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# Quasiperiodic energy pumping in coupled oscillators under periodic forcing

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**Abstract**

The effect of a nonlinear energy sink (NES) with relatively small mass on the dynamics of a coupled system under periodic forcing in the vicinity of a main (1:1) resonance is studied theoretically and experimentally. It is demonstrated that over a range of amplitudes of the external forcing the damped system exhibits a quasiperiodic vibration regime, rather than the steady-state response reported in earlier publications. This regime is related to attraction of the dynamical flow to a damped–forced nonlinear normal mode (NNM) of the system and hysteretic motion of the flow in the vicinity of this mode. A physical experiment using an appropriate electric circuit confirms the above results. The regime of quasiperiodic response is shown to provide more efficient vibration suppression than the best-tuned linear absorber with the same mass.


## 1. Introduction

Recently it has been demonstrated that various systems consisting of linear substructures and strongly nonlinear attachments demonstrate localization and irreversible transient transfer (pumping) of energy to prescribed parts of the structure dependent on initial conditions and external forcing [1–8]. Addition of a relatively small and spatially localized attachment leads to essential changes in the properties of the whole system. Unlike common linear and weakly nonlinear systems, systems with strongly nonlinear elements are able to react efficiently on the amplitude characteristics of the external forcing in a wide range of frequencies [1,6–8]. Thus, the systems under consideration give rise to a new concept of nonlinear energy sink (NES).

Preliminary [1–3] as well as recent [5] investigations devoted to dynamics of various realizations of two-degree-of-freedom (2dof) systems (composed of one dof related to the linear subsystem and one dof related to the nonlinear attachment) have demonstrated that the energy transfer from a linear non-conservative structure to attachment is due to resonance capture. This is a transient dynamical phenomenon that has been theoretically studied since the mid-1970s [9–12]. It occurs in coupled non-conservative oscillators and leads to transient capture of the dynamical flow on a resonance manifold of the system. It has been demonstrated [2,4]

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that the possibility of the energy pumping/resonance capture phenomenon in non-conservative systems can be understood and explained by studying the energy dependence of the nonlinear undamped free periodic solutions (nonlinear normal modes (NNM)) of the corresponding conservative system which are obtained when all damping forces are eliminated. Recent investigation [13] based on the approach of invariant manifolds [14,15] has introduced an asymptotic procedure suitable for explicit inclusion of damping within the framework of NNM.

The goal of the present paper is to investigate the response of the system comprising a large primary mass and small NES to external harmonic forcing in the vicinity of the most dangerous 1:1 resonance. Steady-state responses (i.e. responses with almost constant amplitude) of the primary oscillator with NES attached were studied previously [16] for somewhat different design of the NES than we use. It will be demonstrated that in close vicinity of the main resonance the system with NES can exhibit quasiperiodic rather than steady-state response, leading to qualitatively different dynamical behavior. Analytic predictions concerning the regime of the response will be obtained from an appropriately extended invariant manifold approach taking into account both the damping and the external forcing terms.

The structure of the paper is as follows. In the next section the model of 2dof system with NES under external forcing is presented and elaborated analytically and verified by means of numerical simulation. In the third section the analytic and numeric results are verified by physical experiment with the help of specially designed electric circuits. The fourth section is devoted to discussion of possible use of NES as vibration absorber. The last section contains some concluding remarks.

## 2. Analytic investigation of the model and numeric evidences

Analytic treatment of the model generally follows the procedure developed earlier [5,13] with modifications due to presence of the external forcing.

Let us consider the system, which consists of a linear oscillator and a small strongly nonlinear attachment (pure cubic nonlinearity) and is forced harmonically. The system is described by the following equations:

$$\begin{aligned} &\ddot{y}_1 + \varepsilon\lambda(\dot{y}_1 - \dot{y}_2) + y_1 + 8\varepsilon(y_1 - y_2)^3 = \varepsilon A \cos \omega t, \\ &\varepsilon\ddot{y}_2 + \varepsilon\lambda(\dot{y}_2 - \dot{y}_1) + 8\varepsilon(y_2 - y_1)^3 = 0, \end{aligned} \tag{1}$$

where $y_1$ and $y_2$ are the displacements of the linear oscillator and the attachment respectively, $\varepsilon\lambda$ is the damping coefficient, $\varepsilon A$ is the amplitude of external force and $\omega$ is its frequency. $\varepsilon \ll 1$ is a small parameter which establishes the order of magnitude for coupling, damping, mass of the nonlinear attachment and the amplitude of the external forcing; coefficients $\lambda$ and $A$ are chosen to be of order unity. The rigidity of the nonlinear spring is chosen to be equal to $8\varepsilon$ and the linear frequency of the linear oscillator is chosen as unity. Neither of these restrictions affects the generality of the treatment below, since they may be changed independently by proper rescaling of the variables.

The mass of the attachment is considered to be small compared to the mass of the main oscillator. This condition has obvious motivation from the viewpoint of possible applications: the NES is designed to have a small mass compared to the main system and does not require alternative grounding.

