Available online at www.sciencedirect.com
SCIENCE DIRECT®
Journal of Sound and Vibration 294 (2006) 894–915
JOURNAL OF SOUND AND VIBRATION
www.elsevier.com/locate/jsvi

A diffusion model for a one-dimensional structure, coupled with an auxiliary system

J.-M. Mencik[a,b], A. Berry[a,*]

[a]G.A.US, Département de Génie Mécanique, Université de Sherbrooke, 2500 Université, Sherbrooke, Qc, Canada, J1 K 2R1
[b]Laboratoire Vibrations Acoustique de l'INSA de Lyon, 20 Avenue Albert Einstein, 69621 Villeurbanne Cedex, France

Received 23 April 2004; received in revised form 7 November 2005; accepted 13 December 2005
Available online 10 March 2006

Abstract

This paper extends the concept of a local energy approach to homogeneous structures which are coupled with an auxiliary resonant system. The case of a one-dimensional homogeneous master structure (bar, beam) coupled over its length with a homogeneous auxiliary system composed of resonant arbitrary subsystems is analyzed. It is shown that under specific assumptions, the vibrational energy density of the coupled master structure can be predicted by solving a simple energetic boundary value problem that accounts for the mechanical coupling with the auxiliary subsystem. In the context of vibrational energy propagation, an important question is whether heterogeneity introduced by the auxiliary system enhances the diffusive behavior of the master structure. Numerical results for various types of auxiliary systems show that the effective diffusion coefficient of the coupled system is generally increased compared to the uncoupled master structure.

1. Introduction

Energy methods are well suited to predict the vibratory behavior of complex structures in the mid- and high-frequency range. The statistical energy analysis (SEA) is a global energy approach [1,2] that proposes a description of the vibrational energy in systems consisting of weakly coupled subsystems. In the mid- and high-frequency range, SEA provides an approximation of the spatially averaged energy of these subsystems. The application of SEA is subject to several specific assumptions: white noise excitation over a frequency band $\Delta\omega$; equal distribution of the energy over the various modes of a given subsystem in the frequency band $\Delta\omega$. The second condition is guaranteed for simple structure such as homogeneous plates, but not necessarily for complex structures. One difficulty of SEA is the evaluation of the coupling loss factor between connected subsystems [1,2], which allows evaluation of energy flow between these subsystems.

On the other hand, prediction of local energy in structures is possible using a thermal conductivity analogy [3], allowing a diffusion model of vibrational energy in structures. This approach has several advantages: a diffusion equation is a second-order differential equation which is simpler to solve than a classical displacement equation, the vibratory behavior of complex structures can therefore be approximated using a

*Corresponding author. Tel.: +1 819 821 8000x2148; fax: +1 819 821 7163.
E-mail address: alain.berry@usherbrooke.ca (A. Berry).

0022-460X/$ - see front matter
doi:10.1016/j.jsv.2005.12.036

thermal finite element solver; moreover, the solution of a diffusion equation is spatially smoother than the solution of the displacement problem, and under specific conditions provides a spatial average of the exact energy distribution in structures [4–8]. However, the applicability of a diffusion model for vibrating structures, as proposed by the thermal analogy, is based on two fundamental assumptions [9]: (1) at any point of the structure, the potential and the kinetic energy densities are equal; (2) a Fourier law, stating that the structural intensity is proportional to the gradient of the total energy density is assumed at any point in the structure. In fact, Le Bot et al. [4] have shown that the exact energetic model for a homogeneous Euler–Bernoulli beam is strictly different from a diffusion model. Ichchou et al. [10] have correctly derived the conditions required to formulate a diffusion equation inside a non-loaded homogeneous one-dimensional system (bar or beam). These conditions are: (1) linear and elastic system; (2) steady state conditions with harmonic excitations at frequency ω; (3) small hysteretic damping loss factor; (4) far from singularities, evanescent waves are neglected; (5) interference between progressive waves is not taken into account.

Under these assumptions, Langley [11] has shown that a homogeneous two-dimensional system (membrane, plate), whose displacement field can be decomposed into progressive plane waves, can be modeled as a diffusion equation according to the thermal analogy. Other authors have shown that the fact of neglecting the interference between progressive waves is equivalent to performing a spatial average of the energy quantities over half a wavelength [5–8]. Here again, the systems being studied are homogeneous and free of excitations.

In this paper, we propose to extend the concept of a local energy approach to homogeneous one-dimensional structures which are coupled with an homogeneous auxiliary system composed of resonant subsystems. The underlying question is whether heterogeneity introduced by the auxiliary subsystems enhances the diffusive behavior of the master structure. Diffusion models of a coupled bar and of a coupled beam are successively derived. In each case, it is shown that the vibrational energy density of the coupled master structure can be predicted by solving two energetic boundary value problems that account for the boundary impedance of the auxiliary system. Numerical results are presented for various types of auxiliary subsystems, especially the case of continuous ‘‘fuzzy’’ subsystems, as described in References [9,12]. In the context of energy diffusion, the proposed model has the advantage of characterizing the coupled master structure by a single diffusion coefficient.

2. Energetic model for a bar

The energy behavior of a homogeneous bar (master structure) coupled with a homogeneous and orthotropic auxiliary system composed of identical and independent elastic subsystems is investigated in this section. Both the potential energy density and the kinetic energy density are studied, since these two quantities are not generally equal for a coupled system.

2.1. Energy equation of a bar coupled with an auxiliary system

Let us consider a homogeneous elastic bar (master structure) of length L, as illustrated in Fig. 1, coupled over its length with an auxiliary system, and set in steady-state vibration by an external harmonic excitation of

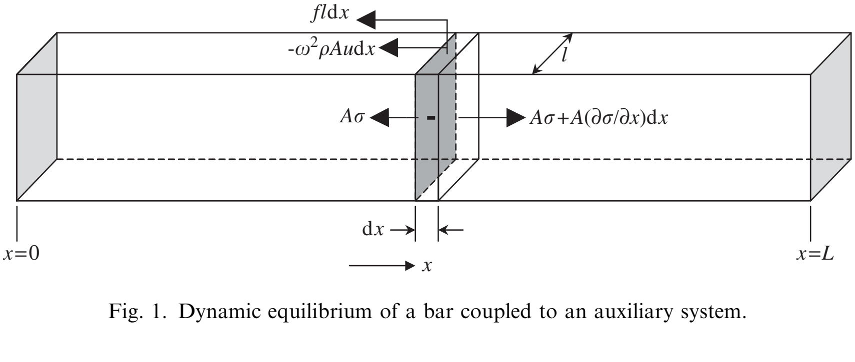

Fig. 1. Dynamic equilibrium of a bar coupled to an auxiliary system.

frequency $\omega/2\pi$. The coupling surface between the bar and the auxiliary system is denoted by Γ. It is assumed that the auxiliary system satisfies the following properties [9]:

(H1) On Γ, the auxiliary system is composed of a large number of identical and independent subsystems, uniformly distributed and coupled in a similar manner over Γ.
(H2) The surface Γ can be discretized into coupling subsurfaces of identical area, such that, on each coupling subsurface, the master structure is coupled with only one subsystem.
(H3) The displacement is constant over each subsurface.
(H4) Relative to each subsurface, each subsystem is excited in a same direction.
(H5) The surface force and surface displacement are collinear at any coupling point.

Specifically, the above assumptions describe an auxiliary system which is homogeneous and orthotropic on Γ, and which is continuously coupled with the master structure. In this context, it has been shown in References [9,12] that the mechanical action f applied by the bar (master structure) to the auxiliary system can be modeled on Γ from a boundary impedance Z:

$$f = \mathrm{i}\omega Z(\omega)u \quad \text{on } \Gamma, \tag{1}$$

where u represents the longitudinal displacement. The dynamic equilibrium of an element dx of the elastic bar leads to [13]

$$A\frac{\partial\sigma}{\partial x} + \omega^2\rho A u - f \times l = 0, \tag{2}$$

where A and ρ represent the cross-sectional area and the mass density of the bar, respectively, σ is the longitudinal stress which is related to the displacement by $\sigma = E\partial u/\partial x$ [13] (E is the complex Young's modulus, defined by $E = E_0(1 + \mathrm{i}\eta)$ where E_0 and η represent the Young's modulus and the loss factor of the bar, respectively). Finally, l represents an arbitrary coupling dimension along the bar width (see Fig. 1), the coupling surface area being given by $|\Gamma| = L \times l$.

Combining Eqs. (1) and (2) results in

$$\frac{\partial^2 u}{\partial x^2} + (a + \mathrm{i}b)u = 0 \quad \text{in}]0,\ L[, \tag{3}$$

where

$$a = \mathrm{Re}\left\{\frac{\omega^2\rho A - \mathrm{i}\omega Z \times l}{EA}\right\}, \quad b = \mathrm{Im}\left\{\frac{\omega^2\rho A - \mathrm{i}\omega Z \times l}{EA}\right\}. \tag{4}$$

It is assumed that the solution of Eq. (3) consists of weakly damped progressive waves, provided that: (1) $a > 0$, (2) $b \neq 0$ and (3) $\eta_{\text{bar}}^{\text{eq}} \ll 1$ where $\eta_{\text{bar}}^{\text{eq}} = |b/a|$ is defined, for $a \neq 0$, as the equivalent damping of the bar coupled with the auxiliary system. Indeed, under assumptions (1)–(3), Eq. (3) takes the familiar form

$$\frac{\partial^2 u}{\partial x^2} + \left(k_{\text{bar}}^{\text{eq}}\right)^2\left(1 - \mathrm{i}\eta_{\text{bar}}^{\text{eq}}\right)u = 0, \tag{5}$$

where $k_{\text{bar}}^{\text{eq}}$ is the wavenumber of the bar coupled with the auxiliary system, $\left(k_{\text{bar}}^{\text{eq}}\right)^2 = a$. The establishment of Eq. (5) is based on the fact that $b < 0$ [9]. The equivalent damping $\eta_{\text{bar}}^{\text{eq}}$ can be formulated from the components of the boundary impedance Z, which can be deterministically formulated by [9,12]

$$\mathrm{i}\omega Z = -\omega^2\left(\mu_0 + R\right) + \mathrm{i}\omega I \quad \text{on } \Gamma, \tag{6}$$

where $\mu_0 + R$ and I stand for the apparent mass per unit area and the apparent damping per unit area of the auxiliary system, respectively. Hence, under assumptions (1)–(3), the equivalent damping $\eta_{\text{bar}}^{\text{eq}}$ writes [9]

$$\eta_{\text{bar}}^{\text{eq}} \approx \eta + \frac{\omega I \times l}{\omega^2\left(\rho A + \left(\mu_0 + R\right) \times l\right)}. \tag{7}$$

The second term of the right-hand side of Eq. (7) represents the damping introduced by the auxiliary system: this term depends on the apparent damping I of the auxiliary system and on the mass per unit length

$\rho A + (\mu_0 + R) \times l$ of the bar coupled with the auxiliary system. Moreover, it can be shown that the second term of the right-hand side of Eq. (7) is always positive [9]: in other words, the auxiliary system always introduces damping. When $Z = 0$, it can be verified that $\eta_{\text{bar}}^{\text{eq}} \approx \eta$. Moreover, $I = 0$ results in $\eta_{\text{bar}}^{\text{eq}} \approx \eta$ which means that the damping of the bar is not modified by added point masses. Finally, $\eta_{\text{bar}}^{\text{eq}} \to \infty$ when $a \to 0$: this behavior will be numerically highlighted in Section 2.3 concerning an auxiliary system composed of linear oscillators.

