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A conjecture on route to chaos in a hard Duffing oscillator by homoclinic entanglement

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Abstract

Response of a weakly damped hard Duffing oscillator, which apparently does not admit any homoclinic entanglement, is analysed. A possibility of homoclinic entanglement is conjectured that may help to understand onset of chaotic behaviour under simple harmonic excitation.

1. Introduction

A good deal of research effort has been spent in the past three decades to understand different routes to chaotic response of a harmonically excited Duffing oscillator. The origin of complex behaviour of the soft-Duffing and so-called Duffing–Holmes oscillator can be explained by homoclinic entanglement using Melnikov's criterion [1]. Common in these two kinds of oscillator is the presence of saddle point in phase curves of free response. In a hard Duffing oscillator, however, such fixed point does not exist, although the response of the oscillator shows similar trends (period doubling route, quasiperiodic route, etc.) as the other two. In this paper a conjecture has been proposed to explain the chaotic behaviour of a hard Duffing oscillator by homoclinic entanglement. The practical difficulty in constructing Melnikov function for this case places an obstacle against giving a concrete proof.

2. Theoretical analysis

Let us consider the following equation of a weakly damped hard Duffing oscillator:

$$\ddot{x} + x + \alpha x^3 + \beta \dot{x} = F \cos \Omega t \quad (\alpha > 0, 1 \gg \beta > 0). \tag{1}$$

The steady-state solution of the oscillator is written as

$$x(t) = A \cos(\Omega t + \phi) + y(t), \tag{2}$$

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where the time function $y(t)$ is the correction term that accommodates all the frequencies including the existing one. The contribution of this term may be as high as that of the former depending upon relationship between various parameters, namely, F, α, Ω and β. It may be seen that the function $y(t)$ satisfies the following equation:

$$\ddot{y} + \left[\left(1 + \frac{3}{2}A^2\alpha\right) + \frac{3}{2}A^2\alpha\cos 2(\Omega t + \phi)\right]y + \beta\dot{y} + 3A\alpha\cos(\Omega t + \phi)y^2 + \alpha y^3 = -\frac{\alpha}{4}A^3\cos 3(\Omega t + \phi), \tag{3}$$

where A and ϕ are obtained solving

$$[A(1-\Omega^2) + \tfrac{3}{4}\alpha A^3]^2 + \beta^2 A^2\Omega^2 = F^2$$

and

$$\tan\phi = -\frac{\beta A\Omega}{A(1-\Omega^2) + \frac{3}{4}\alpha A^3}.$$

Eq. (3) can be written in a simplified form

$$\ddot{y} + (\delta + 2\varepsilon\cos 2(\Omega t + \phi))y + \beta\dot{y} + 3A\alpha\cos(\Omega t + \phi)y^2 + \alpha y^3 = -\frac{\alpha}{4}A^3\cos 3(\Omega t + \phi), \tag{4}$$

where $\delta = 1 + \frac{3}{2}A^2\alpha$ and $\varepsilon = \frac{3}{4}A^2\alpha$. The above equation is an equation of motion of a viscously damped nonlinear oscillator subjected to both external harmonic and parametric excitation. It is well known that if the higher order terms in y (i.e. terms containing y^2 and y^3) are neglected, the homogeneous part of the solution yields unbounded values depending upon the relationship between δ and ε. This instability in $y(t)$ manifests itself as instability of various orders in the response $x(t)$.

The situation is complicated in presence of higher order terms. The unbounded responses of the linear system become bounded owing to the nonlinear terms. Because an analytical result is, in general, not possible the nature of approximate solution is studied using method of slowly varying parameters. The general solution of the equation may be written as

$$y(t) = \sum_{i=1}^{\infty} \tilde{a}_i(t)\cos i(\Omega t + \phi) + \tilde{b}_i(t)\sin i(\Omega t + \phi).$$

In the following an approximate solution is sought neglecting higher order harmonics, and the response is assumed in the following form:

$$y(t) = a(t)\cos(\Omega t + \phi) + b(t)\sin(\Omega t + \phi). \tag{5}$$

It may be said that, as a result of approximation, the correct solution will differ from the approximate values by amount that become larger as the value of A increases.