Change of variables

$$v = y_1 + \varepsilon y_2, \quad w = y_1 - y_2 \tag{2}$$

reduces Eqs. (1) to the following form:

$$\begin{aligned} &\ddot{v} + \frac{v}{1+\varepsilon} + \frac{\varepsilon w}{1+\varepsilon} = \varepsilon A \cos \omega t, \\ &\ddot{w} + \frac{v}{1+\varepsilon} + \frac{\varepsilon w}{1+\varepsilon} + (1+\varepsilon)\lambda \dot{w} + 8(1+\varepsilon)w^3 = \varepsilon A \cos \omega t. \end{aligned} \tag{3}$$

Then a new small parameter is introduced and the dependent variables are rescaled as follows:

$$\chi = \varepsilon^{1/3}, \quad V = \chi^{-1} v, \quad W = w. \tag{4}$$

With account of (4) Eqs. (3) are reduced to the following form (only terms up to order of $O(\chi^2)$ are kept):

$$\ddot{V} + V + \chi^2 W = \chi^2 A \cos \omega t,$$
$$\ddot{W} + \chi V + \lambda \dot{W} + 8W^3 = 0. \tag{5}$$

As stated above, the goal of present investigation is the exploration of nonlinear normal modes of Eqs. (1) in the vicinity of 1:1 resonance. It means that both variables, $V$ and $W$, are supposed to have frequencies close to unity. Besides, the dynamics of the system is analyzed in the vicinity of the most dangerous resonance and therefore the frequency of the external excitation also should be close to unity:

$$\omega = 1 + \chi^3 \sigma, \tag{6}$$

where parameter $\sigma$ describes the detuning of the external force frequency.

Therefore, it may be adopted that both variables are expressed as

$$V = \cos(t + \mu_1(\chi t)) f_1(\chi t) + O(\chi),$$
$$W = \cos(t + \mu_2(\chi t)) f_2(\chi t) + O(\chi), \tag{7}$$

where $\mu_i$, $i = 1, 2$ takes into account phase shift and slow phase drift and $f_i$, $i = 1, 2$—slow amplitude modulation. Only phase trajectories with initial conditions inside the domain of attraction of 1:1 resonance manifold are considered. Eqs. (5) may be reshaped to the following form:

$$\ddot{V} + V + \chi^2 W = \chi^2 A \cos((1 + \chi^3 \sigma)t),$$
$$\ddot{W} + W + \chi(\delta[\lambda \dot{W} + 8W^3 - W] + V) = 0, \tag{8}$$

where $\delta = \chi^{-1}$. If the estimation presented in Eqs. (7) is valid, then one obtains

$$\ddot{W} = -\cos((1 + O(\chi))t + \varphi_2) f_2(\chi t) + O(\chi) = -W + O(\chi).$$

It means that in order to balance power 1 of small parameter $\chi$ in the second equation of Eqs. (8), one must adopt

$$\delta[\lambda \dot{W} + 8W^3 - W] \sim O(1)$$

and therefore expression in square brackets should be of order $\chi$. It is rather natural, as it describes slow modulation and damping of the vibrations with frequency close to unity.

Complex variables are introduced according to following relationship:

$$\varphi_1 \exp(\mathrm{i}t) = \dot{V} + \mathrm{i}V,$$
$$\varphi_2 \exp(\mathrm{i}t) = \dot{W} + \mathrm{i}W.$$

With account of this change of variables, Eqs. (8) are rewritten as

$$\begin{aligned}\dot{\varphi}_1 - \frac{\mathrm{i}\chi^2}{2}(\varphi_2 - \varphi_2^* \exp(-2\mathrm{i}t)) &= \chi^2 \frac{A}{2}(\exp(\mathrm{i}\chi^3\sigma t)\exp(\mathrm{i}t) + \exp(-\mathrm{i}\chi^3\sigma t)\exp(-\mathrm{i}t))\dot{\varphi}_2 \\ &\quad + \chi\left\{\delta\left[\frac{\mathrm{i}}{2}(\varphi_2 - \varphi_2^*\exp(-2\mathrm{i}t)) + \frac{\lambda}{2}(\varphi_2 + \varphi_2^*\exp(-2\mathrm{i}t))\right.\right. \\ &\quad \left.\left. + \mathrm{i}\exp(-\mathrm{i}t)(\varphi_2\exp(\mathrm{i}t) - \varphi_2^*\exp(-\mathrm{i}t))^3\right] - \frac{\mathrm{i}}{2}(\varphi_1 - \varphi_1^*\exp(-2\mathrm{i}t))\right\} \\ &= 0,\end{aligned} \tag{9}$$

where the asterisk denotes complex conjugation.

Multiple scales analysis is performed according to the following expansions:

$$\varphi_k = \varphi_{k0} + \chi\varphi_{k1} + \chi^2\varphi_{k2} + \dots, \quad k = 1, 2$$
$$\tau_l = \chi^l t, \quad \frac{\mathrm{d}}{\mathrm{d}t} = \frac{\partial}{\partial\tau_0} + \chi\frac{\partial}{\partial\tau_1} + \chi^2\frac{\partial}{\partial\tau_2} + \dots. \tag{10}$$

Combination of Eqs. (9) and (10) yields in zero approximation:

$$\frac{\partial \varphi_{k0}}{\partial \tau_0} = 0 \Rightarrow \varphi_{k0} = \varphi_{k0}(\tau_1, \tau_2, \ldots), \quad k = 1, 2. \tag{11}$$

Account of terms having the order of $O(\chi)$ leads to the following equations:

$$\begin{aligned}
&\frac{\partial \varphi_{10}}{\partial \tau_1} + \frac{\partial \varphi_{11}}{\partial \tau_0} = 0, \\
&\frac{\partial \varphi_{20}}{\partial \tau_1} + \frac{\partial \varphi_{21}}{\partial \tau_0} + \delta\left[\frac{\mathrm{i}}{2}(\varphi_{20} - \varphi_{20}^* \exp(-2\mathrm{i}\tau_0)) + \frac{\lambda}{2}(\varphi_{20} + \varphi_{20}^* \exp(-2\mathrm{i}\tau_0)) \right. \\
&\left. \quad + \mathrm{i}\exp(-\mathrm{i}\tau_0)(\varphi_{20}\exp(\mathrm{i}\tau_0) - \varphi_{20}^*\exp(-\mathrm{i}\tau_0))^3\right] - \frac{\mathrm{i}}{2}(\varphi_{10} - \varphi_{10}^* \exp(-2\mathrm{i}\tau_0)) = 0.
\end{aligned} \tag{12}$$