In the following, an energy approach instead of displacement Eq. (3) is used to solve the dynamics of the coupled system. The time-averaged potential energy density U (J/m) and the time-averaged kinetic energy density T (J/m) of the bar (master structure), integrated over the section of the bar, are given by [9,14]

$$U = \frac{E_0 A}{4} \frac{\partial u}{\partial x} \frac{\overline{\partial u}}{\partial x}, \quad T = \frac{E_0 A k_0^2}{4} \; u \; \overline{u}, \tag{8,9}$$

where k_0 is the wavenumber of the uncoupled bar, $k_0^2 = \omega^2 \rho / E_0$, and "—" stands for the complex conjugate. Using Eqs. (3), (8) and (9), it can be shown that these energy densities are solutions of a single fourth-order differential equation:

$$\frac{\partial^4 F}{\partial x^4} + 4a \frac{\partial^2 F}{\partial x^2} - 4b^2 F = 0 \quad \text{in}]0, \; L[. \tag{10}$$

In this equation, F designates either U or T. It is possible to establish (see Appendix A) that under conditions (1)–(3) stated previously, solution $x \mapsto F$ can be expressed as the sum of two functions, $x \mapsto G$ and $x \mapsto H$, which are solutions of the following second-order differential equations:

$$\frac{\partial^2 G}{\partial x^2} - a \left(\eta_{\text{bar}}^{\text{eq}} \right)^2 G = 0 \quad \text{in}]0, \; L[, \tag{11}$$

$$\frac{\partial^2 H}{\partial x^2} + 4aH = 0 \quad \text{in}]0, \; L[, \tag{12}$$

where $\eta_{\text{bar}}^{\text{eq}} = |b/a|$. Eqs. (11) and (12) are a diffusion equation and a wave equation, respectively. The solution of Eq. (11) is analogous to the solution of a heat diffusion equation: it is composed of two exponentially decaying terms and its spatial variations depend on the dissipation phenomena that occur in the coupled bar. The solution of Eq. (12) is composed of oscillatory functions that result from the interference between progressive extensional waves: in the following, it is assumed that this part of the solution can be neglected [10], which means that $\|H\| \ll \|G\| \forall x$. The validity of this hypothesis is discussed in Section 2.3. Hence, under conditions (1)–(3) and under the assumption that the progressive waves are not taken into account in the evaluation of the energies, the potential and kinetic energies of the bar are solutions of a diffusion equation:

$$\frac{\partial^2 F}{\partial x^2} - \left(\gamma_{\text{diff}} \right)^2 F = 0 \quad \text{in}]0, \; L[, \tag{13}$$

where the diffusion coefficient, namely $\gamma_{\text{diff}}(\text{m}^{-1})$, is defined from the equivalent damping of the coupled bar $\eta_{\text{bar}}^{\text{eq}}$ (Eq. (7)):

$$\gamma_{\text{diff}} = \eta_{\text{bar}}^{\text{eq}} \sqrt{a}. \tag{14}$$

The general solution of Eq. (13) is

$$F = \alpha \mathrm{e}^{-\gamma_{\text{diff}} x} + \beta e^{\gamma_{\text{dif}f} x}, \tag{15}$$

where the spatial variations of the solution $x \mapsto F$ are related to the diffusion coefficient γ_{diff}.

2.2. *Energetic boundary conditions*

The determination of the constants α and β in Eq. (15) requires energetic boundary conditions of the system, at $x = 0$ and L. We assume that conditions (1) $a > 0$, (2) $b \neq 0$ and (3) $\eta_{\text{bar}}^{\text{eq}} \ll 1$ previously stated are satisfied. The energetic boundary conditions can be formulated in terms of active power. This approach is interesting in practice because active power at the end of a bar can easily be measured [15]. Active power $P(x,\omega)$ is defined at position $x \in [0, L]$ and at frequency $\omega/2\pi$, from the energy flow [9] entering the bar element illustrated in Fig. 1 [8]:

$$P = \frac{1}{2}\mathrm{Re}\left\{ \mathrm{i}\omega EA \frac{\partial u}{\partial x}\bar{u} \right\}. \tag{16}$$

Considering Eqs. (8) and (9), it can be easily shown (see Reference [9]) that the active power can be expressed either from the potential energy density, or from the kinetic energy density:

$$P = \frac{\omega}{2b(a^2+b^2)}\left((a - b\eta)\frac{\partial^3 U}{\partial x^3} + (4a^2 + 2b^2 - 2ab\eta)\frac{\partial U}{\partial x} \right) \tag{17}$$

or

$$P = \frac{\omega}{2bk_0^2}\left(\frac{\partial^3 T}{\partial x^3} + (4a - 2b\eta)\frac{\partial T}{\partial x} \right). \tag{18}$$

A rigorous derivation of the energetic boundary conditions requires to consider both exponential and oscillatory solutions in the potential and kinetic energies: $U = U_{\mathrm{diff}} + U_{\mathrm{wave}} \forall x \in]0, L[$ where $x \mapsto U_{\mathrm{diff}}$ and $x \mapsto U_{\mathrm{wave}}$ are solutions of diffusion Eq. (11) and wave Eq. (12), respectively; similarly, $T = T_{\mathrm{diff}} + T_{\mathrm{wave}} \forall x \in]0, L[$ where $x \mapsto T_{\mathrm{diff}}$ and $x \mapsto T_{\mathrm{wave}}$ are solutions of diffusion Eq. (11) and wave Eq. (12), respectively.

2.2.1. Potential energy density boundary value problem

Under the assumptions $\eta_{\mathrm{bar}}^{\mathrm{eq}} = |b/a| \ll 1$ and $\eta \ll 1$ (the structural damping of the bar is weak), the active power given by Eq. (17) can be simplified assuming $P = P_{\mathrm{diff}} + P_{\mathrm{wave}}$, where P_{diff} and P_{wave} are expressed for $a, b \neq 0$ (assumptions (1) and (2)) as functions of U_{diff} and U_{wave} from Eqs. (11) and (12):

$$P_{\mathrm{diff}} \approx \frac{2\omega}{b}\frac{\partial U_{\mathrm{diff}}}{\partial x} \tag{19}$$

and

$$P_{\mathrm{wave}} \approx \frac{\omega(b/a + \eta)}{a}\frac{\partial U_{\mathrm{wave}}}{\partial x}. \tag{20}$$

From Eq. (15), solution $G = U_{\mathrm{diff}}$ of diffusion Eq. (11) can be expressed as $G = G^+ + G^-$, such that $\partial G^{\pm}/\partial x = \mp \eta_{\mathrm{bar}}^{\mathrm{eq}}\sqrt{a}\ G^{\pm}$. Similarly, solution $H = U_{\mathrm{wave}}$ of wave Eq. (12) can be expressed from components H^+ and H^- propagating in opposite directions ($H = H^+ + H^-$) and such that $\partial H^{\pm}/\partial x = \mp 2\mathrm{i}\sqrt{a}\ H^{\pm}$. The corresponding expressions for the active power are $P_{\mathrm{diff}}^{\pm} \approx \mp 2\omega\eta_{\mathrm{bar}}^{\mathrm{eq}}\sqrt{a}/b\ U_{\mathrm{diff}}^{\pm}$ and $P_{\mathrm{wave}}^{\pm} \approx \mp 2\mathrm{i}\omega\left(b/a + \eta\right)/\sqrt{a}\ U_{\mathrm{wave}}^{\pm}$, which leads to $\left\|P_{\mathrm{diff}}^{\pm}\right\| \approx 2\omega/\sqrt{a}\left\|U_{\mathrm{diff}}^{\pm}\right\|$ and $\left\|P_{\mathrm{wave}}^{\pm}\right\| \leqslant 2\omega\left(\eta_{\mathrm{bar}}^{\mathrm{eq}} + \eta\right)/\sqrt{a}\left\|U_{\mathrm{wave}}^{\pm}\right\|$. Assuming that $\left\|U_{\mathrm{diff}}^{\pm}\right\|$ and $\left\|U_{\mathrm{wave}}^{\pm}\right\|$ are of the same order $\forall x \in]0, L[$ and considering the fact that $\eta_{\mathrm{bar}}^{\mathrm{eq}} \ll 1$ (assumption (3)) and $\eta \ll 1$, it appears that component $P_{\mathrm{wave}}^{\pm}$ can be neglected compared to component $P_{\mathrm{diff}}^{\pm}$. Hence $P \approx P_{\mathrm{diff}}$. The energy flow resulting from the oscillatory solution is therefore neglected. To summarize, the single quantity being taken into account in the calculation of the active power is the solution of diffusion Eq. (13). The potential energy density of the system is thus obtained by solving an energetic boundary value problem, under the conditions (1) $a > 0$, (2) $b \neq 0$, (3) $\eta_{\mathrm{bar}}^{\mathrm{eq}} \ll 1$ and $\eta \ll 1$:

$$\begin{cases} \dfrac{\partial^2 U(x,\omega)}{\partial x^2} - (\gamma_{\mathrm{diff}}(\omega))^2 U(x,\omega) = 0, \quad x \in]0, \ L[, \\ \left.\dfrac{\partial U(x,\omega)}{\partial x}\right|_{x=0} = \dfrac{b(\omega)}{2\omega}P(0,\omega), \quad \left.\dfrac{\partial U(x,\omega)}{\partial x}\right|_{x=L} = \dfrac{b(\omega)}{2\omega}P(L,\omega). \end{cases} \tag{21}$$

2.2.2. Kinetic energy density boundary value problem

Similarly, under the assumptions $\eta_{\mathrm{bar}}^{\mathrm{eq}} \ll 1$ and $\eta \ll 1$, the active power given by Eq. (18) can be simplified assuming $P = P_{\mathrm{diff}} + P_{\mathrm{wave}}$, where P_{diff} and P_{wave} are expressed for $b \neq 0$ (assumption (2)) as a function of T_{diff} and T_{wave} from Eqs. (11) and (12):

$$P_{\mathrm{diff}} \approx \frac{2\omega a}{bk_0^2}\frac{\partial T_{\mathrm{diff}}}{\partial x} \tag{22}$$

and

$$P_{\mathrm{wave}} = \frac{\omega\eta}{k_0^2}\frac{\partial T_{\mathrm{wave}}}{\partial x}. \tag{23}$$