The time-dependent coefficients $a(t)$ and $b(t)$, which vary slowly enough that the second derivatives could be neglected, are solved from the following first-order differential equations:

$$\beta\dot{a} + 2\Omega\dot{b} + (\delta - \Omega^2 + \varepsilon)a + \beta\Omega b + \tfrac{3}{4}\alpha a(a^2 + b^2) + \tfrac{9}{4}A\alpha a^2 + \tfrac{3}{4}A\alpha b^2 = 0 \tag{6}$$

and

$$-2\Omega\dot{a} + \beta\dot{b} + (\delta - \Omega^2 - \varepsilon)b - \beta\Omega a + \tfrac{3}{4}\alpha b(a^2 + b^2) + \tfrac{3}{2}A\alpha ab = 0. \tag{7}$$

The fixed points of $a(t)$ and $b(t)$ are the solutions of the algebraic equations

$$(\delta - \Omega^2 + \varepsilon)a + \beta\Omega b + \tfrac{3}{4}\alpha a(a^2 + b^2) + \tfrac{9}{4}A\alpha a^2 + \tfrac{3}{4}A\alpha b^2 = 0 \tag{8}$$

and

$$(\delta - \Omega^2 - \varepsilon)b - \beta\Omega a + \tfrac{3}{4}\alpha b(a^2 + b^2) + \tfrac{3}{2}A\alpha ab = 0. \tag{9}$$

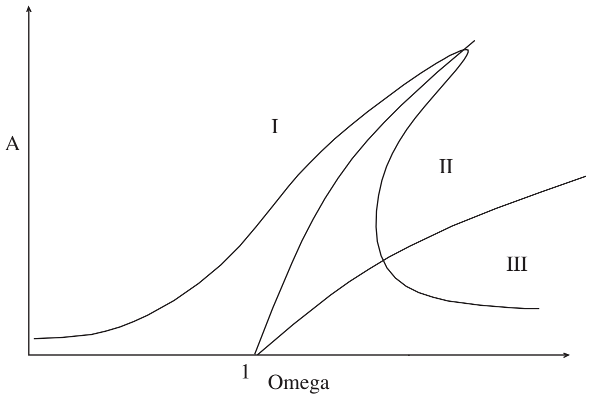

Fig. 1. Stability zones in parametric plane, I: stable focus, II: saddle point, and III: stable focus. The value of $\alpha = 1$.

It is clearly seen that $a = 0$ and $b = 0$ is always a fixed point. The linearized equations around this fixed point become

$$\begin{bmatrix} \beta & 2\Omega \\ -2\Omega & \beta \end{bmatrix} \begin{Bmatrix} \dot{a} \\ \dot{b} \end{Bmatrix} + \begin{bmatrix} \delta - \Omega^2 + \varepsilon & \beta\Omega \\ -\beta\Omega & \delta - \Omega^2 - \varepsilon \end{bmatrix} \begin{Bmatrix} a \\ b \end{Bmatrix} = \mathbf{0}.$$

A simple eigenvalue analysis shows that the fixed point is a saddle point if

$$(\delta - \Omega^2)^2 - \varepsilon^2 + \beta^2\Omega^2 < 0.$$

For undamped oscillator the above condition yields

$$\delta - \varepsilon < \Omega^2 < \delta + \varepsilon,$$

or

$$1 + \tfrac{3}{4}A^2\alpha < \Omega^2 < 1 + \tfrac{9}{4}A^2\alpha.$$

If these curves are drawn on the response plane (i.e., A–Ω plane) of an undamped hard Duffing oscillator they become the backbone curve and the curve joining the points of vertical tangencies, respectively. In presence of small damping the instability region is bounded by two curves joining the points of vertical tangencies [2]. Outside these curves the trivial fixed point is a stable focus as the eigenvalue analysis will clearly reveal. The results are schematically shown in Fig. 1 where the instability regions are shown in the A–Ω plane. The curves in the figure are the curves of vertical tangencies. In the following the stability of the non-trivial fixed points is studied in detail.

2.1. Analysis of non-trivial fixed points

Expressing the non-trivial fixed points as

$$a = r\cos\theta \quad \text{and} \quad b = r\sin\theta,$$

the algebraic equations (8) and (9) after minor manipulation take the following forms:

$$(\delta - \Omega^2) + \varepsilon\cos 2\theta + \tfrac{3}{4}\alpha r^2 + \tfrac{9}{4}A\alpha r\cos\theta = 0 \tag{10}$$

and

$$\varepsilon\sin 2\theta + \beta\Omega + \tfrac{3}{4}A\alpha r\sin\theta = 0, \tag{11}$$

which are to be solved. The closed-form solution is not possible. Hence the equations should be solved numerically. However, an approximate solution can be obtained in a series form for small damping $\beta = \gamma\tilde{\beta}$, with γ as a small parameter (i.e. $\gamma \ll 1$). Using a regular perturbation one can write up to $O(\gamma)$:

$$r = r_0 + \gamma r_1 \quad \text{and} \quad \theta = \theta_0 + \gamma\theta_1,$$

where (r_0, θ_0) and (r_1, θ_1) satisfy the following equations obtained after equating the coefficients of the like powers of γ:

$$\gamma^0:\ (\delta - \Omega^2) + \varepsilon \cos 2\theta_0 + \tfrac{3}{4}\alpha r_0^2 + \tfrac{9}{4}A\alpha r_0 \cos\theta_0 = 0, \tag{12}$$

$$\varepsilon \sin 2\theta_0 + \tfrac{3}{4}A\alpha r_0 \sin\theta_0 = 0 \tag{13}$$

and

$$\gamma^1:\ -4\varepsilon\theta_1 \sin 2\theta_0 + \tfrac{3}{2}\alpha r_0 r_1 + \tfrac{9}{4}A\alpha r_1 \cos\theta_0 - \tfrac{9}{4}A\alpha r_0\theta_1 \sin\theta_0 = 0, \tag{14}$$

$$4\varepsilon\theta_1 \cos 2\theta_0 + \tilde{\beta}\Omega + \tfrac{3}{4}A\alpha r_1 \sin\theta_0 + \tfrac{3}{4}A\alpha r_0\theta_1 \cos\theta_0 = 0. \tag{15}$$

From Eq. (13) one gets either $\sin\theta_0 = 0$ or $2\varepsilon\cos\theta_0 + \frac{3}{4}A\alpha r_0 = 0$.

Case I: $\sin\theta_0 = 0$ or $\cos\theta = \pm 1$.

(a) $\cos\theta_0 = 1$ yields

$$r_0 = -\tfrac{3}{2}A \pm \sqrt{(\tfrac{9}{4}A\alpha)^2 - 3(\delta - \Omega^2 + \varepsilon)\alpha}/(\tfrac{3}{2}\alpha),$$

which has either one positive root or no positive root according as $\delta - \Omega^2 + \varepsilon < 0$ or $\delta - \Omega^2 + \varepsilon > 0$, respectively, or in terms of A and Ω, as $\Omega^2 > 1 + \frac{9}{4}\alpha A^2$ or $\Omega^2 < 1 + \frac{9}{4}\alpha A^2$, respectively.

(b) $\cos\theta_0 = -1$ yields

$$r_0 = \tfrac{3}{2}A \pm \sqrt{(\tfrac{9}{4}A\alpha)^2 - 3(\delta - \Omega^2 + \varepsilon)\alpha}/(\tfrac{3}{2}\alpha),$$

which has one positive root if $\delta - \Omega^2 + \varepsilon < 0$, i.e., $\Omega^2 > 1 + \frac{9}{4}\alpha A^2$ and two positive roots if $\delta - \Omega^2 + \varepsilon > 0$, i.e., $\Omega^2 < 1 + \frac{9}{4}\alpha A^2$ and $(\frac{9}{4}A\alpha)^2 - 3(\delta - \Omega^2 + \varepsilon)\alpha > 0$ or in terms of A and $\Omega, \Omega^2 > 1 + \frac{9}{16}\alpha A^2$.

Case II: $\cos\theta_0 = -\frac{3}{8\varepsilon}A\alpha r_0$.

It can be verified easily by direct substitution into Eq. (12) that this case does not yield any equation in r_0 but merely a relation between various parameters, namely $\delta - \Omega^2 - \varepsilon = 0$.

To summarize, one can have two positive roots for $\Omega^2 > 1 + \frac{9}{4}\alpha A^2$, one with $\cos\theta_0 = 1$ and another with $\cos\theta_0 = -1$. If $\Omega^2 < 1 + \frac{9}{4}\alpha A^2$, one can have either no real root or two positive roots depending upon whether $\Omega^2 < 1 + \frac{9}{16}\alpha A^2$ or $\Omega^2 > 1 + \frac{9}{16}\alpha A^2$, respectively. For the latter case, $\cos\theta_0 = -1$ for both the roots.