Secular terms with respect to time scale $\tau_0$ should be eliminated from Eqs. (12). This condition is satisfied if the following relationships hold (solution for zero order Eq. (11) is also taken into account):

$$\begin{aligned}
&\varphi_{10} = \varphi_{10}(\tau_2, \ldots), \quad \varphi_{11} = \varphi_{11}(\tau_1, \tau_2, \ldots), \\
&\frac{\partial \varphi_{20}}{\partial \tau_1} + \delta\left[\frac{\mathrm{i}}{2}\varphi_{20} + \frac{\lambda}{2}\varphi_{20} - 3\mathrm{i}\varphi_{20}|\varphi_{20}|^2\right] - \frac{\mathrm{i}}{2}\varphi_{10} = 0.
\end{aligned} \tag{13}$$

According to the first equation of Eqs. (13), variable $\varphi_{10}$ does not depend on $\tau_1$; therefore, in the conditions of 1:1 resonance the second equation of this system describes evolution of variable $\varphi_{20}$ with respect to time scale $\tau_1$. Behavior of solutions of Eqs. (13) as $\tau_1 \to \infty$ may be established with the help of Bendixon criterion [17]. By splitting the last equation of Eqs. (13) to real and imaginary part, one gets

$$\begin{aligned}
&\frac{\partial x}{\partial \tau_1} = \frac{\delta}{2}y - \frac{\lambda\delta}{2}x - 3\delta y(x^2 + y^2) - \frac{1}{2}\mathrm{Im}(\varphi_{10}) = P(x, y), \\
&\frac{\partial y}{\partial \tau_1} = -\frac{\delta}{2}x - \frac{\lambda\delta}{2}y + 3\delta x(x^2 + y^2) + \frac{1}{2}\mathrm{Re}(\varphi_{10}) = Q(x, y), \\
&x = \mathrm{Re}(\varphi_{20}), \quad y = \mathrm{Im}(\varphi_{20}).
\end{aligned}$$

One obtains $\partial P/\partial x + \partial Q/\partial y = -\lambda\delta < 0$ for any values of $x$ and $y$. Bendixon criterion leads to conclusion that the solutions of the last equation of Eqs. (13) must end (or begin) at fixed points of the equation and cannot be periodic. Consequently, variable $\varphi_{20}$ evolutes towards equilibrium value defined as follows:

$$\begin{aligned}
&\delta\left[\frac{\mathrm{i}}{2}\tilde{\varphi}_{20} + \frac{\lambda}{2}\tilde{\varphi}_{20} - 3\mathrm{i}\tilde{\varphi}_{20}|\tilde{\varphi}_{20}|^2\right] - \frac{\mathrm{i}}{2}\varphi_{10} = 0, \\
&\tilde{\varphi}_{20}(\tau_2, \ldots) = \lim_{\tau_1 \to \infty} \varphi_{20},
\end{aligned} \tag{14}$$

provided that the limit manifold is stable.

Further evolution of variable $\varphi_1$ is described by the equation of order $\chi^2$:

$$\frac{\partial \varphi_{10}}{\partial \tau_2} + \frac{\partial \varphi_{11}}{\partial \tau_1} + \frac{\partial \varphi_{12}}{\partial \tau_0} - \frac{\mathrm{i}}{2}(\varphi_{20} - \varphi_{20}^* \exp(-2\mathrm{i}\tau_0)) = \frac{A}{2}(\exp(\mathrm{i}\sigma\tau_3) + \exp(-\mathrm{i}\sigma\tau_3)\exp(-2\mathrm{i}\tau_0)). \tag{15}$$

Secular terms with respect to $\tau_0$ are absent if

$$\frac{\partial \varphi_{10}}{\partial \tau_2} + \frac{\partial \varphi_{11}}{\partial \tau_1} - \frac{\mathrm{i}}{2}\varphi_{20} = \frac{A}{2}\exp(\mathrm{i}\sigma\tau_3), \tag{16}$$

$\varphi_{10}$, $\tilde{\varphi}_{20}$ and right-hand side of Eq. (16) do not depend on $\tau_1$. Then in the limit $\tau_1 \to \infty$ the secular terms with respect to $\tau_1$ time scale will be absent if

$$\frac{\partial \varphi_{11}}{\partial \tau_1} \to 0.$$

Therefore, as $\tau_1 \to \infty$ , Eq. (16) is reduced to the form

$$\frac{\partial \varphi_{10}}{\partial \tau_2} - \frac{i}{2}\tilde{\varphi}_{20} = \frac{A}{2}\exp(i\sigma\tau_3). \tag{17}$$

Eqs. (14) and (17) describe the dynamics of the system with respect to time scale $\tau_2$ in the limit $\tau_1 \to \infty$. Right-hand side of Eq. (17) depends only on $\tau_3$ time scale and therefore should be considered as constant. Eq. (14) is in fact algebraic connection between two dependent variables, $\varphi_{10}$ and $\tilde{\varphi}_{20}$. Therefore, with respect to time scale $\tau_2$ the effective dimensionality of the state space of the system is reduced from 5 to 2. Such dynamical regime may be interpreted as damped nonlinear normal mode with invariant manifold ($\tilde{\varphi}_{10}$, $\tilde{\varphi}^*_{10}$) [13]. Eqs. (13) and (16) describe how phase trajectory of the system is attracted to this nonlinear normal mode within time scale $\tau_1$. Due to damping the invariant manifold of the NNM also evolutes with respect to time scale $\tau_2$; this evolution is described by Eq. (17).