As previously established for the potential energy density, it can be shown that active power components $P_{\mathrm{diff}}^{\pm}$ of P_{diff} are related to the kinetic energy density components $T_{\mathrm{diff}}^{\pm}$ of T_{diff} by $P_{\mathrm{diff}}^{\pm} \approx \mp 2\omega\eta_{\mathrm{bar}}^{\mathrm{eq}} a\sqrt{a}/\left(bk_0^2\right)\ T_{\mathrm{diff}}^{\pm}$. Moreover, the active power component $P_{\mathrm{wave}}^{\pm}$ of P_{wave} are related to $T_{\mathrm{wave}}^{\pm}$ by $P_{\mathrm{wave}}^{\pm} \approx \mp 2\mathrm{i}\omega\eta\sqrt{a}/k_0^2\ T_{\mathrm{wave}}^{\pm}$. These results lead to $\left\|P_{\mathrm{diff}}^{\pm}\right\| \approx 2\omega\sqrt{a}/k_0^2\left\|T_{\mathrm{diff}}^{\pm}\right\|$ and $\left\|P_{\mathrm{wave}}^{\pm}\right\| \approx 2\omega\eta\sqrt{a}/k_0^2\left\|T_{\mathrm{wave}}^{\pm}\right\|$. Assuming that $\left\|T_{\mathrm{diff}}^{\pm}\right\|$ and $\left\|T_{\mathrm{wave}}^{\pm}\right\|$ are of the same order $\forall x \in]0, L[$ and considering the fact that $\eta \ll 1$, it appears that components $P_{\mathrm{wave}}^{\pm}$ can be neglected compared to components $P_{\mathrm{diff}}^{\pm}$. Hence $P \approx P_{\mathrm{diff}}$. The kinetic energy density of the bar is thus obtained by solving the following energetic boundary value problem, under the conditions (1) $a > 0$, (2) $b \neq 0$, (3) $\eta_{\mathrm{bar}}^{\mathrm{eq}} \ll 1$ and $\eta \ll 1$:

$$\begin{cases} \dfrac{\partial^2 T(x,\omega)}{\partial x^2} - (\gamma_{\mathrm{diff}}(\omega))^2 T(x,\omega) = 0, \quad x \in]0, \ L[, \\ \left.\dfrac{\partial T(x,\omega)}{\partial x}\right|_{x=0} = \dfrac{b(\omega)k_0^2(\omega)}{2\omega a(\omega)}P(0,\omega), \quad \left.\dfrac{\partial T(x,\omega)}{\partial x}\right|_{x=L} = \dfrac{b(\omega)k_0^2(\omega)}{2\omega a(\omega)}P(L,\omega). \end{cases} \tag{24}$$

It should be noted that $k_0^2 U = aT \forall\, x \in [0, \ L]$. In the case where $a \neq k_0^2$ (bar coupled with an auxiliary system), we observe that $U \neq T$, whereas it is commonly assumed that $U = T$ when interference of progressive waves is neglected. This may be explained by the fact that in our case the homogeneous bar contains excitation sources (action of the coupled system), whereas the equality of the potential energy and kinetic energy holds for a system which does not contain excitation sources [10]. In the case of an uncoupled bar $\left(a = k_0^2\right)$, the above energetic boundary value problems are identical to the results found in the literature [10]: in this case, $\gamma_{\mathrm{diff}}^2 \approx \eta^2 w^2/c_g^2$, where c_g represents the group velocity of the waves propagating in the bar, $c_g = (E_0/\rho)^{1/2}$. It can also be verified that $\partial U/\partial x = \partial T/\partial x \approx -(\eta\omega/2c_g^2)P\ \ \forall x$. The two boundary value problems (21) and (24) are then equivalent which implies that $U = T\ \forall x$.

2.3. Numerical results

In the following, the numerical solutions of the diffusion Eq. (13), obtained by solving energetic boundary value problems (21) and (24), are compared to the exact energy densities, obtained by solving the exact displacement Eq. (3) and using Eqs. (8) and (9) to derive the exact potential and kinetic energy of the coupled bar.

2.3.1. A bar coupled with a deterministic auxiliary system

The case under study consists of a free–clamped homogenous bar of length $L = 1\,\mathrm{m}$, coupled with a homogenous resonant auxiliary system composed of $N = 100$ identical linear oscillators uniformly distributed over the length of the bar. An element of the coupled bar is illustrated in Fig. 2 (the bar is free at $x = 0$ and clamped at $x = L$). The characteristics of the bar are: Young's modulus $E_0 = 2.1 \times 10^{11}\,\mathrm{Pa}$, density $\rho = 7800\,\mathrm{kg/m^3}$, cross-sectional area $A = 10^{-4}\,\mathrm{m^2}$ and loss factor $\eta = 5 \times 10^{-3}$. The bar is excited at the free end ($x = 0$) by a harmonic force of amplitude $F = 1000\,\mathrm{N}$. The energetic boundary conditions of the system

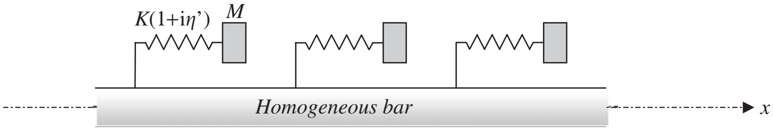

Fig. 2. Homogeneous elastic bar coupled with identical linear oscillators.

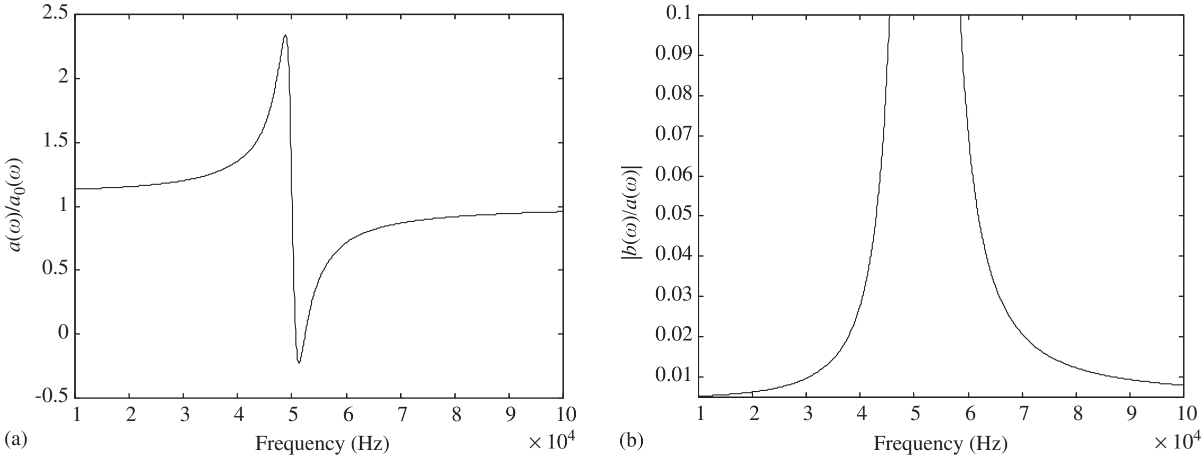

Fig. 3. Plot of the functions: (a) $\omega \mapsto a(\omega)/a_0(\omega)$ and (b) $\omega \mapsto |b(\omega) \mapsto /a(\omega)|$ in the case of the homogenous bar coupled with the linear oscillators.

are $P(0,\omega) = P_{\text{inj}}(\omega)$ (free end), where P_{inj} is the injected power calculated from the solution of Eq. (3),

$$P_{\text{inj}}(\omega) = -\tfrac{1}{2}\text{Re}\{\text{i}\omega F\overline{u(0,\omega)}\} \tag{25}$$

and $P(L,\omega) = 0$ (clamped end).

Each oscillator has mass $M = 10^{-3}$ kg (the ratio between the mass of the auxiliary system and the mass of the bar is about 13%), natural frequency $\Omega/2\pi = 5 \times 10^4$ Hz and loss factor $\eta' = 5 \times 10^{-2}$. The boundary impedance Z of the auxiliary system, which models its action on the bar (master structure) can be formulated at frequency $\omega/2\pi$ [9]:

$$\text{i}\omega Z(\omega) = -\omega^2 R_{\text{oscil}} + \text{i}\omega I_{\text{oscil}} \quad \text{on } \Gamma, \tag{26}$$

where R_{oscil} represents the apparent mass per unit area of the oscillators:

$$R_{\text{oscil}} = \frac{M}{S}\frac{(\Omega/\omega)^2((\Omega/\omega)^2(1+\eta^2)-1)}{((\Omega/\omega)^2-1)^2+\eta^2(\Omega/\omega)^4} \tag{27}$$

and I_{oscil} represents the apparent damping per unit area of the oscillators:

$$I_{\text{oscil}} = \frac{M}{S}\frac{\omega(\Omega/\omega)^2\eta}{((\Omega/\omega)^2-1)^2+\eta^2(\Omega/\omega)^4}, \tag{28}$$

with $S = |\Gamma|/N$.

It is first useful to verify the conditions (1) $a>0$, (2) $b\neq 0$ and (3) $\eta_{\text{bar}}^{\text{eq}}|b/a| \ll 1$ (a and b are defined by Eq. (4)) required for the validity of energetic boundary value problems (21) and (24): for this purpose, the functions $\omega \mapsto a/a_0$ (where $a_0 = a_{Z(\omega)=0} \approx k_0^2$) and $\omega \mapsto \eta_{\text{bar}}^{\text{eq}}$, which represents the equivalent damping of the coupled bar, are plotted in the frequency range centered on the natural frequency $\Omega/2\pi$ of the oscillators (Fig. 3). At 10^4 Hz, the uncoupled bar contains almost two extensional wavelengths, whereas at 10^5 Hz, the uncoupled bar contains almost 19 wavelengths.

Below the natural frequency $\Omega/2\pi$, the action of the oscillators on the bar is an added mass and we observe that $a/a_0 > 1 \forall \omega < \Omega$. Above the natural frequency $\Omega/2\pi$, the action of the oscillators is an added stiffness and $a/a_0 < 1 \forall \omega > \Omega$. Furthermore, the function $\omega \mapsto a/a_0$ takes negative values (hence condition (1) is violated) in the frequency range $[5 \times 10^4\,\mathrm{Hz}, 5.4 \times 10^4\,\mathrm{Hz}]$. On the other hand, the equivalent damping of the coupled bar $\eta_{\mathrm{bar}}^{\mathrm{eq}} = |b/a|$ is strictly positive in the frequency band under consideration, which implies that condition (2) $b \neq 0$ is satisfied. Moreover, $\eta_{\mathrm{bar}}^{\mathrm{eq}} \mapsto \infty$ when a/a_0 vanishes. Therefore, condition (3) $\eta_{\mathrm{bar}}^{\mathrm{eq}} \ll 1$ is violated close to the natural frequency $\Omega/2\pi$ of the oscillators. Far enough from the natural frequency, the equivalent damping of the coupled bar is approximately equal to the loss factor of the bar, $\eta = 5 \times 10^{-3}$. Near the natural frequency $\Omega/2\pi$, a strong energy transfer occurs from the bar to the resonant oscillators and the resulting equivalent damping cannot be considered negligible.

