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Short Communication

# Solutions of nonlinear oscillators with fractional powers by an iteration procedure

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**Abstract**

A modified iteration procedure is applied to nonlinear oscillations with fractional powers. With the procedure, the excellent approximate frequencies and the corresponding periodic solutions can be easily obtained.


Consider a nonlinear oscillator modeled by the equation:

$$\ddot{x} + g(x) = 0, \quad x(0) = A > 0, \quad \dot{x}(0) = 0, \tag{1}$$

where $g(x)$ is a nonlinear function of $x$ and has the property:

$$g(-x) = -g(x).$$

If $g(x)$ does not have for small $x$ a dominant term proportional to $x$, then Eq. (1) is said to be a "truly nonlinear oscillator" (TNO) [1]. One example of such equations is

$$\ddot{x} + \operatorname{sgn}(x)|x|^p = 0, \quad x(0) = A, \quad \dot{x}(0) = 0, \tag{2}$$

where $p < 1$ and $\operatorname{sgn}(x)$ is the sign function, equal to $+1$ if $x > 0$, 0 if $x = 0$, and $-1$ if $x < 0$. Recently, Lim and Wu [2] proposed a modified iteration procedure for Eq. (1). Mickens [1] generalized this procedure for the following equation:

$$\ddot{x} + g(x) = \varepsilon f(x, \dot{x}), \quad x(0) = A, \quad \dot{x}(0) = 0, \tag{3}$$

where

$$f(-x, -\dot{x}) = -f(x, \dot{x}).$$

But they did not give the details as how to carry out the iteration scheme to deal with Eq. (2). The main purpose of this communication is to use an iteration procedure to determine analytical approximations to the periodic solutions of Eq. (2).

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To begin, let the natural frequency of Eq. (1) be $\omega$, which is unknown to be further determined. Then Eq. (1) can be rewritten as [1–5]

$$\ddot{x} + \omega^2 x = \omega^2 x - g(x) =: G(x), \quad x(0) = A, \quad \dot{x}(0) = 0. \tag{4}$$

The linearized equation of Eq. (1) is

$$\ddot{x} + \omega^2 x = 0, \quad x(0) = A, \quad \dot{x}(0) = 0. \tag{5}$$

Comparing Eq. (1) with Eq. (5), we see that even though $g(x)$ is not "small", the function $G(x) = \omega^2 x - g(x)$ is "small". Therefore, the left-hand side of Eq. (4) is linear and the term $G(x)$ on the right-hand side is a "small" function. This is the reason that we prefer Eq. (4) to Eq. (1).

The iteration scheme is [3]

$$\ddot{x}_{k+1} + \omega^2 x_{k+1} = G(x_k), \quad x_k(0) = A, \quad \dot{x}_k(0) = 0, \quad k = 0, 1, 2, \ldots, \tag{6}$$

where the input or starting function is

$$x_0(t) = A\cos\theta = A\cos\omega t. \tag{7}$$

Usually $x_1$ can easily be obtained from Eq. (6). Timoshenko et al. [6] have applied this technique to the Duffing equation, but they only gave the first iteration result. When $k \geqslant 1$, we have

$$G(x_k) = G[x_{k-1} + (x_k - x_{k-1})] \approx G(x_{k-1}) + G_x(x_{k-1})(x_k - x_{k-1}), \tag{8}$$

where

$$G_x(x) = \frac{\mathrm{d}G}{\mathrm{d}x}. \tag{9}$$

Therefore, Eq. (6) can be rewritten as [1,2]

$$\ddot{x}_{k+1} + \omega^2 x_{k+1} = G(x_{k-1}) + G_x(x_{k-1})(x_k - x_{k-1}),$$

$$x_k(0) = A, \quad \dot{x}_k(0) = 0, \quad k = 0, 1, 2, \ldots, \tag{10}$$

where $x_{-1}(t) = x_0(t)$ [1,2]. Instead of Eq. (8) we may also have

$$G(x_k) = G[x_0 + (x_k - x_0)] \approx G(x_0) + G_x(x_0)(x_k - x_0). \tag{11}$$

Now Eq. (6) can be written as

$$\ddot{x}_{k+1} + \omega^2 x_{k+1} = G(x_0) + G_x(x_0)(x_k - x_0),$$

$$x_k(0) = A, \quad \dot{x}_k(0) = 0, \quad k = 0, 1, 2, \ldots\ . \tag{12}$$

In what follows, we will use formula (12) to solve Eq. (2). In this case, formula (12) becomes