After extracting $\varphi_{10}$ from Eq. (14) and substituting to Eq. (17) one obtains

$$\frac{\partial \tilde{\varphi}_{20}}{\partial \tau_2}\left(\frac{i}{2} + \frac{\lambda}{2} - 6i|\varphi_{20}|^2\right) - 3i\tilde{\varphi}_{20}^2\frac{\partial \tilde{\varphi}^*_{20}}{\partial \tau_2} = \frac{iA}{4\delta}\exp(i\sigma\tau_3) - \frac{1}{4\delta}\tilde{\varphi}_{20}. \tag{18}$$

Complex conjugation of Eq. (18) and simple manipulations allow extracting the derivative from the equation:

$$\frac{\partial \tilde{\varphi}_{20}}{\partial \tau_2} = \frac{A(\exp(i\sigma\tau_3) + i\lambda\exp(i\sigma\tau_3) + 6\tilde{\varphi}_{20}^2\exp(-i\sigma\tau_3) - 12|\tilde{\varphi}_{20}|^2\exp(i\sigma\tau_3)) + i\tilde{\varphi}_{20} - \lambda\tilde{\varphi}_{20} - 18i|\tilde{\varphi}_{20}|^2\tilde{\varphi}_{20}}{2\delta(1 + \lambda^2 - 24|\tilde{\varphi}_{20}|^2 + 108|\tilde{\varphi}_{20}|^4)}. \tag{19}$$

Eq. (19) describes evolution of two-dimensional invariant manifold of forced-damped NNM which attracts the five-dimensional phase flow of the initial system. Unlike initial Eqs. (1), this equation is singular. The singularity appears in the limit $\delta \to$ (or, equivalently, $\varepsilon \to 0$) and is a consequence of projecting the five-dimensional phase dynamics of the system on two-dimensional subspace.

In order to explore the behavior of $\tilde{\varphi}_{20}(\tau_2)$ in accordance with Eq. (19), one takes $\tau_3 = 0$, $\tilde{\varphi}_{20} = N\exp(i\theta)$, where $N$ and $\theta$ are real functions of $\tau_2$. The latter change of variables reduces Eq. (19) to the following system:

$$\begin{aligned}
\frac{\partial N}{\partial \tau_2} &= \frac{1}{2\delta D}(A(\cos\theta + \lambda\sin\theta - 12N^2\cos\theta) - \lambda N),\\
\frac{\partial \theta}{\partial \tau_2} &= \frac{1}{2\delta DN}(A(-\sin\theta + \lambda\cos\theta + 18N^2\sin\theta) + N - 18N^3),\\
D &= 1 + \lambda^2 - 24N^2 + 108N^4.
\end{aligned} \tag{20}$$

Detailed investigation of Eqs. (20) and related systems will be published elsewhere [18]. Explanation of possible quasiperiodic response does not require complete analysis and is presented below.

First of all, for $0<\lambda<1/\sqrt{3}$ Eqs. (20) exhibit singularities due to nullification of the denominator at two values of $N$:

$$N_{1,2} = \frac{2 \pm \sqrt{1 - 3\lambda^2}}{18}. \tag{21}$$

Besides, Eqs. (20) have obvious fixed point at $N = A, \theta = \pi/2$. This fixed point corresponds to steady-state response of the system described by Eqs. (1) to the harmonic forcing. It is easy to check that for $A<N_1$ and $A>N_2$ this fixed point is a stable focus. However for $N_1<A<N_2$ the fixed point is of saddle type. Therefore the steady-state regime is unstable, thus giving rise to quasiperiodic response. For $\lambda>1/\sqrt{3}$ the fixed point is always stable and thus no quasiperiodic response is expected.

In paper [13] it is demonstrated that for the system without forcing ($A = 0$) the first equation of Eqs. (20) does not depend on $\theta$ and has the following implicit solution:

$$54Z^2 - 24Z + (1 + \lambda^2)\log Z = C - \lambda\tau_2/\delta, \;\; Z = N(\tau_2)^2. \tag{22}$$

For $0<\lambda<1/\sqrt{3}$ the above solution gives rise to three-branch structure with two stable and one unstable branch. The quasiperiodic response regime thus may be interpreted as "jumps" between two stable branches. For $\lambda>1/\sqrt{3}$ there is only one stable branch of and no "jumps" are possible.