The potential energy and kinetic energy densities at position $x = L/2$ are presented in Fig. 4 as a function of frequency. The diffusion coefficient of the bar coupled with the oscillators γ_{diff} (Eq. (14)) is compared to the diffusion coefficient of the uncoupled bar ($Z = 0$) in Fig. 5. In Fig. 4, the modal behavior is well captured by

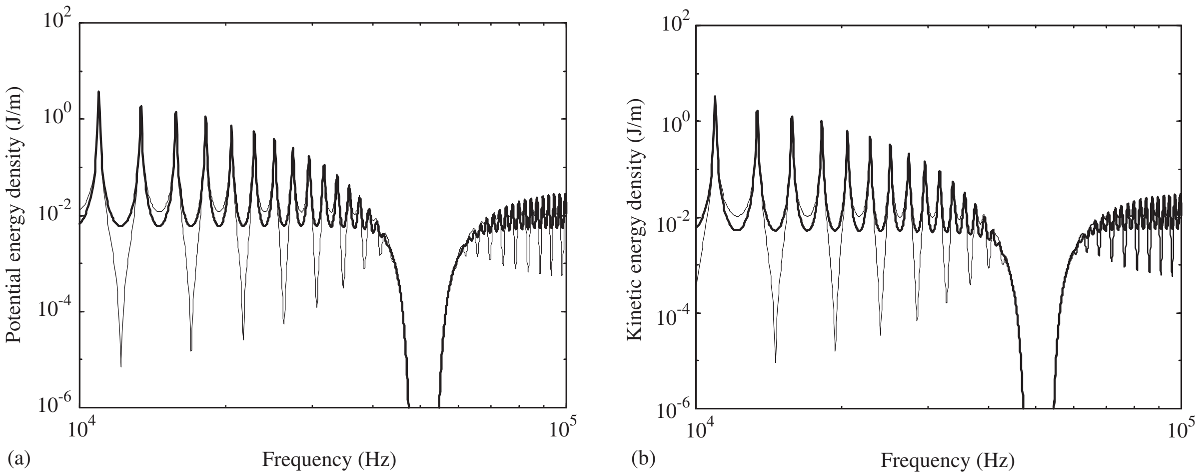

Fig. 4. (a) Potential energy density and (b) kinetic energy density of the homogeneous bar coupled with the linear oscillators, at position $x = L/2$: ——, exact values; ——, solutions of the diffusion equation.

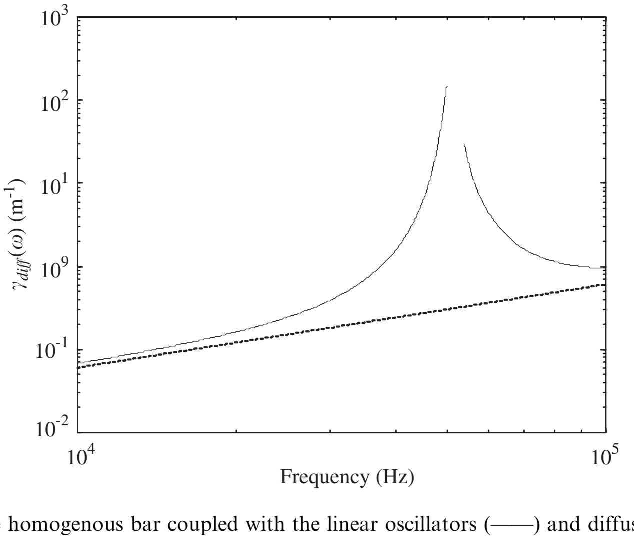

Fig. 5. Diffusion coefficient of the homogenous bar coupled with the linear oscillators (——) and diffusion coefficient of the uncoupled homogeneous bar (————).

the diffusion model because the boundary condition in terms of input power is exactly calculated. The deviation between the solutions of the diffusion equation and the exact energy densities results from the interference between progressive waves. This deviation decreases close to the resonance of the oscillators, as both the response of the oscillators and the diffusion coefficient increases: the diffusive behavior of the bar is enhanced by the energy transfer in the auxiliary system. The appropriateness of the diffusion model to describe the system dynamics is limited by the following two aspects: (1) far from resonance of the oscillators (small diffusion coefficient), the interference between progressive waves introduces important differences between energy-based prediction and exact values; (2) in the frequency range [4.5×10^4 Hz, 5.7×10^4 Hz], the diffusion coefficient and equivalent damping of the system become too large to be compatible with the assumptions of the diffusion equation model.

To highlight the above results, the spatial variations of the potential and kinetic energies along the bar are plotted at a frequency closed to the natural frequency of the oscillators, $\omega_1/2\pi = 4.4 \times 10^4$ Hz (Fig. 6), and at a frequency far from the natural frequency of the oscillators, $\omega_2/2\pi = 7 \times 10^4$ Hz (Fig. 7). At these frequencies, the diffusion coefficients are $\gamma_{\text{diff}}(\omega_1) \approx 4.3\,\text{m}^{-1}$ and $\gamma_{\text{diff}}(\omega_2) \approx 1.6\,\text{m}^{-1}$, respectively. Close to the resonance of the auxiliary system ($\omega_1/2\pi = 4.4 \times 10^4$ Hz), the spatial variation of the energy in the bar is well

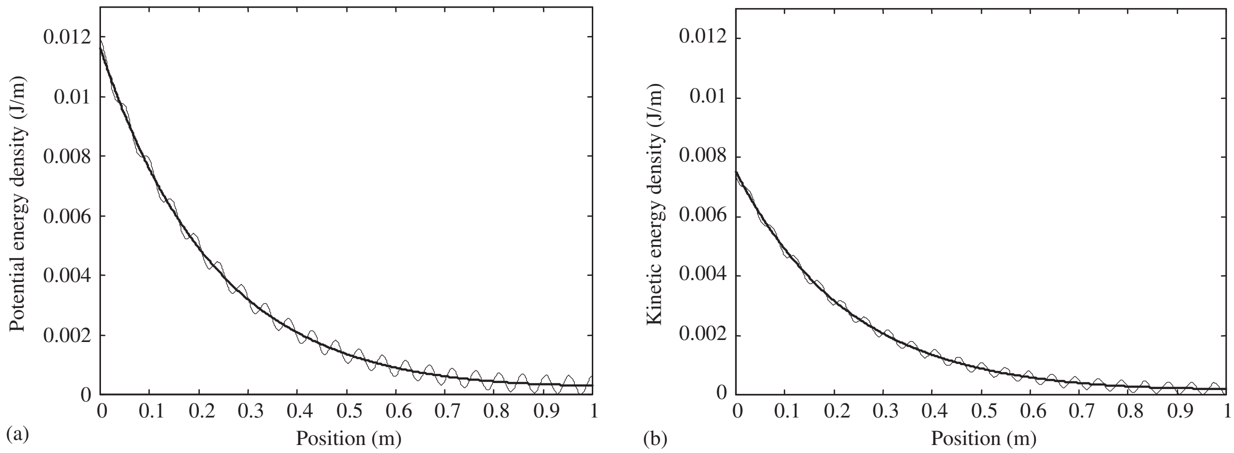

Fig. 6. Spatial variations of (a) potential energy density and (b) kinetic energy density of the homogeneous bar coupled with the linear oscillators, at frequency $\omega_1/2\pi = 4.4 \times 10^4$ Hz: ——, exact values; ——, solutions of the diffusion equation.

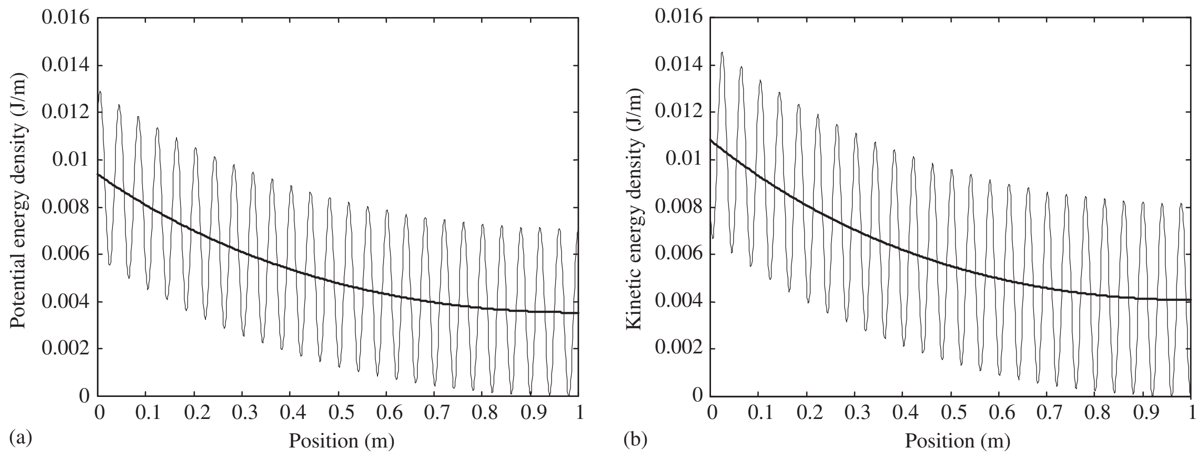

Fig. 7. Spatial variations of (a) potential energy density and (b) kinetic energy density of the homogeneous bar coupled with the linear oscillators, at frequency $\omega_1/2\pi = 7 \times 10^4$ Hz: ——, exact values; ——, solutions of the diffusion equation.

approximated by the solutions of the energetic boundary value problem: Fig. 6 shows a small contribution of the oscillatory contribution, as well as a strong decay of the energy along the bar. On the other hand, far from the resonance ($\omega_2/2\pi = 7 \times 10^4$ Hz), the interference between progressive waves introduces strong differences between exact values and energetic predictions.

It is worth noting that the coupling action of the auxiliary system results in non-equal values of the potential and kinetic energy densities in general. When $\omega < \Omega$, the added inertia introduced by the auxiliary system results in a smaller kinetic energy density. When $\omega > \Omega$, the added stiffness of the auxiliary system results in a smaller potential energy density.

2.3.2. A bar coupled with a fuzzy auxiliary system

The case investigated here consists of the same bar as before, coupled with a homogenous structural fuzzy composed of elastic bars whose exact geometrical parameters are assumed unknown and are described statistically (Fig. 8). A probabilistic model of the boundary impedance for such a fuzzy has been proposed in References [9,12]. The case of a fuzzy auxiliary system addresses the diffusive behavior in a master structure coupled with a non-deterministic appendage that has a continuous distribution of natural frequencies. The bar is excited at the free end ($x = 0$) by a harmonic force of amplitude $F = 1000$ N.