$$\ddot{x}_{k+1} + \omega^2 x_{k+1} = \omega^2 x_k - \mathrm{sgn}(x_0)|x_0|^p - p\,\mathrm{sgn}(x_0)|x_0|^p (x_k - x_0)/x_0,$$

$$x_k(0) = A, \quad \dot{x}_k(0) = 0, \quad k = 0, 1, 2, \ldots, \tag{13}$$

where use has been made of the relation

$$\frac{\mathrm{d}}{\mathrm{d}x}[\mathrm{sgn}(x)|x|^p] = p\,\mathrm{sgn}(x)|x|^p/x. \tag{14}$$

The derivation of this relation is given in Appendix A.

First, let [7]

$$\begin{aligned}\mathrm{sgn}(x_0)|x_0|^p &= A^p \mathrm{sgn}[(A\cos\theta)]|\cos\theta|^p \\ &= A^p(a_{1p}\cos\theta + a_{3p}\cos 3\theta + a_{5p}\cos 5\theta + \cdots),\end{aligned} \tag{15}$$

where

$$a_{1p} = \frac{4}{\pi}\int_0^{\pi/2} (\cos\theta)^p \cos\theta \, d\theta = \frac{4\Gamma(1+p/2)}{\sqrt{\pi}(p+1)\Gamma((p/2)+\frac{1}{2})}, \tag{16a}$$

$$a_{3p} = \frac{4}{\pi}\int_0^{\pi/2} (\cos\theta)^p \cos 3\theta \, d\theta = \frac{(P-1)a_{1p}}{P+3}, \tag{16b}$$

$$a_{5p} = \frac{4}{\pi}\int_0^{\pi/2} (\cos\theta)^p \cos 5\theta \, d\theta = \frac{(p-1)(p-3)a_{1p}}{(p+3)(p+5)}. \tag{16c}$$

Here $\Gamma(p)$ is the Gamma function [8]. The computations of $(a_{1p}, a_{3p}, a_{5p})$ are given in detail in Appendix B. Obviously, Eqs. (16) are identical to the results in Ref. [5] when $p = \frac{1}{3}$. Substituting Eq. (7) into Eq. (13) and taking into account Eq. (15), we have

$$\ddot{x}_1 + \omega^2 x_1 = (\omega^2 A - a_{1p}A^p)\cos\theta - a_{3p}A^p\cos 3\theta - a_{5p}A^p \cos 5\theta,$$
$$x_1(0) = A, \quad \dot{x}_1(0) = 0. \tag{17}$$

The requirement of no secular terms in $x_1(t)$ implies that

$$\omega = \omega_1 = \frac{\sqrt{a_{1p}}}{A^{(1-p)/2}}. \tag{18}$$

The corresponding approximate periodic solution $x_1(t)$ becomes

$$x_1(t) = A\cos\omega t + b_3(\cos\omega t - \cos 3\omega t) + b_5(\cos\omega t - \cos 5\omega t), \tag{19}$$

where $\omega$ is given by Eq. (18) and

$$b_3 = -\frac{a_{3p}A^p}{8\omega_1^2} = -\frac{a_{3p}A}{8a_{1p}} = \frac{(1-p)A}{8(p+3)}, \tag{20a}$$

$$b_5 = -\frac{a_{5p}A^p}{24\omega_1^2} = -\frac{a_{5p}A}{24a_{1p}} = \frac{(1-p)(p-3)A}{24(p+3)(p+5)}. \tag{20b}$$

If $k = 1$, Eq. (13) becomes

$$\ddot{x}_2 + \omega^2 x_2 = \omega^2 x_1 - \mathrm{sgn}(x_0)|x_0|^p - p\,\mathrm{sgn}(x_0)|x_0|^p(x_1 - x_0)/x_0,$$
$$x_2(0) = A, \quad \dot{x}_2(0) = 0. \tag{21}$$