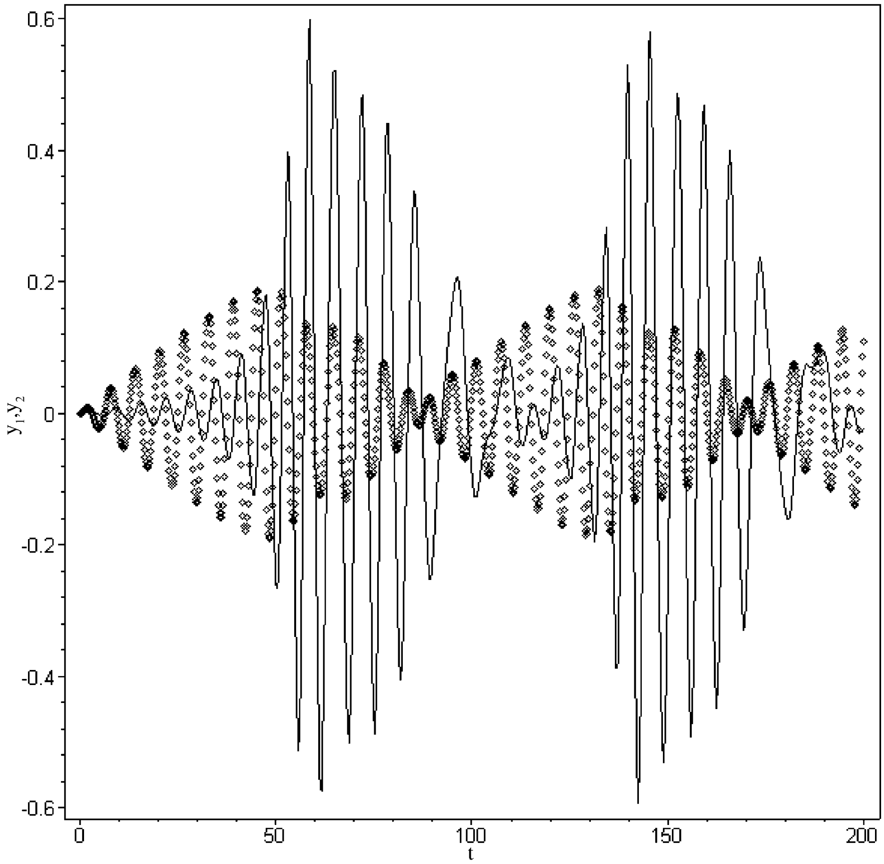


Fig. 1. Quasiperiodic response of system (1) for set of parameters $A = 0.2$, $\varepsilon = 0.05$, $\lambda = 0.2$, $\omega = 1.01$. Dotted line denotes the displacement of the linear oscillator and the solid-line denotes displacement of the NES versus time.

In order to verify the above conclusions, Eqs. (1) are integrated numerically by standard Runge–Kutta method with zero initial conditions and the following set of parameters (Fig. 1):

$$A = 0.2, \quad \varepsilon = 0.05, \quad \lambda = 0.2, \quad \sigma = 0.2. \tag{23}$$

The plot demonstrates typical quasiperiodic behavior of both responses ($y_1$ and $y_2$). This behavior is stable and was revealed also for long-time simulations (up to 20 000 time units or about 200 periods). In order to verify the suggested explanation of this phenomenon (breakdowns of motion on 1:1 resonance invariant manifold due to singularities accompanied by successive attractions of the phase trajectory to the other stable branch of the same manifold) the internal coordinate of the system ($y_1 - y_2$) is plotted versus time and compared to critical values of this function which correspond to singularities of Eqs. (20) (Fig. 2).

It is easy to see that the modulation of the internal coordinate follows the critical values with reasonable accuracy and therefore the system primarily moves in the quasiperiodic regime with successive "jumps" and "relaxations". The other argument in favor of the suggested scenario is that no quasiperiodic response has been revealed for values of $\lambda$ above the range $0 < \lambda < 1/\sqrt{3}$.

## 3. Physical experiment using electric circuit

Experimental verification of analytic and numeric findings described in the previous section was performed by means of appropriately designed electric circuit. Such experiment allows estimating the robustness of the quasiperiodic response regime as the electric circuit inevitably contains additional damping and other factors not accounted in the analytic and numeric models.

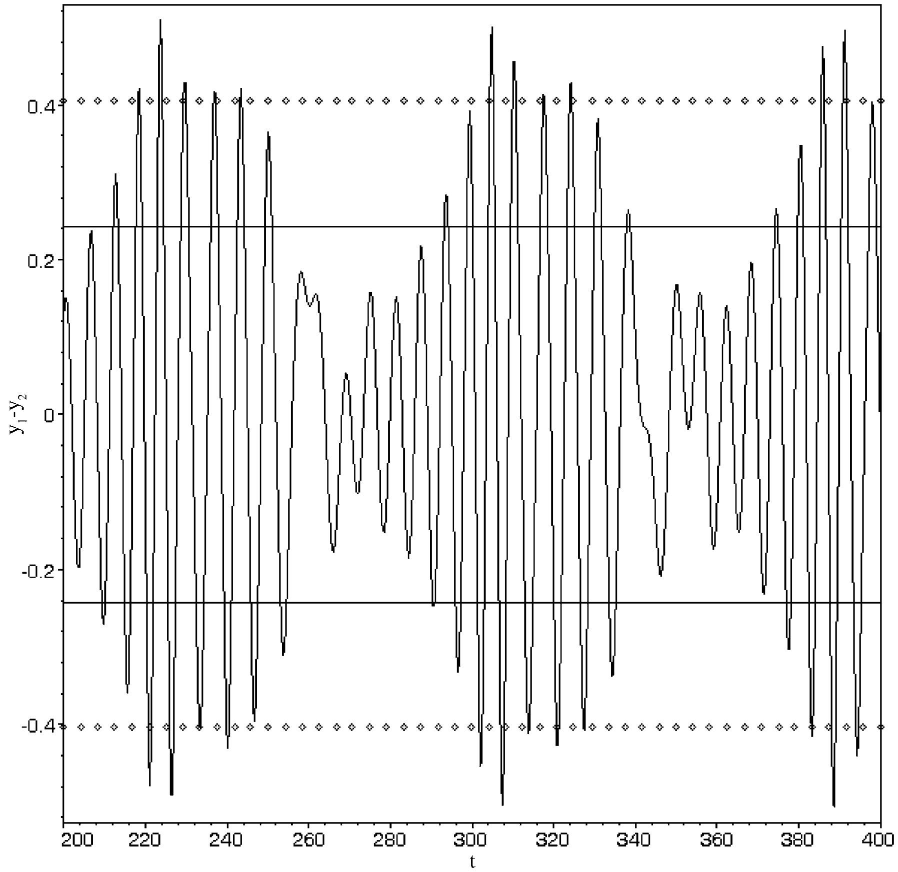


Fig. 2. Plot of the NES internal deformation $y_1(t) - y_2(t)$ versus time for set of parameters $A = 0.2$, $\varepsilon = 0.05$, $\lambda = 0.2$, $\omega = 1.01$. Horizontal lines correspond to singularities of Eqs. (20), described by Eq. (21). Dotted line corresponds to $N_2$ and solid line to $N_1$.