The fuzzy consists in $N = 50$ identical subsystems. Each fuzzy subsystem is composed of 1000 clamped–free bars whose length Λ and section area Σ are defined by two continuous independent normalized random variables [9,12], with mean values $\Lambda = 0.1$ m (length) and $\Sigma = 5 \times 10^{-10}$ m^2 (section area) and dispersion parameters $\lambda_1 = 0.6$ and $\lambda_2 = 0.4$, respectively. The other parameters of the fuzzy are: mass density $\rho' = 31{,}400$ kg/m^3, Young's modulus $E'_0 = 2.1 \times 10^{11}$ Pa, loss factor $\eta' = 5 \times 10^{-2}$, modal density $n \approx 10^{-2}(\text{rad s}^{-1})^{-1}$, fundamental natural frequency $\Omega_1/2\pi \approx 4 \times 10^3$ Hz. The mean mass of the fuzzy corresponds approximately to 10% of the mass of the master structure.

The underlying assumptions of the diffusion model, (1) $a > 0$, (2) $b \neq 0$ and (3) $\eta_{\text{bar}}^{\text{eq}} = |b/a| \ll 1$, are first numerically verified in the case of a fuzzy auxiliary system. At each frequency in the interval [10^2 Hz, 5×10^5 Hz], the mathematical expectation of the boundary impedance Z is calculated by numerical integration over [$\Omega_1/4\pi$, 10^6 Hz] [12]. The functions $\omega \mapsto a/a_0$ and $\omega \mapsto \eta_{\text{bar}}^{\text{eq}}$ are plotted in Fig. 9. The conditions (1)–(3) are respected. The functions $\omega \mapsto a/a_0$ and $\omega \mapsto \eta_{\text{bar}}^{\text{eq}}$ reach their maximum approximately at the natural frequency of the fuzzy $\Omega_1/2\pi \approx 4 \times 10^3$ Hz. Close to $\Omega_1/2\pi$, the action of the fuzzy on the bar is both inertial and dissipative. Below $\Omega_1/2\pi$, the dominant effect is added inertia and above $\Omega_1/2\pi$, the action of the fuzzy is mostly to increase the equivalent damping in the bar (with an asymptotic value of the equivalent damping being that of the bar in high frequency).

The potential energy and kinetic energy densities at position $x = L/2$ are presented in Fig. 10 as a function of frequency. The diffusion coefficient of the bar coupled with the fuzzy γ_{diff} (Eq. (14)) is compared to the diffusion coefficient of the uncoupled bar ($Z = 0$) in Fig. 11. The diffusion model adequately picks up the main trends in the frequency dependence of the potential and kinetic energies: the action of the fuzzy on the bar is apparent in the frequency range [$\Omega_1/2\pi$, 10^5 Hz] starting at the fundamental frequency of the fuzzy. In this frequency range, the response of the system is considerably damped. This is also apparent on the diffusion coefficient γ_{diff} (Fig. 11) which is significantly increased in [$\Omega_1/2\pi$, 10^5 Hz]. The fuzzy clearly enhances the diffusive behavior of the system in the frequency range starting at the fundamental frequency of the fuzzy.

Fig. 8. Homogeneous bar coupled to a structural fuzzy.

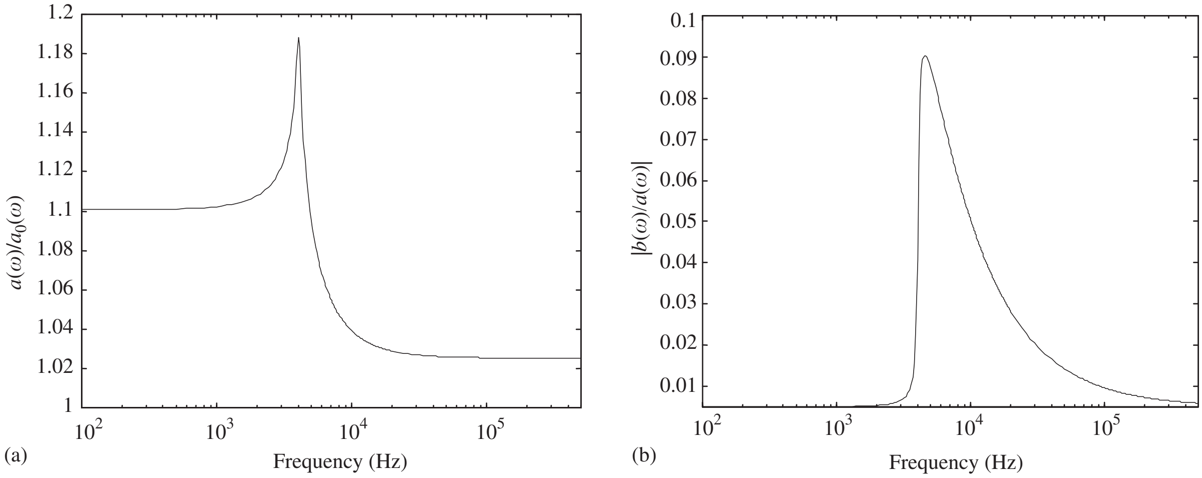

Fig. 9. Plot of the functions (a) $\omega \mapsto a(\omega)/a_0(\omega)$ and (b) $\omega \mapsto |b(\omega) \mapsto /a(\omega)|$ in the case of the homogenous bar coupled with a structural fuzzy.

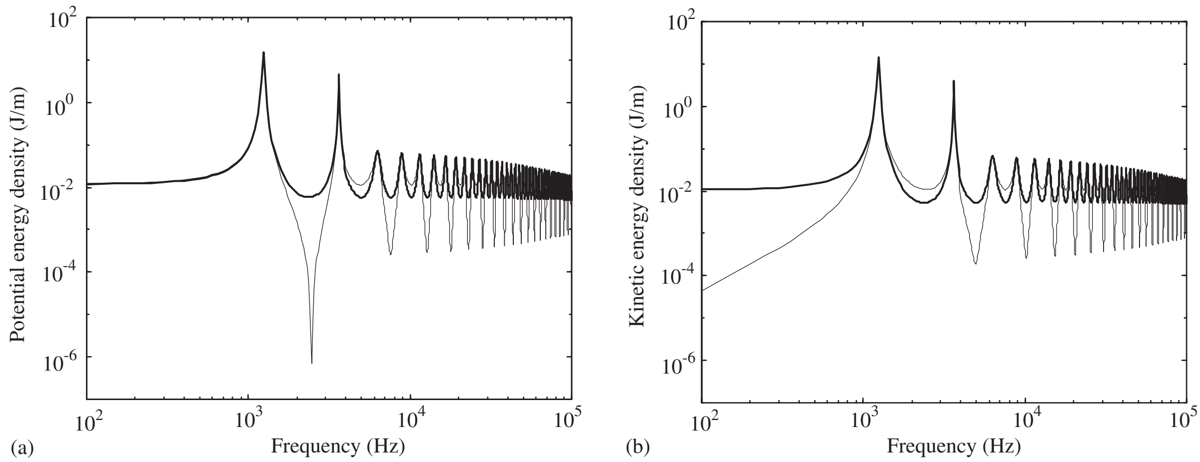

Fig. 10. (a) Potential energy density and (b) kinetic energy density of the homogeneous bar coupled with the fuzzy, at position $x = L/2$: ——, exact values; ——, solutions of the diffusion equation.

3. Energetic model for a beam

The case of a bending beam is now analyzed: the energy behavior of a homogeneous beam (master structure) coupled with a homogeneous and orthotropic auxiliary system composed of identical and independent elastic subsystems is investigated in this section.

3.1. Energy equation of a beam coupled with an auxiliary system

We consider a homogeneous Euler–Bernoulli beam (master structure) of length L, coupled over its length with a homogeneous and orthotropic auxiliary system. Assumptions (H1)–(H5) of Section 2.1 regarding the auxiliary system still hold. The disturbance on the coupled system is a transverse external harmonic excitation of frequency $\omega/2\pi$. Under pure bending assumption, the forces and moments applied to a beam element dx located at position x are illustrated in Fig. 12.

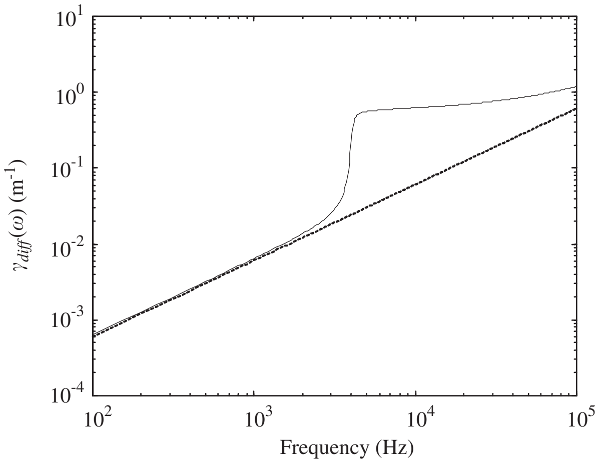

Fig. 11. Diffusion coefficient of the homogenous bar coupled with the fuzzy (——) and diffusion coefficient of the uncoupled homogeneous bar (————).

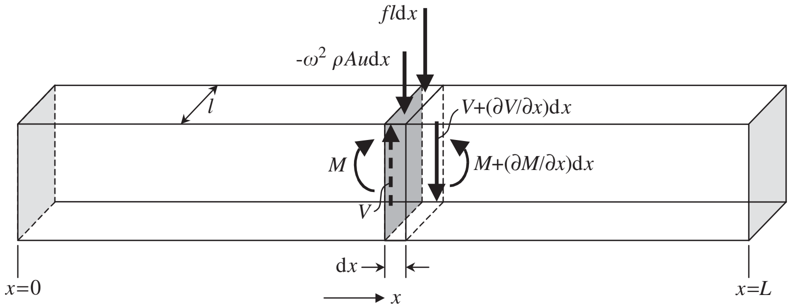

Fig. 12. Dynamic equilibrium of a beam coupled to an auxiliary system.

The force equilibrium of the beam element is [13]

$$\frac{\partial V}{\partial x} - \omega^2 \rho A u + f \times l = 0 \tag{29}$$

and the moment equilibrium condition gives [13]

$$-V + \frac{\partial M}{\partial x} \approx 0. \tag{30}$$

Here, u represents the transverse displacement, V is the shearing force, and M is the bending moment, defined by $M = EI\partial^2 u/\partial x^2$ (EI: complex bending rigidity, $E = E_0(1 + \mathrm{i}\eta)$: complex Young's modulus, E_0: Young's modulus and η: loss factor of the beam). It has been shown in References [9,12] that assumptions (H1)–(H5) of Section 2.1 allow to write the surface force f applied by the master structure to the auxiliary system as a function of a boundary impedance Z on the coupling surface (say Γ),$f = \mathrm{i}\omega Z u$. In Eq. (29), A and ρ represent the cross-sectional area and mass density of the beam, respectively, and l represents an arbitrary coupling dimension along the bar width, the coupling surface area being given by $|\Gamma| = L \times l$.

