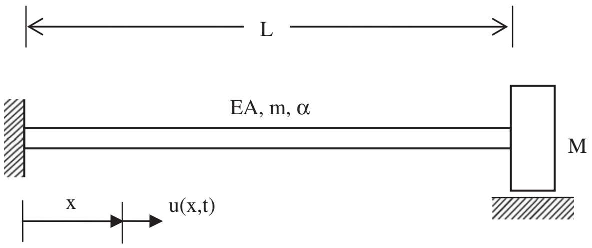

Fig. 1. Axially vibrating viscoelastic rod with a tip mass.

rigidity, length, mass per unit length and viscoelastic constant of the rod material are EA, L, m and α, respectively.

Equation of motion of the viscoelastic rod can be found in the literature [1] as

$$EA\,u''(x,t) + \alpha A u''^{\bullet}(x,t) - m\ddot{u}(x,t) = 0, \tag{1}$$

where $u(x,t)$ represents the axial displacement at the location x and t is time. Primes and dots denote partial derivatives with respect to position coordinate x and time t, as usual. The corresponding boundary conditions are shown to be

$$u(0,t) = 0, \tag{2}$$

$$EA\,u'(L,t) + \alpha A u''^{\bullet}(L,t) + M\ddot{u}(L,t) = 0, \tag{3}$$

where the first is obvious and the second expresses the force balance at the tip.

Assuming a solution in the form

$$u(x,t) = U(x)e^{\lambda t}, \tag{4}$$

where $U(x)$ and λ denote the amplitude function and an eigenvalue, both being complex in general, and substitution into Eqs. (1)–(3) leads to the following characteristic equation:

$$\bar{\lambda}(e^{\bar{\lambda}L} + e^{-\bar{\lambda}L}) + \frac{M\lambda^2}{EA + \alpha A\lambda}(e^{\bar{\lambda}L} - e^{-\bar{\lambda}L}) = 0, \tag{5}$$

where

$$\bar{\lambda}^2 = m\lambda^2/(EA + \alpha A\lambda). \tag{6}$$

The characteristic equation in Eq. (5) can be written as

$$e^{\bar{\bar{\lambda}}} + e^{-\bar{\bar{\lambda}}} + \alpha_M\bar{\bar{\lambda}}(e^{\bar{\bar{\lambda}}} - e^{-\bar{\bar{\lambda}}}) = 0, \tag{7}$$

where

$$\bar{\bar{\lambda}} = \bar{\lambda}L, \tag{8}$$

$$\alpha_M = M/mL \tag{9}$$

are introduced.

The characteristic equation above can further be brought into one of the three alternative forms given below:

$$\cosh\bar{\bar{\lambda}} + \alpha_M\bar{\bar{\lambda}}\sinh\bar{\bar{\lambda}} = 0, \tag{10}$$

$$1 + e^{2\bar{\bar{\lambda}}} + \alpha_M\bar{\bar{\lambda}}(e^{2\bar{\bar{\lambda}}} - 1) = 0, \tag{11}$$

$$1 + \alpha_M\bar{\bar{\lambda}}\tanh\bar{\bar{\lambda}} = 0. \tag{12}$$

After having obtained the parameter $\bar{\bar{\lambda}}$ by solving numerically one of the characteristic equations above, in order to obtain the eigenvalue λ, relationship (6) has to be used in conjunction with Eq. (8), which leads to the following quadratic equation for $\lambda/\bar{\omega}_0$:

$$(\lambda/\bar{\omega}_0)^2 - \bar{\alpha}\bar{\bar{\lambda}}^2(\lambda/\bar{\omega}_0) - \bar{\bar{\lambda}}^2 = 0. \tag{13}$$

Here,

$$\bar{\omega}_0^2 = EA/mL^2, \tag{14}$$

$$\bar{\alpha} = \alpha A/(mL^2\bar{\omega}_0) \tag{15}$$

are introduced where $\bar{\alpha}$ represents the nondimensional viscoelastic constant of the rod, in short, the nondimensional damping parameter.