Next, the values of r_1 and θ_1 are expressed in terms of r_0 and θ_0 as

$$r_1 = -\frac{\tilde{\beta}\Omega}{\frac{3}{4}A\alpha\sin\theta_0 + K(\frac{3}{2}\alpha r_0 + \frac{9}{4}A\alpha\cos\theta_0)},$$

$$\theta_1 = \left[\frac{\frac{3}{2}\alpha r_0 + \frac{9}{4}A\alpha\cos\theta_0}{4\varepsilon\sin 2\theta_0 + \frac{9}{4}A\alpha r_0 \sin\theta_0}\right] r_1,$$

where

$$K = \frac{4\varepsilon\cos 2\theta_0 + \frac{3}{4}A\alpha r_0\cos\theta_0}{4\varepsilon\sin 2\theta_0 + \frac{3}{4}A\alpha r_0\sin\theta_0}$$

if $\sin\theta_0 \neq 0$. However, for the given situation $(\sin\theta_0 = 0)$, it can be readily seen that $r_1 = 0$, $\theta_1 = -\tilde{\beta}\Omega/(4\varepsilon + \frac{3}{4}A\alpha r_0\cos\theta_0)$.

2.2. *Stability analysis of nonlinear fixed points*

The stability of any fixed point $(\overline{a}, \overline{b})$ is ascertained after linearizing Eqs. (6) and (7) around the fixed points. The linearized equations become

$$\mathbf{A}\dot{\mathbf{x}} + \mathbf{B}\mathbf{x} = \mathbf{0},$$

where

$$\mathbf{A}=\begin{bmatrix}\tilde{\beta} & 2\Omega\\ -2\Omega & \tilde{\beta}\end{bmatrix},\quad \mathbf{B}=\begin{bmatrix}b_{11} & b_{12}\\ b_{21} & b_{22}\end{bmatrix},\quad \mathbf{x}=\begin{Bmatrix}\xi\\ \eta\end{Bmatrix}$$

and

$$\begin{aligned}
b_{11}&=(\delta-\Omega^2+\varepsilon)+\tfrac{9}{4}\alpha\overline{a}^2+\tfrac{3}{4}\alpha\overline{b}^2+\tfrac{9}{2}\alpha A\overline{a},\\
b_{12}&=\tfrac{3}{2}\alpha\overline{a}\overline{b}+\tfrac{3}{2}\alpha A\overline{b}+\tilde{\beta}\Omega,\\
b_{21}&=\tfrac{3}{2}\alpha\overline{a}\overline{b}+\tfrac{3}{2}\alpha A\overline{b}-\tilde{\beta}\Omega,\\
b_{22}&=(\delta-\Omega^2-\varepsilon)+\tfrac{9}{4}\alpha\overline{b}^2+\tfrac{3}{4}\alpha\overline{a}^2+\tfrac{3}{2}\alpha A\overline{a}.
\end{aligned}$$

In the above expressions $\overline{a}=r\cos\theta$ and $\overline{b}=r\sin\theta$. The characteristic equation is obtained after substituting $\mathbf{x}=\mathbf{X}e^{-\lambda t}$ in the linearized equation and becomes

$$\mathrm{Det}[\mathbf{B}-\lambda\mathbf{A}]=0,$$

or in expanded form

$$\lambda^2(\tilde{\beta}^2+4\Omega^2)+\lambda(-\tilde{\beta}b_{11}-\tilde{\beta}b_{22}-2\Omega b_{12}+2\Omega b_{21})+(b_{11}b_{22}-b_{12}b_{21})=0. \tag{16}$$

The fixed point $(\overline{a},\overline{b})$ is stable or unstable if the real part of the root of the above characteristic equation is positive or negative, respectively. For $\tilde{\beta}=0$, it may be easily verified that when $\cos\theta_0=-1$, the fixed point is a saddle point if $(r_0-\frac{3}{2}A)(2A-r_0)>0$. This condition is satisfied for the fixed point $r_0=\frac{3}{2}A+\sqrt{(\frac{9}{4}A\alpha)^2-3(\delta-\Omega^2+\varepsilon)\alpha/(\frac{3}{2}\alpha)}$ only if $\Omega^2<1+\frac{3}{4}\alpha A^2$. When $\cos\theta_0=1$, the fixed point $r_0=-\frac{3}{2}A+\sqrt{(\frac{9}{4}A\alpha)^2-3(\delta-\Omega^2+\varepsilon)\alpha/(\frac{3}{2}\alpha)}$ is always a saddle point because the condition becomes then $(r_0+\frac{3}{2}A)(2A+r_0)>0$, which is always true. The stability of the other fixed points (which are non-hyperbolic in nature for an undamped system) cannot be determined without introducing a small damping. It may be mentioned, however, that in presence of damping the nature of the saddle points does not change.