Combining Eqs. (29) and (30) results in

$$\frac{\partial^4 u}{\partial x^4} - (a + \mathrm{i}b)u = 0 \quad \mathrm{in}]0, \;\; L[, \tag{31}$$

where

$$a = \mathrm{Re}\left\{\frac{\omega^2 \rho A - \mathrm{i}\omega Z \times l}{EI}\right\}, \quad b = \mathrm{Im}\left\{\frac{\omega^2 \rho A - \mathrm{i}\omega Z \times l}{EI}\right\}. \tag{32}$$

Following an approach similar to the case of the bar, it is assumed that the solution of Eq. (31) is composed of weakly damped progressive and evanescent waves, provided: (1) $a>0$, (2) $b\neq 0$ and (3) $\eta_{\mathrm{beam}}^{\mathrm{eq}} \ll 1$ where $\eta_{\mathrm{beam}}^{\mathrm{eq}} = |b/a|$ is defined, for $a\neq 0$, as the equivalent damping of the beam coupled with the auxiliary system. Assumptions (1)–(3) allow to write Eq. (31) in the familiar form

$$\frac{\partial^2 u}{\partial x^2} - \left(k_{\mathrm{beam}}^{\mathrm{eq}}\right)^4\left(1 - \mathrm{i}\eta_{\mathrm{beam}}^{\mathrm{eq}}\right)u = 0, \tag{33}$$

where $k_{\mathrm{beam}}^{\mathrm{eq}}$ is the equivalent wavenumber of the beam coupled with the auxiliary system, $(k_{\mathrm{beam}}^{\mathrm{eq}})^4 = a$. According to Eqs. (32) and (4), it appears that the expressions of the equivalent damping for the beam and for the bar are rigorously similar for auxiliary systems described by the same boundary impedance operator

$$\eta_{\mathrm{beam}}^{\mathrm{eq}} = \eta_{\mathrm{bar}}^{\mathrm{eq}}. \tag{34}$$

In particular, under assumptions (1)–(3), the equivalent damping of the beam $\eta_{\mathrm{beam}}^{\mathrm{eq}}$ is expressed from Eq. (7).

In the following, an energy approach instead of displacement Eq. (31) is used to solve the dynamics of the coupled system. The time-averaged potential energy density U (J/m) and the time-averaged kinetic energy density T (J/m) of the beam (master structure), integrated over the section of the beam [9], are given by [14]

$$U = \frac{E_0 I}{4}\frac{\partial^2 u}{\partial x^2}\overline{\frac{\partial^2 u}{\partial x^2}}, \quad T = \frac{E_0 I k_0^4}{4} u\bar{u}, \tag{35,36}$$

where $k_0^4 = \omega^2 \rho A / E_0 I$ is the wavenumber in the uncoupled bar. These energy densities are solutions of two coupled, eighth-order differential equations (Appendix B):

$$\frac{\partial^8}{\partial x^8}\begin{pmatrix} U \\ T \end{pmatrix} - 4\begin{pmatrix} a & 3(a^2+b^2)/k_0^4 \\ 3k_0^4 & a \end{pmatrix}\frac{\partial^4}{\partial x^4}\begin{pmatrix} U \\ T \end{pmatrix} + 4\begin{pmatrix} 2a^2+b^2 & -2a(a^2+b^2)/k_0^4 \\ -2ak_0^4 & 2a^2+b^2 \end{pmatrix}\begin{pmatrix} U \\ T \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{37}$$

The resolution of such a system was proposed by Le Bot et al. [4] in the case of an uncoupled homogeneous beam. A similar approach can be applied in the present case: we suppose that the solutions of system (37) are of the form

$$\begin{pmatrix} U_i \\ T_i \end{pmatrix} = \begin{pmatrix} A_i(\omega) \\ B_i(\omega) \end{pmatrix} \mathrm{e}^{\lambda_i(\omega)x}. \tag{38}$$

Under the conditions (1) $a>0$, (2) $b\neq 0$ and (3) $\eta_{\mathrm{beam}}^{\mathrm{eq}} = |b/a| \ll 1$, the 16 eigenvalues $\{\lambda_i\}_{i=1,\ldots,16}$ are given by (Appendix B):

$$\begin{aligned}\{\lambda_i\} = \Bigg\{ & \frac{\sqrt[4]{a}}{2}\eta_{\mathrm{beam}}^{\mathrm{eq}}, -\frac{\sqrt[4]{a}}{2}\eta_{\mathrm{beam}}^{\mathrm{eq}}, \mathrm{i}\frac{\sqrt[4]{a}}{2}\eta_{\mathrm{beam}}^{\mathrm{eq}}, -\mathrm{i}\frac{\sqrt[4]{a}}{2}\eta_{\mathrm{beam}}^{\mathrm{eq}}, \\ & 2\sqrt[4]{a}, -2\sqrt[4]{a}, 2\mathrm{i}\sqrt[4]{a}, -2\mathrm{i}\sqrt[4]{a}, \\ & (1+\mathrm{i})\sqrt[4]{a}\sqrt[4]{1+\eta_{\mathrm{beam}}^{\mathrm{eq}}}, -(1+\mathrm{i})\sqrt[4]{a}\sqrt[4]{1+\eta_{\mathrm{beam}}^{\mathrm{eq}}}, (1+\mathrm{i})\sqrt[4]{a}\sqrt[4]{1-\eta_{\mathrm{beam}}^{\mathrm{eq}}}, -(1+\mathrm{i})\sqrt[4]{a}\sqrt[4]{1-\eta_{\mathrm{beam}}^{\mathrm{eq}}}, \\ & (1-\mathrm{i})\sqrt[4]{a}\sqrt[4]{1+\eta_{\mathrm{beam}}^{\mathrm{eq}}}, -(1-\mathrm{i})\sqrt[4]{a}\sqrt[4]{1+\eta_{\mathrm{beam}}^{\mathrm{eq}}}, (1-\mathrm{i})\sqrt[4]{a}\sqrt[4]{1-\eta_{\mathrm{beam}}^{\mathrm{eq}}}, -(1-\mathrm{i})\sqrt[4]{a}\sqrt[4]{1-\eta_{\mathrm{beam}}^{\mathrm{eq}}}\Bigg\}.\end{aligned} \tag{39}$$

These solutions are distinct since it is supposed that $\eta_{\mathrm{beam}}^{\mathrm{eq}} \neq 0$ $(b\neq 0)$. Hence, the potential and kinetic energies take the general form

$$\begin{pmatrix} U \\ T \end{pmatrix} = \sum_i \begin{pmatrix} U_i \\ T_i \end{pmatrix}. \tag{40}$$

The various eigenvalues can be physically interpreted as follows:

- Solutions $\mp \mathrm{i}(a^{1/4}/2) \times \eta^{\mathrm{eq}}_{\mathrm{beam}}$ are associated to weakly fluctuating energy densities along the structure (according to the condition $\eta^{\mathrm{eq}}_{\mathrm{beam}} \ll 1$) which result from the interference between two evanescent waves. In the high frequency range, such evanescent waves are however confined to the boundaries and excitation point in the beam. It will be assumed that these waves, and hence the energy density associated to their interference, are neglected far from the singularities [16].
- Solutions $\pm 2a^{1/4}, \pm(1 \pm \mathrm{i})a^{1/4}(1 \pm \eta^{\mathrm{eq}}_{\mathrm{beam}})^{1/4}$ are associated to strongly fluctuating energy densities along the structure. The solutions $\pm(1 \pm \mathrm{i})a^{1/4}(1 \pm \eta^{\mathrm{eq}}_{\mathrm{beam}})^{1/4}$ result from the interference between progressive and evanescent waves. Similarly, since evanescent waves are confined to the boundaries and excitation point in the beam, the above solutions will be neglected in terms of energy density along the beam.
- Finally, solutions $\pm 2\mathrm{i}a^{1/4}$ are associated to standing waves which result from the interference between two progressive waves. Following Ichchou et al. [10] in the case of an uncoupled homogeneous structure, these solutions can also be neglected. The validity of this assumption will be discussed in Section 3.3.

To summarize, under the assumption that evanescent waves are confined to boundaries and driving point and that the energy resulting from interference between progressive waves can be neglected, the energy densities of the structure can be obtained from the remaining two eigenvalues $\pm(a^{1/4}/2)\eta^{\mathrm{eq}}_{\mathrm{beam}}$; the energy in the system can be derived from the diffusion equation:

$$\frac{\partial^2 F}{\partial x^2} - (\gamma_{\mathrm{diff}})^2 F = 0 \quad \mathrm{in}]0, \ L[, \tag{41}$$

where F designates either U or T, and γ_{diff} stands for the diffusion coefficient (m^{-1}):

$$\gamma_{\mathrm{diff}} = \eta^{\mathrm{eq}}_{\mathrm{beam}} \frac{a^{1/4}}{2}. \tag{42}$$

The general solution of Eq. (41) is

$$F = \alpha \mathrm{e}^{-\gamma_{\mathrm{diff}} x} + \beta e^{\gamma_{\mathrm{diff}} x}, \tag{43}$$

where the spatial variations of the solution $x \mapsto F$ are related to the diffusion coefficient γ_{diff}.

It can be observed that, although the equivalent damping of the beam $\eta^{\mathrm{eq}}_{\mathrm{beam}}$ is formally identical to the equivalent damping of the bar $\eta^{\mathrm{eq}}_{\mathrm{bar}}$, the expressions of the diffusion coefficients for the two structures are different (see Eqs. (14) and (42)). That means that the energy densities in the two systems do not exhibit the same spatial variations. This result seems natural if we consider that the equations of diffusion for bar and beam are derived from strongly different energetic relationships (see Eqs. (10) and (37)).

3.2. Energetic boundary conditions

The determination of the constants α and β in Eq. (43) requires energetic boundary conditions of the system, at $x = 0$ and L. Similarly to the case of a bar, the energetic boundary conditions are expressed in terms of active power in the beam. The assumptions (1) $a > 0$, (2) $b \neq 0$ and (3) $\eta^{\mathrm{eq}}_{\mathrm{beam}} \ll 1$ previously stated still hold. Active power $P(x,\omega)$ is defined at position $x \in [0, L]$ and at frequency $\omega / 2\pi$ as [8]

$$P = -\frac{1}{2}\mathrm{Re}\left\{ \mathrm{i}\omega EI \frac{\partial^3 u}{\partial x^3} \bar{u} \right\} + \frac{1}{2}\mathrm{Re}\left\{ \mathrm{i}\omega EI \frac{\partial^2 u}{\partial x^2} \overline{\frac{\partial u}{\partial x}} \right\}. \tag{44}$$

Similarly, reactive power $Q(x,\omega)$ is defined by [8]

$$Q = -\frac{1}{2}\mathrm{Im}\left\{ \mathrm{i}\omega EI \frac{\partial^3 u}{\partial x^3} \bar{u} \right\} + \frac{1}{2}\mathrm{Im}\left\{ \mathrm{i}\omega EI \frac{\partial^2 u}{\partial x^2} \overline{\frac{\partial u}{\partial x}} \right\}. \tag{45}$$