Obviously,

$$p\,\mathrm{sgn}(x_0)|x_0|^p \frac{x_1 - x_0}{x_0} = pA^{-1}\mathrm{sgn}(x_0)|x_0|^p\left[\frac{b_3(\cos\theta - \cos 3\theta)}{\cos\theta} + \frac{b_5(\cos\theta - \cos 5\theta)}{\cos\theta}\right]$$
$$= 2pA^{-1}\,\mathrm{sgn}(x_0)|x_0|^p[b_3(1-\cos 2\theta) + b_5(\cos 2\theta - \cos 4\theta)], \tag{22}$$

where use has been made of the relations

$$(\cos\theta - \cos 3\theta)/\cos\theta = 2(1 - \cos 2\theta), \tag{23a}$$

$$(\cos\theta - \cos 5\theta)/\cos\theta = 2(\cos 2\theta - \cos 4\theta). \tag{23b}$$

Substituting Eq. (15) into Eq. (22) and simplifying the resulting expression results in

$$p\,\mathrm{sgn}(x_0)|x_0|^p \frac{x_1 - x_0}{x_0} = \frac{B(p^2+4p+11)}{p+5}\cos\theta + \frac{B(p^3+35p^2+291p+321)}{24(p+5)}\cos 3\theta$$
$$+ \frac{B(3p^2-40p+45)}{24}\cos 5\theta + \text{ higher order harmonics}, \tag{24}$$

where

$$B = \frac{p(1-p)a_{1p}A^p}{(p+3)^2(p+5)}. \tag{25}$$

Substituting Eqs. (15), (19) and (24) into Eq. (21) and making some arithmetical manipulations gives

$$\ddot{x}_2 + \omega^2 x_2 = c_1' \cos\theta + c_3' \cos 3\theta + c_5' \cos 5\theta, \quad x_2(0) = A, \quad \dot{x}_2(0) = 0, \tag{26}$$

where

$$c_1' = \frac{\omega^2 A}{6(p+3)(p+5)}(5p^2 + 46p + 93) - \frac{a_{1p}A^p}{(p+3)^2(p+5)^2}(13p^3 + 87p^2 + 251p + 225), \tag{27a}$$

$$c_3' = \frac{p-1}{8(p+3)}\left[\omega^2 A + \frac{a_{1p}A^p}{3(p+3)(p+5)^2}(p^4 + 11p^3 - 21p^2 - 999p - 1800)\right], \tag{27b}$$

$$c_5' = \frac{p-1}{24(p+3)(p+5)}\left[(p-3)\omega^2 A + \frac{a_{1p}A^p}{p+3}(3p^3 - 64p^2 + 45p + 216)\right]. \tag{27c}$$

The requirement of no secular terms in $x_2(t)$ implies that the coefficient of the $\cos\omega t$ is zero. Letting $c_1' = 0$ and solving for the frequency yields

$$\begin{aligned}\omega = \omega_2 &= \left[\frac{6(13p^3 + 87p^2 + 251p + 225)a_{1p}}{(p+3)(p+5)(5p^2 + 46p + 93)A^{1-p}}\right]^{1/2} \\ &= \left[\frac{24(13p^3 + 87p^2 + 251p + 225)\Gamma(1 + p/2)}{\sqrt{\pi}(p+1)(p+3)(p+5)(5p^2 + 46p + 93)\Gamma((p/2) + \frac{1}{2})A^{1-p}}\right]^{1/2}.\end{aligned} \tag{28}$$

The corresponding approximate periodic solution $x_2(t)$ is

$$x_2(t) = A\cos\omega t + c_3(\cos\omega t - \cos 3\omega t) + c_5(\cos\omega t - \cos 5\omega t), \tag{29}$$

where

$$c_3 = \frac{c_3'}{8\omega_2^2} = \frac{(p-1)A}{64(p+3)}\left[1 + \frac{(5p^2 + 46p + 93)(p^4 + 11p^3 - 21p^2 - 999p - 1800)}{18(p+5)(13p^3 + 87p^2 + 251p + 225)}\right], \tag{30}$$