The solution of the quadratic equation in Eq. (13) yields the nondimensional eigenvalue of the vibrational system as

$$(\lambda/\bar{\omega}_0)_{1,2} = \frac{\bar{\alpha}\bar{\bar{\lambda}}^2}{2} \pm \sqrt{\frac{\bar{\alpha}^2\bar{\bar{\lambda}}^4}{4} + \bar{\bar{\lambda}}^2}. \tag{16}$$

As a verification of the characteristic equations obtained, it is in order to consider the special case $\bar{\alpha} = 0$, i.e., the undamped rod-case. In the undamped case, it is reasonable to expect the eigenvalue λ in the form

$$\lambda = \pm \mathrm{i}\omega, \tag{17}$$

where ω represents the eigenfrequency of the undamped system and i denotes the imaginary unit, as usual. In this case, using Eqs. (6) and (8), it can be shown that the parameter $\bar{\bar{\lambda}}$ reduces to

$$\bar{\bar{\lambda}} = \mathrm{i}\bar{\beta}, \tag{18}$$

where the nondimensional frequency parameter $\bar{\beta}$ is introduced via

$$\bar{\beta}^2 = \omega^2 mL^2/EA. \tag{19}$$

Substitution of Eq. (18) into Eq. (10) gives the frequency equation in the following form:

$$\cosh(\mathrm{i}\bar{\beta}) + \alpha_M(\mathrm{i}\bar{\beta})\sinh(\mathrm{i}\bar{\beta}) = 0. \tag{20}$$

Making use of the formulas [2]

$$\cosh(i\,y) = \cos\,y,\ \sinh(i\,y) = i\,\sin\,y$$

results in the well-known frequency equation of an axially vibrating elastic rod, carrying a tip mass [3]

$$\tan\,\bar{\beta} = \frac{1}{\alpha_M\bar{\beta}}, \tag{21}$$

which justifies formula (16) with Eqs. (10)–(12), at least for the special case of $\bar{\alpha} = 0$.

As in Refs. [4,5] one can think of representing the vibrational system in Fig. 1 by an "equivalent" single degree-of-freedom spring-damper-mass system. It is reasonable to make use of the simplified model in Fig. 2 for this purpose, where

$$k = EA/L,\ \ d = \alpha A/L. \tag{22}$$

δ, is the ratio of the mass to be added to the tip, to the mass of the rod itself.

The constant δ is not yet known and will be determined requiring that the "first" eigenvalue of the continuous system in Fig. 1 is equal to the eigenvalue of the model in Fig. 2. Before proceeding further, it is in order to represent the system in Fig. 2 in terms of the nondimensional parameters as in Fig. 3, where α_M and $\bar{\alpha}$ were already defined in Eqs. (9) and (15), respectively.

The characteristic equation of the model in Fig. 3 is simply

$$(\alpha_M + \delta)\lambda''^2 + \bar{\alpha}\lambda'' + 1 = 0, \tag{23}$$

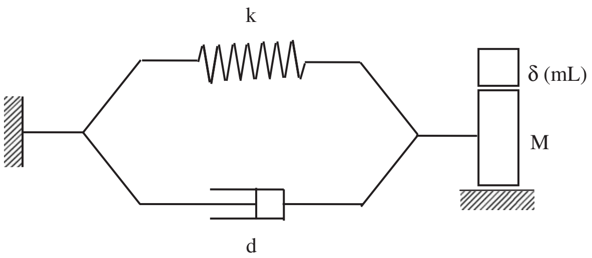

Fig. 2. Equivalent spring-damper-mass system for obtaining the "first" eigenvalue of the system in Fig. 1.