It may be noticed form the above analysis that for a small damping, $\overline{b}=O(\gamma)$ and therefore the terms b_{12} and b_{21} are both of order $O(\gamma)$. Neglecting, as before, terms containing higher orders of γ, the roots of the characteristic polynomial become

$$\lambda=\left[\tilde{\beta}(b_{11}+b_{22}+4\Omega^2)\pm\sqrt{-16\Omega^2 b_{11}b_{22}}\right]\Big/8\Omega^2.$$

It is obvious that the fixed points are stable or unstable foci if $b_{11}b_{22}>0$ and $b_{11}+b_{22}+4\Omega^2$ is positive or negative, respectively. The latter condition is equivalent to $(r/A)^2+2(r/A)\cos\theta+1+(2\Omega^2/3A^2\alpha)$ being positive or negative, respectively. With very little effort it can be proved that this always holds good. In fact, the expression may be rewritten as $((r/A)+\cos\theta)^2+(1-\cos^2\theta)+(2\Omega^2/3A^2\alpha)$, which proves the statement. Therefore, the other fixed points are stable foci.

The summary of the entire analysis is presented in Fig. 2. In A–Ω plane, within region I, bounded by curved $\Omega^2=1+\frac{9}{16}A^2\alpha$ and $1+\frac{3}{4}A^2\alpha$ and in region III where $\Omega^2>1+\frac{9}{4}A^2\alpha$ there exist two stable foci (including $a=0$ and $b=0$) and a saddle point. Also in region II, lying between $\Omega^2=1+\frac{3}{4}A^2\alpha$ and $1+\frac{9}{4}A^2\alpha$, there are two stable foci and one saddle point, the difference being that in the latter case, $a=0$ and $b=0$ is a saddle point.

Turning attention, once again, to Eq. (5), one finds that $y(t)$ reaches a stable limit cycle whose amplitude is given by the fixed points of a and b. If a stroboscopic map is drawn for $y(t)$ with time frame $T=2\pi/\Omega$, the points should follow the phase curves of the dynamic system given by Eqs. (6) and (7), if only the first harmonic is considered. In Eq. (4), the right-hand side which may be roughly called the forcing term, has higher frequency. Therefore the phase curves, mentioned above, get distorted as A is increased which can happen for proper combinations of F and Ω. For some values of A the possibility of a homoclinic

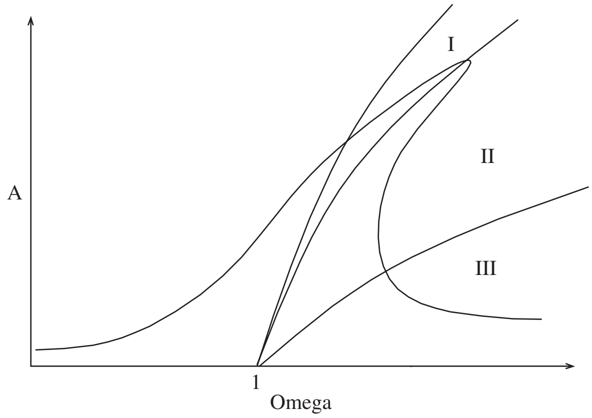

Fig. 2. Nature of fixed points of a and b in A–Ω plane, I, III: two stable foci (including $a = 0,\ b = 0$) and a saddle point, II: two stable foci and one saddle ($a = 0,\ b = 0$). The value of $\alpha = 1$.

entanglement leading to chaotic response cannot be ruled out. Construction of Melnikov function, which is a measure of proximity of homoclinic entanglement, is a very difficult task and has not been attempted for this problem. Note that as a and b behave chaotically so does the overall response of $y(t)$ and hence $x(t)$. But at Ω a sharp peak is expected in the frequency spectra of response in chaotic regime. That, such a peak exists is well known in literature.

It should be mentioned that only first-order instability of the nonlinear parametrically excited system (Eq. (4)) is considered in this paper. Instability of the higher orders may show similar complexities.

3. Conclusion

Response of a weakly damped hard Duffing oscillator is conventionally derived by various techniques [3–5] whereby approximate results are obtained neglecting some terms. It is shown here that this exclusion does not always give correct pictures of the responses. In this paper, the effects of the terms neglected in harmonic balance method are studied. It has been shown that the dynamics is complicated and, if carefully studied, may give some insight into onset of chaotic oscillation in such a system. Although, the equation of motion of a hard Duffing oscillator does not immediately show any possibility of homoclinic entanglement this is possible according to the analysis presented here. The analysis, however, remains only at the level of conjecture because the associated Melnikov function has not been constructed.

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