Table 1
Comparison of the approximate frequency $\omega_2$ with the exact frequency $\omega_e$

| $p$ | $\omega_e A^{(1-p)/2}$ [9] | $\omega_2 A^{(1-p)/2}$ Eq. (28) | Error (%) |
|---|---|---|---|
| 3/4 | 1.024957 | 1.024974 | 0.0017 |
| 2/3 | 1.033652 | 1.033680 | 0.0027 |
| 3/5 | 1.040749 | 1.040784 | 0.0034 |
| 1/2 | 1.051637 | 1.051678 | 0.0039 |
| 3/7 | 1.059596 | 1.059631 | 0.0033 |
| 1/3 | 1.070451 | 1.070453 | 0.0002 |
| 1/5 | 1.086126 | 1.086000 | −0.0116 |
| 1/6 | 1.090133 | 1.089953 | −0.0166 |
| 1/8 | 1.095194 | 1.094926 | −0.0245 |
| 1/9 | 1.096894 | 1.096592 | −0.0275 |
| 1/10 | 1.098258 | 1.097927 | −0.0302 |
| 0 | 1.110721 | 1.110030 | −0.0622 |

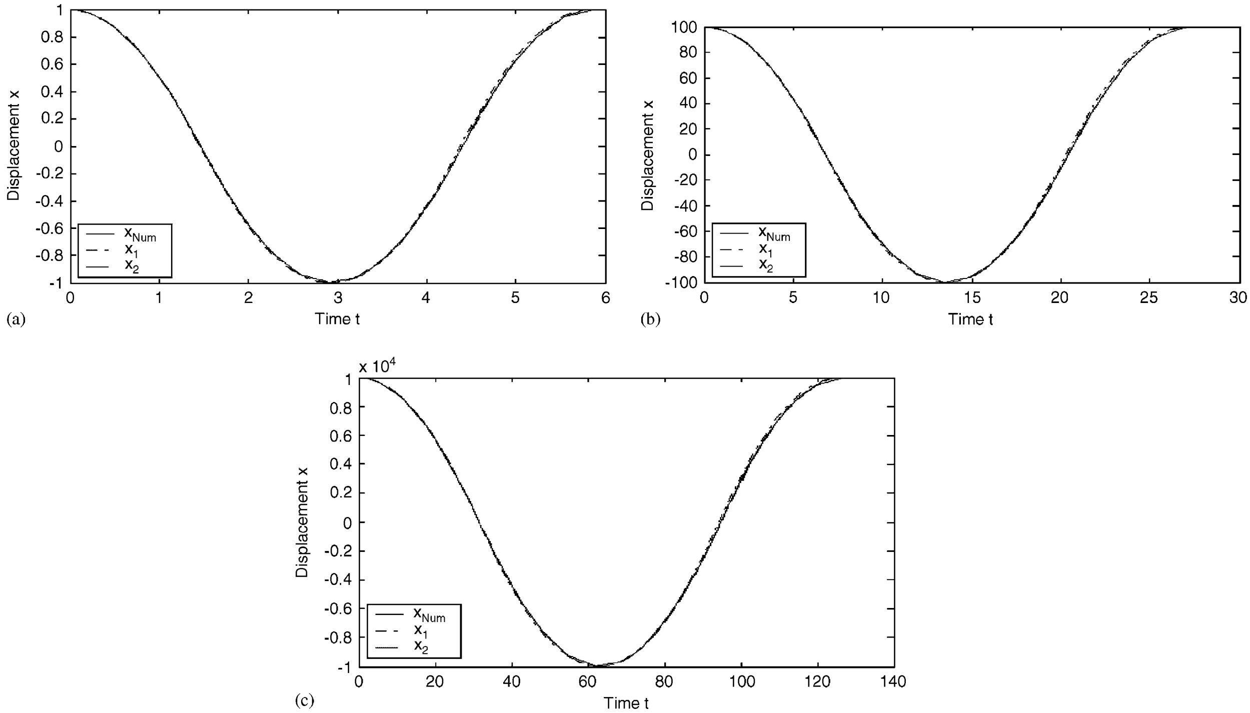


Fig. 1. Comparison of the approximate solutions with the numerical solutions to Eq. (32) for: (a) $A = 1$; (b) $A = 100$; (c) $A = 10,000$.