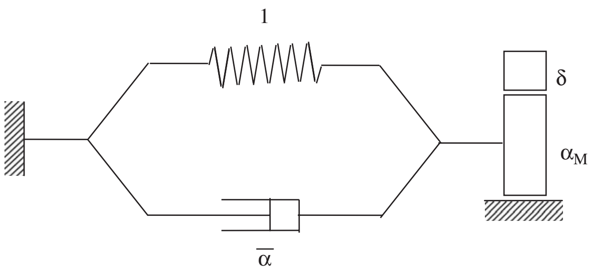

Fig. 3. Nondimensionalized version of the system in Fig. 2.

which gives the solution

$$\lambda'' = \frac{-\bar{\alpha} \pm \sqrt{\bar{\alpha}^2 - 4(\alpha_M + \delta)}}{2(\alpha_M + \delta)}. \tag{24}$$

Requiring

$$\lambda'' = \lambda', \tag{25}$$

where the dimensionless "first" eigenvalue of the continuous system, i.e., $\lambda/\bar{\omega}_0$ in Eq. (16) is denoted shortly as λ', leads to

$$\alpha_M + \delta = -\frac{1 + \bar{\alpha}\lambda'}{\lambda'^2}. \tag{26}$$

Numerical evaluations reveal that the right side is independent of $\bar{\alpha}$ such that it can be set $\bar{\alpha} = 0$, corresponding to the undamped case. On the other hand, in the undamped case, one sees from Eqs. (16) and (18) that

$$\lambda' = \lambda/\bar{\omega}_0 = \pm\bar{\bar{\bar{\lambda}}} = \pm \mathrm{i}\bar{\beta}_1, \tag{27}$$

where $\bar{\beta}_1$ is to be determined by solving Eq. (21) numerically. Hence, Eq. (26) can be reformulated as

$$\delta = \frac{1}{\bar{\beta}_1^2} - \alpha_M. \tag{28}$$

In conjunction with Fig. 3, expression (28) can be interpreted in the manner that the factor δ by which the own mass of the axially vibrating rod must be multiplied in order to be taken into account is the same as in the undamped case. $\bar{\beta}_1$ denotes the first root of Eq. (21) with respect to $\bar{\beta}$, for the corresponding α_M value.

Making use of the approach in Ref. [4] for larger α_M values, the first root of Eq. (21) can be approximated by

$$\bar{\beta}_1 \approx 1/\sqrt{\alpha_M + 1/3}, \tag{29}$$

which, when substituted into Eq. (28) yields

$$\delta = 1/3. \tag{30}$$

Let us return to the eigenvalue pair given in Eq. (24) which can be rewritten as

$$\lambda'' = \frac{-\bar{\alpha}}{2(\alpha_M + \delta)} \pm \mathrm{i}\,\frac{\sqrt{4(\alpha_M + \delta) - \bar{\alpha}^2}}{2(\alpha_M + \delta)}. \tag{31}$$

Recalling the fact noticed previously, that δ does not depend on $\bar{\alpha}$, it is seen from above that the real part of the eigenvalues of the damped system depends upon $\bar{\alpha}$ linearly, whereas the imaginary part decreases, as $\bar{\alpha}$ gets larger.

3. Numerical evaluations

In Table 1, for a wide range of the nondimensional mass parameter α_M, the corresponding $\bar{\bar{\lambda}}$ values are listed which are the ‘‘first’’ solutions of the transcendental equations (10)–(12). For the sake of completeness and also for the benefit of design engineers working in this field, in Table 2, the first dimensionless eigenfrequency

Table 1
The ‘‘first’’ roots of the transcendental equations (10)–(12) for a wide range of the nondimensional mass parameter α_M

α_M	$\bar{\bar{\lambda}}$
0	±1.570796i
0.001	±1.569227i
0.002	±1.567661i
0.003	±1.566098i
0.004	±1.564538i
0.005	±1.562982i
0.006	±1.561428i
0.007	±1.559878i
0.008	±1.558330i
0.009	±1.556786i
0.01	±1.555245i
0.02	±1.540006i
0.03	±1.525076i
0.04	±1.510452i
0.05	±1.496129i
0.06	±1.482103i
0.07	±1.468370i
0.08	±1.454924i
0.09	±1.441759i
0.1	±1.428870i
0.2	±1.313838i
0.3	±1.219952i
0.4	±1.142227i
0.5	±1.076874i
0.6	±1.021114i
0.7	±0.972911i
0.8	±0.930757i
0.9	±0.893519i
1	±0.860334i
2	±0.653271i
3	±0.547161i
4	±0.480094i
5	±0.432841i
6	±0.397248i
7	±0.369197i
8	±0.346354i