Appendix C establishes that P and Q can be expressed from the third and seventh derivatives of the energy densities:

$$\begin{pmatrix} P \\ Q \end{pmatrix} = \frac{\omega}{2k_0^4 b^3}\left(-a\begin{pmatrix} k_0^4(2-(b/a)\eta) & a(2+(b/a)^2-(b/a)\eta) \\ k_0^4(2\eta+(b/a)) & a(2\eta+(b/a)^2\eta+(b/a)) \end{pmatrix}\frac{\partial^7}{\partial x^7}\begin{pmatrix} U \\ T \end{pmatrix} + 4a^2 \right.$$
$$\left. \times \begin{pmatrix} k_0^4(8+3(b/a)^2-4(b/a)\eta) & a(8+7(b/a)^2-4(b/a)\eta-3(b/a)^3\eta) \\ k_0^4(8\eta+3(b/a)^2\eta+4(b/a)) & a(8\eta+7(b/a)^2\eta+4(b/a)+3(b/a)^3) \end{pmatrix}\frac{\partial^3}{\partial x^3}\begin{pmatrix} U \\ T \end{pmatrix}\right). \quad (46)$$

This expression generalizes the one proposed by Le Bot et al. [4] in the case of an uncoupled homogenous beam. Considering Eqs. (38), (40) and (46), it is possible to write

$$\begin{pmatrix} P \\ Q \end{pmatrix} = \sum_i \begin{pmatrix} P_i \\ Q_i \end{pmatrix} = \sum_{i=1}^{16}\begin{pmatrix} C_i(\omega) \\ D_i(\omega) \end{pmatrix} \mathrm{e}^{\lambda_i(\omega)x}. \quad (47)$$

Following Le Bot et al [4], and provided that evanescent waves do not transmit energy separately [16], the interference between progressive and evanescent waves can be neglected in the evaluation of the power. This means that the terms $\{P_i\}_i$ associated to solutions $\lambda_i = \pm(1\pm \mathrm{i})a^{1/4}(1\pm\eta_{\mathrm{beam}}^{\mathrm{eq}})^{1/4}(\lambda_i^4 \approx -4a)$ can be neglected in Eq. (47). Under the conditions $\eta_{\mathrm{beam}}^{\mathrm{eq}} \ll 1$ and $\eta \ll 1$, the remaining term $\{P_i\}_i$ can be evaluated in Eq. (47). For $i\in\{1,2,3,4\}$, $\lambda_i^4 = (a/16){\eta_{\mathrm{beam}}^{\mathrm{eq}}}^4$, and

$$\begin{pmatrix} P_i \\ Q_i \end{pmatrix} \approx \frac{8\omega a^2}{k_0^4 b^3}\begin{pmatrix} 2k_0^4 & 2a \\ k_0^4(2\eta+(b/a)) & a(2\eta+(b/a)) \end{pmatrix}\frac{\partial^3}{\partial x^3}\begin{pmatrix} U_i \\ T_i \end{pmatrix}, \quad \forall i \in \{1,2,3,4\}. \quad (48)$$

For $i\in\{5,6,7,8\}$, $\lambda_i^4 = 16a$, and

$$\begin{pmatrix} P_i \\ Q_i \end{pmatrix} \approx \frac{6\omega a^2 {\eta_{\mathrm{beam}}^{\mathrm{eq}}}^2}{k_0^4 b^3}\begin{pmatrix} k_0^4 & a \\ k_0^4\eta & a(\eta+(b/a)) \end{pmatrix}\frac{\partial^3}{\partial x^3}\begin{pmatrix} U_i \\ T_i \end{pmatrix} \quad \forall i \in \{5,6,7,8\}. \quad (49)$$

It can be assumed that the terms $\{P_i\}_{i=5,6,7,8}$ obtained from Eq. (49) are small compared than those obtained from Eq. (48) and are therefore neglected. This means, among others, that the energy flow resulting from the interference between progressive waves is neglected [16]. The validity of this assumption is based on the fact that $\eta_{\mathrm{beam}}^{\mathrm{eq}} \ll 1$ and that, for $i\in\{1,2,3,4\}$ and $j\in\{5,6,7,8\}$, $\left\|k_0^4\partial^3 U_i/\partial x^3 + a\partial^3 T_i/\partial x^3\right\|$ and $\left\|k_0^4\partial^3 U_j/\partial x^3 + a\partial^3 T_j/\partial x^3\right\|$ are assumed to be of the same order. Finally, it will be assumed that the evanescent waves, and hence the active power associated to their interference, are neglected far from the singularities [16]. Therefore, the active power is simply expressed as a function of the remaining two eigenvalues $\lambda_i(\omega) = \pm(a^{1/4}/2)|b|/a$. In this case the active and reactive components of structural power are given by

$$\begin{pmatrix} P_i \\ Q_i \end{pmatrix} \approx \frac{2\omega\sqrt{a}}{k_0^4 b}\begin{pmatrix} 2k_0^4 & 2a \\ k_0^4(2\eta+(b/a)) & a(2\eta+(b/a)) \end{pmatrix}\frac{\partial}{\partial x}\begin{pmatrix} U_i \\ T_i \end{pmatrix}, \quad i = 1,2, \quad (50)$$

where U_i and T_i are the solutions of diffusion Eq. (41). It is possible to decouple P_i and Q_i in Eq. (50) if we assume that the relation $k_0^4 U = aT$ holds for a beam (the equivalent relation $k_0^2 U = aT$ has already been established for a bar as a consequence of Eqs. (21) and (24)). In the case of an uncoupled beam ($a = k_0^4$), the relation $U = T \quad \forall x$ has been verified under the above assumptions by Ichchou et al. [10]. Hence, the active power P can be expressed from the spatial derivative of either the potential energy density, or the kinetic energy density:

$$P \approx \frac{8\omega\sqrt{a}}{b}\frac{\partial U}{\partial x}, \quad P \approx \frac{8\omega a\sqrt{a}}{k_0^4 b}\frac{\partial T}{\partial x}. \quad (51,52)$$

To summarize, the potential energy density of the beam is obtained by solving the following energetic boundary value problem, formulated from diffusion Eq. (41) and relationships (51) and (52), under the conditions (1) $a>0$, (2) $b\neq 0$, (3) $\eta_{\text{beam}}^{\text{eq}} = |b/a| \ll 1$ and $\eta \ll 1$:

$$\begin{cases} \dfrac{\partial^2 U(x,\omega)}{\partial x^2} - (\gamma_{\text{diff}}(\omega))^2 U(x,\omega) = 0, \quad x \in]0;\ L[, \\ \left.\dfrac{\partial U(x,\omega)}{\partial x}\right|_{x=0} = \dfrac{b(\omega)}{8\omega\sqrt{a(\omega)}} P(0,\omega), \quad \left.\dfrac{\partial U(x,\omega)}{\partial x}\right|_{x=L} = \dfrac{b(\omega)}{8\omega\sqrt{a(\omega)}} P(L,\omega). \end{cases} \tag{53}$$

Similarly the kinetic energy is obtained by solving the energetic boundary value problem

$$\begin{cases} \dfrac{\partial^2 T(x,\omega)}{\partial x^2} - (\gamma_{\text{diff}}(\omega))^2 T(x,\omega) = 0, \quad x \in]0;\ L[, \\ \left.\dfrac{\partial T(x,\omega)}{\partial x}\right|_{x=0} = \dfrac{k_0(\omega)^4 b(\omega)}{8\omega a(\omega)\sqrt{a(\omega)}} P(0,\omega), \quad \left.\dfrac{\partial T(x,\omega)}{\partial x}\right|_{x=L} = \dfrac{k_0(\omega)^4 b(\omega)}{8\omega a(\omega)\sqrt{a(\omega)}} P(L,\omega). \end{cases} \tag{54}$$

In the general case where $a \neq k_0^2$ (beam coupled with an auxiliary system), we observe that $U \neq T$. In the case of an uncoupled beam, we simply verify that the energetic problems formulated above are identical to the results obtained in the literature [10]: in this case, $\gamma_{\text{diff}}^2 \approx \eta^2\omega^2/c_g^2$, where c_g represents the group velocity of the waves propagating in the beam, $c_g = 2(\omega^2 E_0 I/\rho A)^{1/4}$; furthermore, $\partial U/\partial x = \partial T/\partial x \approx -(\eta\omega/2c_g^2) \times P\ \forall x$, and $U = T \forall x$.

3.3. Numerical results

The numerical solutions of energetic boundary value problems (53) and (54) are compared to the exact energy densities, obtained by solving the exact displacement equation (31) and using Eqs. (35) and (36) to derive the exact potential and kinetic energy of the coupled bar.

We consider a free–free elastic homogeneous Euler–Bernoulli beam. The characteristics of the beam are: bending rigidity $E_0 I = 175\,\text{N}\text{m}^2$, density $\rho = 7800\,\text{kg/m}^3$, length $L = 1\,\text{m}$, cross-sectional area $A = 10^{-4}\,\text{m}^2$ and loss factor $\eta = 5 \times 10^{-3}$. The beam is excited at $x = 0$ by a harmonic force of amplitude $F = 100\,\text{N}$. The energetic boundary conditions of the system are $P(0,\omega) = P_{\text{inj}}(\omega)$, where P_{inj} is the injected power calculated from the exact displacement solution,

$$P_{\text{inj}}(\omega) = -\tfrac{1}{2}\text{Re}\{\text{i}\omega F\overline{u(0,\omega)}\} \tag{55}$$

and $P(L,\omega) = 0$. The auxiliary system is a homogenous structural fuzzy composed of elastic bars whose exact geometrical parameters are assumed unknown and are described statistically (Fig. 13). The fuzzy couples to the transverse displacement of the beam, and is identical to the one simulated in the case of a bar (Section 2.3), therefore the boundary impedance Z is identical. At 10^3 Hz and at 10^5 Hz, the uncoupled beam contains approximately three bending wavelengths and 33 bending wavelengths over its length, respectively.

The underlying assumptions of the diffusion model, (1) $a>0$, (2) $b\neq 0$ and (3) $\eta_{\text{beam}}^{\text{eq}} \ll 1$, are first numerically verified in this case of a fuzzy auxiliary system. The functions $\omega \mapsto a/a_0$ and $\omega \mapsto \eta_{\text{beam}}^{\text{eq}}$ (equivalent damping) are

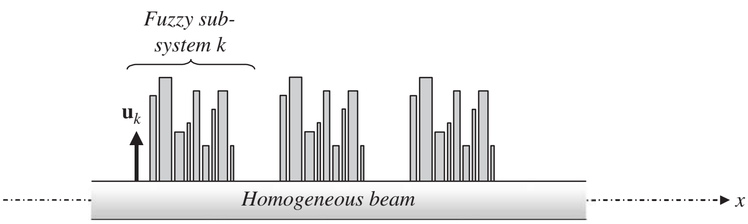

Fig. 13. Homogeneous beam coupled to a structural fuzzy.