The experiment was performed with the help of installation presented in Fig. 3. Scheme of the electric circuit used is presented in Fig. 4.

Parameters of the electric scheme used for the experiment correspond to the following values of coefficients in Eqs. (1): $\varepsilon = 0.065, \lambda = 0.1, A = 0.192, \sigma = 0$. The external periodic forcing is performed by a generator.

The results of the experiment (compared with the results of appropriate numerical simulation) are presented in Fig. 5.

The above experimental results demonstrate that the electric circuit clearly exhibits the energy pumping from "large" to "small" mass by mechanism of 1:1 resonance, accompanied by generation of quasiperiodic relaxation vibrations. This process is robust to the parameter uncertainties of the electric circuit and external generator used.

## 4. NES as vibration absorber

Experimental results presented in the previous section demonstrate that the quasiperiodic response regime associated with energy transfer to NES can be realized in physical systems. This fact makes the discussion of possible applications of this phenomenon reasonable.

One possible application is related to absorption of vibrations in mechanical systems. System (1) resembles classical example of 1dof mechanical system under action of periodic force with vibration absorber attached [19–21]. The only difference from many previous studies is that strong nonlinearity is suggested. As it was demonstrated above, this system may exhibit types of motion unavailable for linear vibration absorbers.

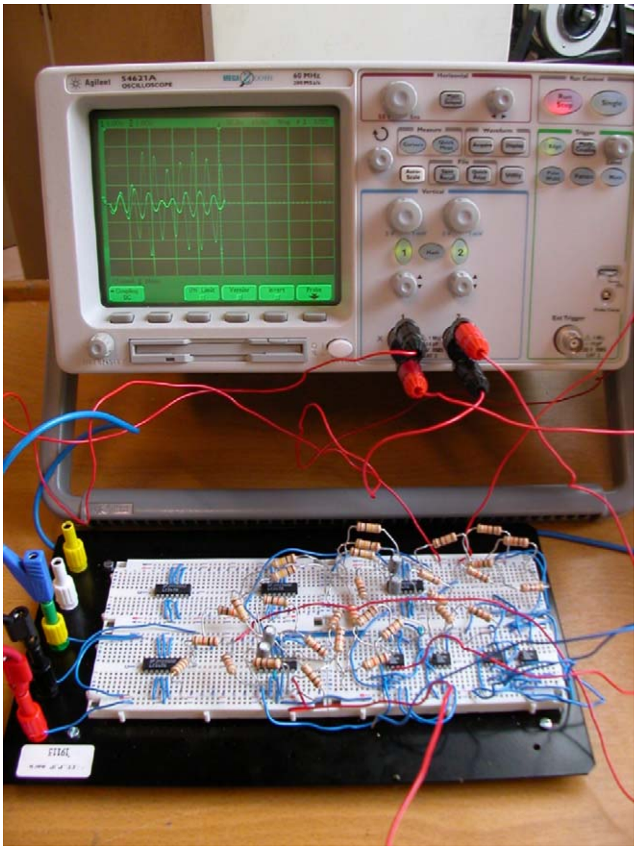

Fig. 3. General view of the electric circuit and oscilloscope with the results of simulation.

The energy of vibrations is transferred to the NES and damped out in quasiperiodic regime and thus attenuation of vibrations of the primary mass is achieved.

In order to assess the efficiency of this method of vibration suppression the performance of the NES is compared to that of properly tuned linear absorber having the same mass and the same damping coefficient (the damping coefficient value corresponds to proper tuning of the linear absorber). The dynamic of the linear absorber is described by the following system of equations:

$$\ddot{y}_1 + \varepsilon\lambda(\dot{y}_1 - \dot{y}_2) + y_1 + \varepsilon k(y_1 - y_2) = \varepsilon A \cos(1 + \varepsilon\sigma)t,$$
$$\varepsilon\ddot{y}_2 + \varepsilon\lambda(\dot{y}_2 - \dot{y}_1) + \varepsilon k(y_2 - y_1) = 0. \quad (24)$$

Both systems are considered in the point of the most dangerous resonance (maximum of the response amplitude) corresponding to slightly different values of frequency shift $\varepsilon\sigma$. The criterion chosen for comparison is energy of vibrations stored in the system at every moment of time. This energy is computed from Hamiltonians of systems described by Eqs. (1) and (24) without forcing and damping terms. Parameters of properly tuned linear absorber were computed by straightforward procedure of minimizing the response with the help of MAPLE software.

The results of comparison are presented in Fig. 6.

From Fig. 6 it is clear that in the regime of quasiperiodic response the NES ensures better suppression of oscillations than the best linear absorber with the same mass. The average energy stored in the vibrating system with NES is about 50% less than in the system with the linear absorber.