Table 1 (*continued*)

α_M	$\bar{\bar{\lambda}}$
9	$\pm 0.327285i$
10	$\pm 0.311053i$
15	$\pm 0.255365i$
20	$\pm 0.221760i$
25	$\pm 0.198676i$
30	$\pm 0.181566i$
35	$\pm 0.168230i$
40	$\pm 0.157458i$
45	$\pm 0.148521i$
50	$\pm 0.140952i$
55	$\pm 0.134433i$
60	$\pm 0.128742i$
65	$\pm 0.123718i$
70	$\pm 0.119239i$
75	$\pm 0.115214i$
80	$\pm 0.111571i$
85	$\pm 0.108253i$
90	$\pm 0.105214i$
95	$\pm 0.102418i$
100	$\pm 0.099834i$
200	$\pm 0.070652i$
300	$\pm 0.057703i$
400	$\pm 0.049979i$
500	$\pm 0.044706i$
600	$\pm 0.040813i$
700	$\pm 0.037787i$
800	$\pm 0.035348i$
900	$\pm 0.033327i$
1000	$\pm 0.031618i$

Table 2
The first roots of transcendental equation (21) for the same range of α_M as in Table 1

α_M	$\bar{\beta}_1$
0	1.570796
0.001	1.569227
0.002	1.567661
0.003	1.566098
0.004	1.564538
0.005	1.562982
0.006	1.561428
0.007	1.559878
0.008	1.558330
0.009	1.556786
0.01	1.555245
0.02	1.540006
0.03	1.525076
0.04	1.510452
0.05	1.496129
0.06	1.482103
0.07	1.468370
0.08	1.454924
0.09	1.441759
0.1	1.428870
0.2	1.313838

Table 2 (*continued*)

α_M	$\bar{\beta}_1$
0.3	1.219952
0.4	1.142227
0.5	1.076874
0.6	1.021114
0.7	0.972911
0.8	0.930757
0.9	0.893519
1	0.860334
2	0.653271
3	0.547161
4	0.480094
5	0.432841
6	0.397248
7	0.369197
8	0.346354
9	0.327285
10	0.311053
15	0.255365
20	0.221760
25	0.198676
30	0.181566
35	0.168230
40	0.157458
45	0.148521
50	0.140952
55	0.134433
60	0.128742
65	0.123718
70	0.119239
75	0.115214
80	0.111571
85	0.108253
90	0.105214
95	0.102418
100	0.099834
200	0.070652
300	0.057703
400	0.049979
500	0.044706
600	0.040813
700	0.037787
800	0.035348
900	0.033327
1000	0.031618

parameters $\bar{\beta}_1$ of the undamped system, i.e., the first roots of Eq. (21) are collected for the same α_M range. It is not surprising that $\bar{\beta}_1$ values are, due to Eq. (18), simply the $\bar{\bar{\lambda}}$ values in Table 1, without the imaginary unit, where the minus signs are omitted.

The eigenvalues of the continuous system in Fig. 1 and discrete model in Fig. 3 are given in Table 3 for $\bar{\alpha} = 0.1$ and 1, respectively, which are complex, due to the presence of damping.

The complex numbers $\lambda'_{0.1}$ in the second column are eigenvalues of the continuous system in Fig. 1 for $\bar{\alpha} = 0.1$, determined from Eq. (16). The complex numbers $\lambda''_{0.1}$ which are the roots of the characteristic equation (23), given in Eq. (24), i.e., eigenvalues of the discrete model, are exactly the same as $\lambda'_{0.1}$. Therefore, they are not repeated in a separate column. The corresponding eigenvalues for $\bar{\alpha} = 1$ are denoted as λ'_1 and λ''_1,

Table 3
Collection of the "first" eigenvalues of the system in Figs. 1–3 and δ values for the same range of α_M as previously; $\bar{\alpha} = 0.1$ and $\bar{\alpha} = 1$