$$c_5 = \frac{c_5'}{24\omega_2^2} = \frac{(p-1)A}{576(p+3)}\left[\frac{p-3}{p+5} + \frac{(5p^2+46p+93)(3p^3-64p^2+45p+216)}{6(13p^3+87p^2+215p+225)}\right]. \tag{31}$$

Eq. (18) is identical to Eq. (12) in Ref. [7]. The comparison of the approximate frequency $\omega_2$ given by Eq. (28) with the exact frequency $\omega_e$ of Gottlieb [9] is presented in Table 1. Table 1 shows that $\omega_2$ can give very excellent approximate frequencies. Comparing Table 1 in this paper with Table 1 in Ref. [7], we can see that $\omega_2$ is more accurate than $\omega_{2p}$ in Ref. [7].

Now we compare the approximate periodic solutions with the numerical solutions. Without loss of generality, let $p = \frac{1}{3}$. In this case $a_{1p} = 1.15960$ [5] and Eq. (2) becomes [5,10,11]

$$\ddot{x} + x^{1/3} = 0, \quad x(0) = A, \quad \dot{x}(0) = 0. \tag{32}$$

The numerical solution $x_{\text{Num}}(t)$ of Eq. (32) obtained by using Runge–Kutta (R–K) method, and the corresponding approximate solutions $x_1(t)$ and $x_2(t)$ computed by Eq. (19) and Eq. (29) ($p = 1/3$), respectively, are plotted on Fig. 1 for $A = 1$, 100, and 10,000. It can be seen from Fig. 1 that even for $A = 10,000$, $x_1(t)$ and $x_2(t)$ are nearly identical to the numerical solution.

In summary, a modified iteration method, which is described by Eq. (12), has been applied to the nonlinear oscillator modeled by Eq. (2). The approximate frequency $\omega_1$ given in Eq. (18) is identical to the result in Ref. [7]. The $\omega_2$ obtained by the second iteration gives very accurate results. The approximate periodic solutions $x_1(t)$ and $x_2(t)$ are in good agreement with the numerical solutions to Eq. (32). Although formula (12) is identical to formula (10) for the first and second iterations, formula (12) is more convenient than formula (10) if the third iteration is required. This is because computing the expression $p|x_1|^{p-1} = p\,\text{sgn}(x_1)|x_1|^p/x_1$ in formula (10) is not an easy task.

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## Appendix A. The derivation of relation (14)

Let

$$y(x) = \mathrm{sgn}(x)|x|^p = \begin{cases} \mathrm{sgn}(x)x^p & (x \geqslant 0), \\ \mathrm{sgn}(x)(-x)^p & (x < 0). \end{cases} \tag{A.1}$$

Obviously,

$$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}x} &= \begin{cases} p\,\mathrm{sgn}(x)x^{p-1} & (x \geqslant 0), \\ p\,\mathrm{sgn}(x)(-x)^{p-1} & (x < 0), \end{cases} \\ &= p[\mathrm{sgn}(x)]^2|x|^{p-1} = p|x|^{p-1}. \end{aligned} \tag{A.2}$$

If $x \neq 0$, Eq. (A.2) can also be rewritten as

$$\frac{\mathrm{d}y}{\mathrm{d}x} = p\,\mathrm{sgn}(x)\mathrm{sgn}(x)x|x|^{p-1}/x = p\,\mathrm{sgn}(x)|x|^p/x. \tag{A.3}$$

## Appendix B. The computations of $(a_{1p}, a_{3p}, a_{5p})$

Using the relation [12]

$$\int_0^{\pi/2} \cos^n x\,\mathrm{d}x = \int_0^{\pi/2} \sin^n x\,\mathrm{d}x = \frac{\sqrt{\pi}\Gamma((n/2)+\frac{1}{2})}{2\Gamma((n/2)+1)} \quad (n > -1) \tag{B.1}$$

and the relation [8]

$$\Gamma(p+1) = p\Gamma(p), \tag{B.2}$$

we have

$$a_{1p} = \frac{4}{\pi}\int_0^{\pi/2} (\cos\theta)^p \cos\theta\,\mathrm{d}\theta = \frac{2\Gamma((p/2)+1)}{\sqrt{\pi}\Gamma((p/2)+\frac{3}{2})} = \frac{4\Gamma((p/2)+1)}{\sqrt{\pi}(p+1)\Gamma((p/2)+\frac{1}{2})}. \tag{B.3}$$