The above result demonstrates that the quasiperiodic response regime may be advantageous from the viewpoint of vibration absorption. Possible reason of this effect is that the nonlinear attachment in the

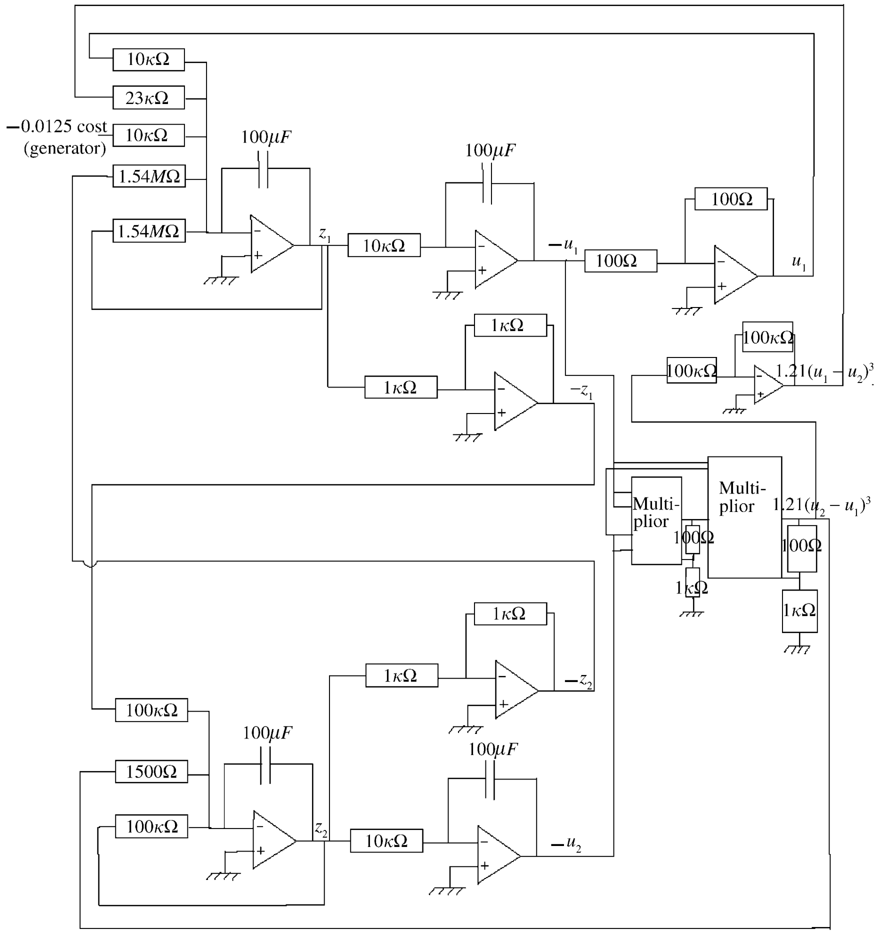


Fig. 4. Electric scheme of the experimental circuit.

quasiperiodic response regimes vibrates with relatively large amplitude and in multiple-frequency regime. Both these factors facilitate the damping of energy and give rise to better absorption capability.

It should be mentioned that the parameters used for the above simulation are different from those listed in set (23). The reason is that the quasiperiodic regime is more profound for set (23) (the modulation of fast frequency is very deep) and the advantage of the system with the NES with respect to the vibration suppression (as compared to the best-tuned linear absorber) is more obvious for parameters used in current section. This discrepancy raises a question concerning optimization of the system performance, which will be the object of further research.

The system under consideration exhibits this sort of response only for certain amplitude range of the external forcing. Outside this amplitude range the response will be approximately steady state, similarly to one described in Ref. [16], provided that no more complicated resonances will exhibit themselves. Our simulations demonstrate that no significant energetic advantage in the suppression of the oscillations is achieved in steady-state regime, although some broadening of the suppression frequency range may be demonstrated. The goal for further research in this field is to achieve broader range of quasiperiodic response of the NES via design of more suitable potential functions.

## 5. Concluding remarks and discussion

Quasiperiodic response of a strongly nonlinear energy sink with small mass attached to a harmonically forced oscillator has been revealed for the first time. The response regime is related to hysteretic motion of the phase trajectory of the system, in the vicinity of the invariant manifold of a damped-forced nonlinear normal

