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Short Communication

Solution of a Duffing-harmonic oscillator by the method of harmonic balance

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Abstract

The first-order harmonic balance method via first Fourier coefficient is used to construct an approximate frequency–amplitude relation for a Duffing-harmonic oscillator. This relation is in agreement with the result obtained by the Ritz procedure.

Conservative nonlinear oscillatory systems can often be modeled by potentials having a rational form for the potential energy [1–3]. An example is the nonlinear oscillator modeled by

$$\frac{d^2y}{d\tau^2} + \frac{\alpha y^3}{\beta + \gamma y^2} = 0 \tag{1}$$

for which the parameters (α,β,γ) are non-negative. Defining $y = \sqrt{\beta/\gamma}x$ and $\tau = \sqrt{\gamma/\alpha}t$, Eq. (1) is reduced to the following non-dimensional equation [3]:

$$\ddot{x} + x^3(1 + x^2)^{-1} = 0, \tag{2}$$

where overdots denote differentiation with respect to time t. For small x, the equation of motion (2) is that of a Duffing-type nonlinear oscillator (i.e., $\ddot{x} + x^3 \approx 0$), while for large x, the equation of motion (2) approximates that of a linear harmonic oscillator (i.e., $\ddot{x} + x \approx 0$). Hence, Eq. (2) is referred as the Duffing-harmonic oscillator [3]. The restoring force in the equation is the same for both negative and positive amplitudes. Eq. (2) can be rewritten as

$$(1 + x^2)\ddot{x} + x^3 = 0. \tag{3}$$

An analytic approximation to the periodic solution of Eq. (3) can be calculated by the use of the method of harmonic balance [4]. Assume that the angular frequency of the Duffing-harmonic oscillator is ω, which is unknown to be further determined. The first approximation is

$$x(t) = A \cos \omega t, \tag{4}$$

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which satisfies the initial conditions

$$x(0) = A, \quad \dot{x}(0) = 0. \tag{5}$$

Substitution of Eq. (4) into Eq. (3) gives

$$\left[-\omega^2\left(1+\frac{3}{4}A^2\right)+\frac{3}{4}A^2\right]A\cos\omega t + \text{higher-order harmonics} = 0. \tag{6}$$

Setting the coefficient of $\cos\omega t$ equal to zero and solving for ω gives [3]

$$\omega = \sqrt{\frac{3}{4}A^2\left(1+\frac{3}{4}A^2\right)^{-1}}. \tag{7}$$

The main objective of this short communication is to solve Eq. (2) instead of Eq. (3) by applying the method of harmonic balance. The method of harmonic balance is a procedure for determining analytic approximations to the periodic solutions of differential equations by using a truncated Fourier series representation [4]. It can be easily shown that

$$\frac{(A\cos\theta)^3}{1+(A\cos\theta)^2} = b_1\cos\theta + b_3\cos 3\theta + \cdots. \tag{8}$$

Here,

$$\begin{aligned} b_1 &= \frac{2}{\pi}\int_0^\pi \frac{(A\cos\theta)^3\cos\theta\,\mathrm{d}\theta}{1+(A\cos\theta)^2} = \frac{2A}{\pi}\int_0^\pi\left(\cos^2\theta - \frac{\cos^2\theta}{1+A^2\cos^2\theta}\right)\mathrm{d}\theta \\ &= A - \frac{2}{\pi A}\int_0^\pi\left(1-\frac{1}{1+A^2\cos^2\theta}\right)\mathrm{d}\theta = A - \frac{2}{A} + \frac{2}{\pi A}\int_0^\pi \frac{\mathrm{d}\theta}{(1+A^2/2)+\frac{A^2}{2}\cos 2\theta} \\ &= A - \frac{2}{A} + \frac{1}{\pi A}\int_0^{2\pi}\frac{\mathrm{d}x}{(1+A^2/2)+\frac{A^2}{2}\cos x} = A + \frac{2}{A}[(1+A^2)^{-1/2}-1], \end{aligned} \tag{9}$$

where use has been made of the relation [5]

$$\int_0^{2\pi}\frac{\mathrm{d}x}{a+b\cos x} = \frac{2\pi}{\sqrt{a^2-b^2}}, \quad a>|b|. \tag{10}$$

Substituting of Eq. (4) into Eq. (2) and taking into account Eq. (8), we have

$$(-\omega^2 A + b_1)\cos\omega t + \text{higher-order harmonics} = 0. \tag{11}$$

Setting the coefficient of $\cos\omega t$ equal to zero and solving for ω gives

$$\omega = \sqrt{\frac{b_1}{A}} = \sqrt{1+\frac{2}{A^2}\left(\frac{1}{\sqrt{1+A^2}}-1\right)}. \tag{12}$$

Eq. (12) is in agreement with the result in Ref. [6], which is obtained by using the Ritz procedure [7]. It has been shown that Eq. (12) is more accurate than Eq. (7) [6].

In summary, this short communication provides an approximate frequency–amplitude relation (12) for the equation of motion (2) by using the first-order harmonic balance approach via first Fourier coefficient. It is very interesting to note that substitution of Eq. (4) into Eq. (3) and substitution of Eq. (4) into Eq. (2) do not give the same result! Why does this phenomenon occur? This is because Eq. (3) includes the cubic nonlinearity only, whereas Eq. (2) includes all odd powers of nonlinearity. Therefore, substituting the first-order harmonic approximation into Eq. (3) produces only the first and the third harmonics, whereas substituting the same anzats into Eq. (2) produces the infinite set of higher harmonics (see Eq. (8)). By applying the method of harmonic balance with higher harmonics, the two procedures will give more accurate results. In the limit of including all the harmonics, they must give exactly the same solution, since Eq. (2) is equivalent to Eq. (3). But in fact, it is difficult to construct higher-order analytical approximations to the solution by the method of harmonic balance, especially for Eq. (2). Therefore, in this paper we restrict our investigation by the first harmonic only.

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