α_M	$\lambda'_{0.1} = \lambda''_{0.1}$	$\lambda'_1 = \lambda''_1$	δ
0	−0.123370 ± 1.565944i	−1.233701 ± 0.972309i	0.405285
0.001	−0.123124 ± 1.564389i	−1.231237 ± 0.972897i	0.405096
0.002	−0.122878 ± 1.562838i	−1.228781 ± 0.973478i	0.404907
0.003	−0.122633 ± 1.561289i	−1.226332 ± 0.974050i	0.404720
0.004	−0.122389 ± 1.559744i	−1.223890 ± 0.974614i	0.404533
0.005	−0.122146 ± 1.558201i	−1.221456 ± 0.975170i	0.404348
0.006	−0.121903 ± 1.556662i	−1.219029 ± 0.975718i	0.404163
0.007	−0.121661 ± 1.555126i	−1.216609 ± 0.976258i	0.403978
0.008	−0.121420 ± 1.553593i	−1.214197 ± 0.976791i	0.403795
0.009	−0.121179 ± 1.552063i	−1.211792 ± 0.977315i	0.403612
0.01	−0.120939 ± 1.550536i	−1.209394 ± 0.977831i	0.403430
0.02	−0.118581 ± 1.535434i	−1.185809 ± 0.982586i	0.401653
0.03	−0.116293 ± 1.520636i	−1.162928 ± 0.986638i	0.399949
0.04	−0.114073 ± 1.506138i	−1.140732 ± 0.990048i	0.398315
0.05	−0.111920 ± 1.491937i	−1.119201 ± 0.992870i	0.396747
0.06	−0.109832 ± 1.478028i	−1.098315 ± 0.995155i	0.395243
0.07	−0.107806 ± 1.464407i	−1.078055 ± 0.996949i	0.393798
0.08	−0.105840 ± 1.451069i	−1.058402 ± 0.998293i	0.392410
0.09	−0.103933 ± 1.438008i	−1.039335 ± 0.999226i	0.391077
0.1	−0.102083 ± 1.425219i	−1.020835 ± 0.999783i	0.389795
0.2	−0.086308 ± 1.311000i	−0.863085 ± 0.990583i	0.379317
0.3	−0.074414 ± 1.217680i	−0.744141 ± 0.966714i	0.371915
0.4	−0.065234 ± 1.140363i	−0.652341 ± 0.937621i	0.366470
0.5	−0.057983 ± 1.075312i	−0.579829 ± 0.907445i	0.362324
0.6	−0.052134 ± 1.019782i	−0.521337 ± 0.877999i	0.359073
0.7	−0.047328 ± 0.971759i	−0.473278 ± 0.850037i	0.356463
0.8	−0.043315 ± 0.929748i	−0.433154 ± 0.823824i	0.354324
0.9	−0.039919 ± 0.892627i	−0.399188 ± 0.799391i	0.352541
1	−0.037009 ± 0.859537i	−0.370087 ± 0.776666i	0.351034
2	−0.021338 ± 0.652923i	−0.213382 ± 0.617439i	0.343220
3	−0.014969 ± 0.546956i	−0.149692 ± 0.526286i	0.340182
4	−0.011525 ± 0.479956i	−0.115245 ± 0.466057i	0.338570
5	−0.009368 ± 0.432739i	−0.093676 ± 0.422583i	0.337572
6	−0.007890 ± 0.397170i	−0.078903 ± 0.389333i	0.336894
7	−0.006815 ± 0.369134i	−0.068153 ± 0.362852i	0.336402
8	−0.005998 ± 0.346302i	−0.059981 ± 0.341121i	0.336030
9	−0.005356 ± 0.327241i	−0.053558 ± 0.322873i	0.335738
10	−0.004838 ± 0.311015i	−0.048377 ± 0.307268i	0.335503
15	−0.003261 ± 0.255344i	−0.032606 ± 0.253275i	0.334792
20	−0.002459 ± 0.221747i	−0.024589 ± 0.220393i	0.334431
25	−0.001974 ± 0.198667i	−0.019736 ± 0.197694i	0.334214
30	−0.001648 ± 0.181559i	−0.016483 ± 0.180816i	0.334068
35	−0.001415 ± 0.168224i	−0.014151 ± 0.167634i	0.333964
40	−0.001240 ± 0.157453i	−0.012397 ± 0.156969i	0.333886
45	−0.001103 ± 0.148517i	−0.011029 ± 0.148111i	0.333825
50	−0.000993 ± 0.140948i	−0.009934 ± 0.140601i	0.333776
55	−0.000904 ± 0.134430i	−0.009036 ± 0.134129i	0.333736
60	−0.000829 ± 0.128739i	−0.008287 ± 0.128475i	0.333702
65	−0.000765 ± 0.123715i	−0.007653 ± 0.123481i	0.333674
70	−0.000711 ± 0.119237i	−0.007109 ± 0.119027i	0.333650
75	−0.000664 ± 0.115212i	−0.006637 ± 0.115023i	0.333629
80	−0.000622 ± 0.111569i	−0.006224 ± 0.111397i	0.333610
85	−0.000586 ± 0.108251i	−0.005859 ± 0.108094i	0.333594
90	−0.000554 ± 0.105213i	−0.005535 ± 0.105069i	0.333580
95	−0.000524 ± 0.102417i	−0.005245 ± 0.102284i	0.333567
100	−0.000498 ± 0.099832i	−0.004983 ± 0.099709i	0.333555
200	−0.000250 ± 0.070651i	−0.002496 ± 0.070608i	0.333444