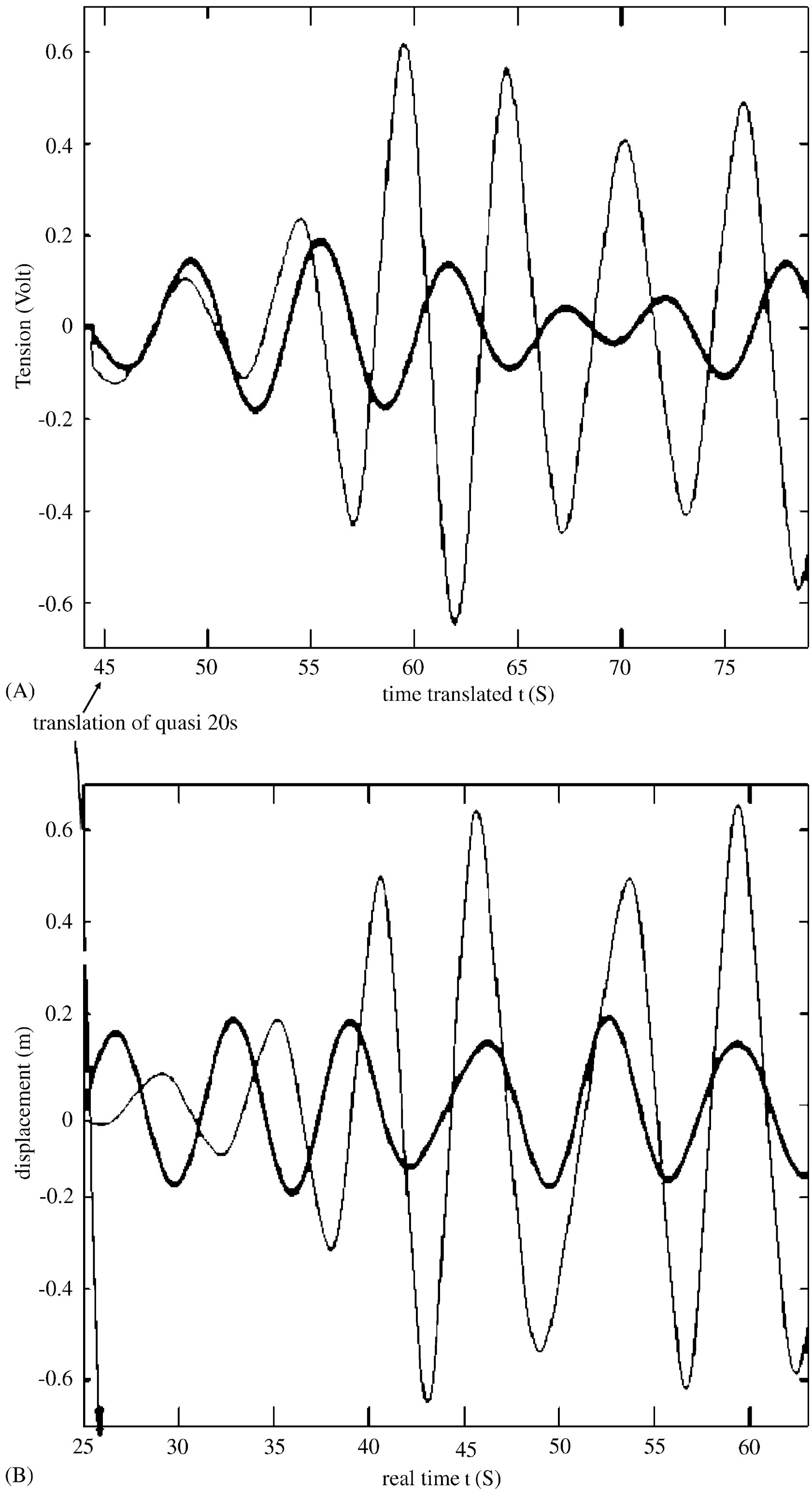


Fig. 5. Comparison between experimental (plot A) and numeric (plot B) results. Thick line represents the displacement of the primary linear oscillator and the thin line represents the displacement of the attachment. The electric scheme is calibrated in a way that quasi "shift" of about 20 s is created due to insufficient synchronization between external generator and the oscilloscope.

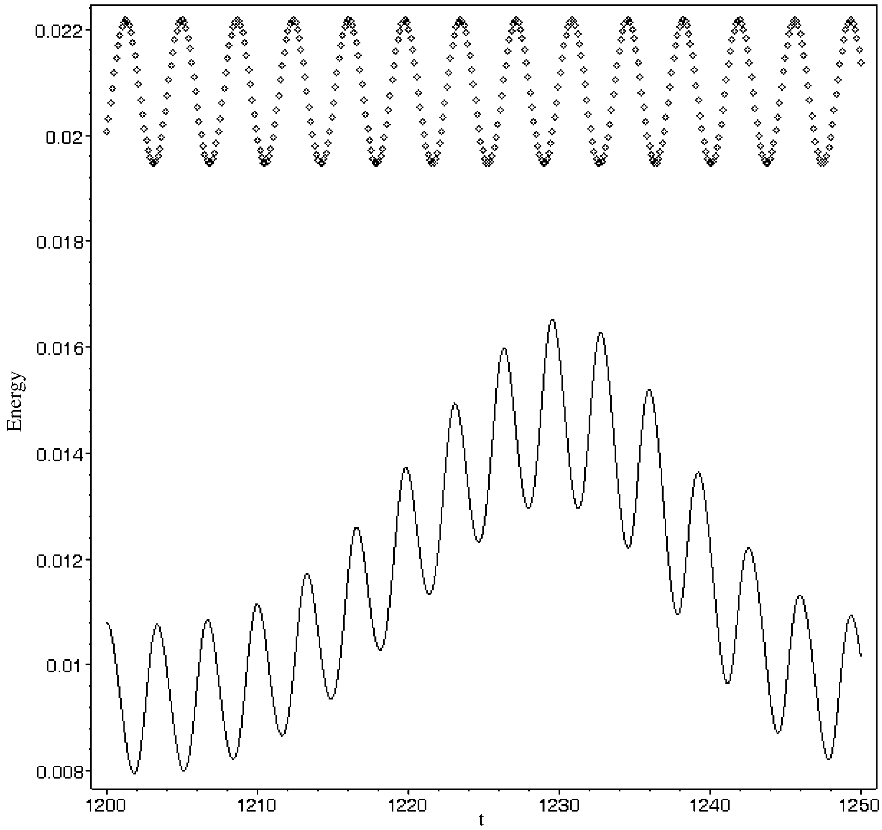


Fig. 6. Comparison of energy stored in the system for nonlinear (solid line and properly tuned linear (dotted line)) vibration absorbers. Parameters used are $A = 0.3$, $\varepsilon = 0.1$, $\lambda = 0.3$, $\omega = 0.95$ (nonlinear absorber), $\omega = 0.85$ (linear absorber), $k = 0.9$. The values of detuning were chosen in order to provide the highest amplitude of the response in each case.

mode. The real possibility of a quasiperiodic response regime was verified using an appropriately designed electric circuit. This response regime has been shown to have possible advantages from the viewpoint of suppressing the vibrations of the primary mass, for a certain range of amplitudes of the external forcing.

The above result leaves many questions open. In fact, vibration suppression in the regime of quasiperiodic response has been recorded for a weakly nonlinear vibration absorber [21] but no comparison of efficiency has been performed. The criterion used above for such comparison (energy stored in the system) is different from the commonly accepted criterion of vibration suppression + reduction of amplitude of the primary oscillator within a certain frequency band of the external forcing. The relation between these two criteria remains to be established. The other issues for further research are exact boundaries for the quasiperiodic response, optimization of the absorber performance and robustness of this method for vibration suppression.

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