Similarly,

$$\begin{aligned} a_{3p} &= \frac{4}{\pi}\int_0^{\pi/2} (\cos\theta)^p \cos 3\theta\,\mathrm{d}\theta = \frac{4}{\pi}\int_0^{\pi/2} (\cos\theta)^p (4\cos^3\theta - 3\cos\theta)\,\mathrm{d}\theta \\ &= 4 \times \frac{2\Gamma((p/2)+2)}{\sqrt{\pi}\Gamma((p/2)+\frac{5}{2})} - 3a_{1p} = \frac{4(p+2)a_{1p}}{p+3} - 3a_{1p} = \frac{(p-1)a_{1p}}{p+3}, \end{aligned} \tag{B.4}$$

$$\begin{aligned} a_{5p} &= \frac{4}{\pi}\int_0^{\pi/2} (\cos\theta)^p \cos 5\theta\,\mathrm{d}\theta = \frac{4}{\pi}\int_0^{\pi/2} (\cos\theta)^p (16\cos^5\theta - 20\cos^3\theta + 5\cos\theta)\,\mathrm{d}\theta \\ &= 16 \times \frac{2\Gamma((p/2)+3)}{\sqrt{\pi}\Gamma((p/2)+\frac{7}{2})} - 20 \times \frac{2\Gamma((p/2)+2)}{\sqrt{\pi}\Gamma((p/2)+\frac{5}{2})} + 5a_{1p} \\ &= \frac{16(p+2)(p+4)a_{1p}}{(p+3)(p+5)} - \frac{20(p+2)a_{1p}}{p+3} + 5a_{1p} = \frac{(p-1)(p-3)a_{1p}}{(p+3)(p+5)}. \end{aligned} \tag{B.5}$$

## References

[1] R.E. Mickens, A generalized iteration procedure for calculating approximations to periodic solutions of "truly nonlinear oscillators", *Journal of Sound and Vibration* 287 (2005) 1045–1051.

[2] C.W. Lim, B.S. Wu, A modified Mickens procedure for certain non-linear oscillators, *Journal of Sound and Vibration* 257 (2002) 202–206.
[3] R.E. Mickens, Iteration procedure for determining approximate solutions to non-linear oscillator equation, *Journal of Sound and Vibration* 116 (1987) 185–188.
[4] H. Hu, A modified method of equivalent linearization that works even when the non-linearity is not small, *Journal of Sound and Vibration* 276 (2004) 1145–1149.
[5] H. Hu, A convolution integral method for certain strongly nonlinear oscillators, *Journal of Sound and Vibration* 285 (2005) 1235–1241.
[6] S. Timoshenko, D.H. Yang, W. Weaver Jr., *Vibration Problems in Engineering*, 4th ed., Wiley, New York, 1974.
[7] C.W. Lim, B.S. Wu, Accurate higher-order approximations to frequencies of nonlinear oscillators with fractional powers, *Journal of Sound and Vibration* 281 (2005) 1157–1162.
[8] I.N. Bronshtein, K.A. Semendyayev, *Handbook of Mathematics*, Van Nostrand Reinhold Company, New York, 1985.
[9] H.P.W. Gottlieb, Frequencies of oscillators with fractional-power non-linearities, *Journal of Sound and Vibration* 261 (2003) 557–566.
[10] R.E. Mickens, Oscillations in an $x^{4/3}$ potential, *Journal of Sound and Vibration* 246 (2001) 375–378.
[11] R.E. Mickens, Analysis of non-linear oscillators having non-polynomial elastic terms, *Journal of Sound and Vibration* 255 (2002) 789–792.
[12] R.S. Burington, *Handbook of Mathematical Tables and Formulas*, fifth ed., McGraw-Hill, New York, 1973.