Table 3 (*continued*)

α_M	$\lambda'_{0.1} = \lambda''_{0.1}$	$\lambda'_1 = \lambda''_1$	δ
300	−0.000166±0.057703i	−0.001665±0.057679i	0.333407
400	−0.000125±0.049979i	−0.001249±0.049964i	0.333389
500	−0.000100±0.044706i	−0.000999±0.044695i	0.333378
600	−0.000083±0.040813i	−0.000833±0.040805i	0.333370
700	−0.000071±0.037787i	−0.000714±0.037781i	0.333365
800	−0.000062±0.035348i	−0.000625±0.035342i	0.333361
900	−0.000056±0.033327i	−0.000555±0.033323i	0.333358
1000	−0.000050±0.031617i	−0.000500±0.031614i	0.333356

respectively, and given in the third column of Table 3. The nondimensional factor δ in the last column is calculated from Eq. (28).

The exact agreement of λ' and λ'' values in each column justifies the fact that the single degree-of-freedom model in Fig. 3 yields the ''first'' eigenvalue of the continuous system in Fig. 1 exactly.

As stated in the previous section, the δ value is the same, irrespective of $\bar{\alpha}$. It is clearly seen that δ approaches 1/3 if α_M tends to infinity. An inspection of the second and third columns reveals further that the absolute values of the real parts of the eigenvalues in the third column are ten times of those in the second column, as expected. Further, the imaginary parts of the corresponding eigenvalues in case of $\bar{\alpha} = 1$ are less than those for $\bar{\alpha} = 0.1$, this trend being more apparent for smaller α_M values.

4. Conclusion

The present note is concerned first with the derivation of the characteristic equation of an axially vibrating viscoelastic rod obeying the Kelvin–Voigt model, carrying a tip mass. Then, the eigenvalues of the mentioned system are calculated and tabulated for a wide range of the nondimensional mass parameter. Further, it is attempted to represent the original continuous system by an ''equivalent'' spring-damper-mass system. It is hoped that especially the characteristic equation derived and then the tables given can be helpful to design engineers working in this area.

References

[1] A.A. Shabana, *Theory of Vibration. Discrete and Continuous Systems*, Vol. II, Springer, New York, 1991.
[2] I.N. Bronstein, K.A. Semendjajew, G. Musiol, H. Mühlig, *Taschenbuch der Mathematik*, Verlag Harri Deutsch, Thun, 1995.
[3] J.L. Humar, *Dynamics of Structures*, Prentice-Hall, Englewood Cliffs, NJ, 1990.
[4] Y. Yamamoto, Spring's effective mass in spring mass system free vibration, *Journal of Sound and Vibration* 220 (1999) 564–570.
[5] M. Gürgöze, On the representation of a cantilevered beam carrying a tip mass by an equivalent spring-mass system, *Journal of Sound and Vibration* 282 (2005) 538–542.