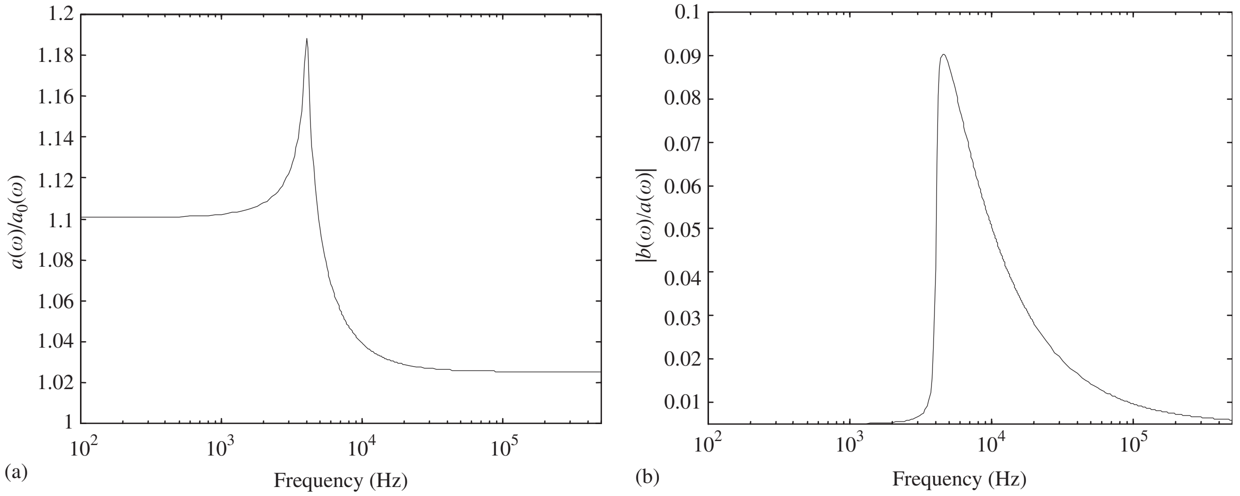

Fig. 14. Plot of the functions (a) $\omega\mapsto a(\omega)/a_0(\omega)$ and (b) $\omega\mapsto |b(\omega)\mapsto/a(\omega)|$ in the case of the homogenous beam coupled with a structural fuzzy.

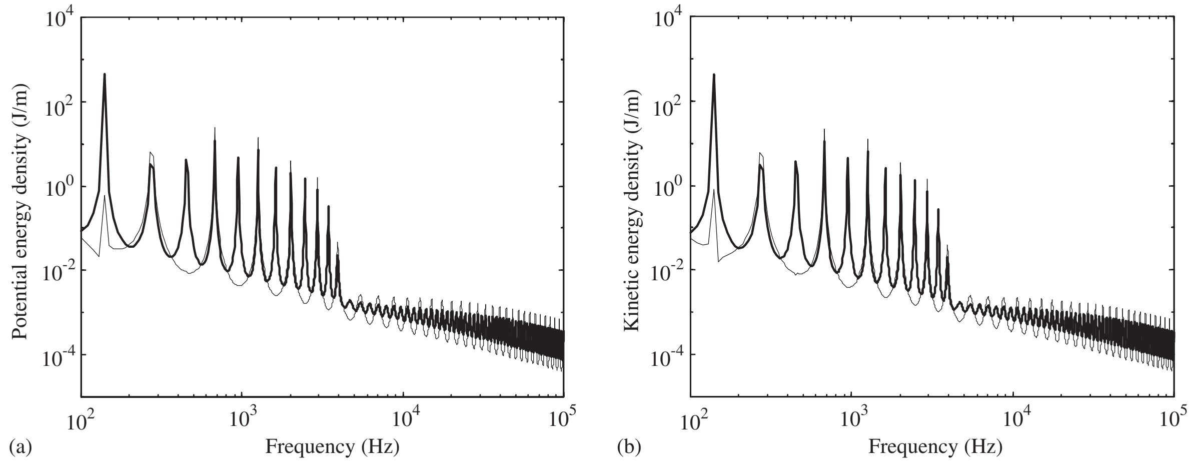

Fig. 15. (a) Potential energy density and (b) kinetic energy density of the homogeneous beam coupled with the fuzzy, at position $x = L/2$: ——, exact values; ——, solutions of the diffusion equation.

plotted in Fig. 14. According to Eq. (34) the equivalent damping of the beam $\eta_{\text{beam}}^{\text{eq}}$ is formally identical to the equivalent damping of the bar $\eta_{\text{bar}}^{\text{eq}}$ (see Fig. 9). The dissipation introduced by the fuzzy auxiliary system into the beam is significant at frequencies above the fundamental mode of the fuzzy, $\Omega_1/2\pi \approx 4\times 10^3$ Hz. The function $\omega\mapsto a/a_0$ is very similar to that found in the case of the bar (Fig. 9).

The potential energy and kinetic energy densities at position $x = L/2$ are shown in Fig. 15 as a function of frequency. The diffusion coefficient of the beam coupled with the fuzzy γ_{diff} (Eq. (42)) is compared to the diffusion coefficient of the uncoupled beam ($Z = 0$) in Fig. 16. The exact energy values in Fig. 15 do not show the resonance of antisymmetric beam modes at the observation point ($x = L/2$) since according to Eqs. (35) and (36), the corresponding value of the potential and kinetic energy is zero at this point. In contrast, the solution of the diffusion equation shows peaks on resonance of the antisymmetric modes because the injected power boundary condition $P(0,\omega) = P_{\text{inj}}(\omega)$ (derived from the exact displacement solution) has a maximum at these frequencies. Similar to the case of a bar, the action of the fuzzy on the beam is apparent above the fundamental frequency of the fuzzy. In this frequency range, the response of the system is considerably

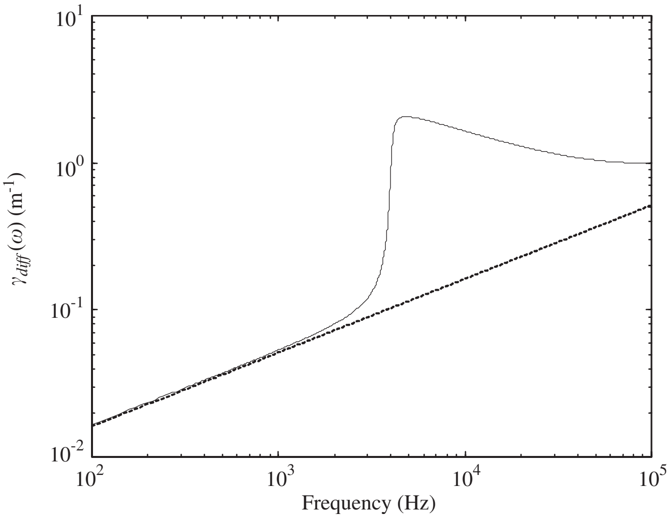

Fig. 16. Diffusion coefficient of the homogenous bar coupled with the fuzzy (——) and diffusion coefficient of the uncoupled homogeneous bar (————).

damped. This is also apparent on the diffusion coefficient γ_{diff} (Fig. 11) which is significantly increased in the frequency range $[\Omega_1/2\pi,\ 10^5\,\text{Hz}]$, as compared to the uncoupled beam. The fuzzy enhances the diffusive behavior of the system above the fundamental frequency of the fuzzy.

4. Conclusion

In this paper, the energetic behavior of a one-dimensional structure coupled with an auxiliary system composed of resonant arbitrary subsystems has been formulated on the basis of a diffusion model. The kinetic and potential energy densities are obtained by solving two energetic boundary-value problems whose solutions are smooth exponential functions of the spatial coordinate along the system. In the context of energy diffusion, the proposed model has the advantage of characterizing the coupled system by a single diffusion coefficient. Solutions predicted by the diffusion model show that the potential and kinetic energy of the system are not generally equal. In the two cases of auxiliary subsystems investigated (discrete oscillators and continuous fuzzy subsystems), it has been shown that the local heterogeneity introduced by the auxiliary system increases the effective diffusion coefficient of the global system.

Appendix A. Derivation of Eqs. (11) and (12)

The characteristic equation $X^4 + 4aX^2 - 4b^2 = 0$ of Eq. (10), has solutions such that

$$X^2 = -2a \pm 2a\sqrt{1 + \left(\frac{b}{a}\right)^2}. \tag{A.1}$$

Under the small damping condition (3) $\eta_{\text{bar}}^{\text{eq}} = |b/a| \ll 1$,

$$X^2 \approx -2a \pm 2a\left(1 + \frac{1}{2}\left(\frac{b}{a}\right)^2\right), \tag{A.2}$$

or

$$\{X^2\} = \left\{\frac{b^2}{a}; -4a\right\}. \tag{A.3}$$

Finally, the roots of the characteristic polynomial are

$$\{X\} = \left\{\frac{b}{\sqrt{a}}, -\frac{b}{\sqrt{a}}, 2i\sqrt{a}, -2i\sqrt{a}\right\}. \tag{A.4}$$

Thus, the general solution F of Eq. (10) can be expressed as the sum of two functions $x \mapsto G$ and $x \mapsto H$, solutions of second-order differential equations with roots $\{X\} = \{b/\sqrt{a}, -b/\sqrt{a}\}$ and $\{X\} = \{2\mathrm{i}\sqrt{a}, -2\mathrm{i}\sqrt{a}\}$, respectively:

$$\frac{\partial^2 G}{\partial x^2} - a\left(\frac{b}{a}\right)^2 G = 0 \quad \text{in}]0,\ L[, \tag{A.5}$$

$$\frac{\partial^2 H}{\partial x^2} + 4aH = 0 \quad \text{in}]0,\ L[. \tag{A.6}$$

Appendix B. Derivation and solution of Eq. (37)

The potential energy density U and the kinetic energy density T is a beam are given by

$$U = \frac{E_0 I}{4}\frac{\partial^2 u}{\partial x^2}\overline{\frac{\partial^2 u}{\partial x^2}}, \quad T = \frac{E_0 I k_0^4}{4} u\bar{u}, \tag{B.1,B.2}$$

where $k_2^0 = \omega^2 \rho A / E_0 I$. Using Eqs. (B.1), (B.2) and (31), the fourth-order space derivatives of U and T can be written

$$\frac{\partial^4 U}{\partial x^4} = 2aU + 6\frac{(a^2+b^2)}{k_0^4}T + E_0 I\left((a+ib)\frac{\partial u}{\partial x}\overline{\frac{\partial^3 u}{\partial x^3}} + \overline{(a+\mathrm{i}b)}\frac{\partial^3 u}{\partial x^3}\overline{\frac{\partial u}{\partial x}}\right), \tag{B.3}$$

$$\frac{\partial^4 T}{\partial x^4} = 2aT + 6k_0^4 U + E_0 I k_0^4\left(\frac{\partial^3 u}{\partial x^3}\overline{\frac{\partial u}{\partial x}} + \frac{\partial u}{\partial x}\overline{\frac{\partial^3 u}{\partial x^3}}\right). \tag{B.4}$$

It is possible to eliminate the displacement u in Eqs. (B.3) and (B.4) by differentiating again four times with respect to x. Finally, after some algebra,

$$\frac{\partial^8}{\partial x^8}\begin{pmatrix} U \\ T \end{pmatrix} - \begin{pmatrix} 4a & 12(a^2+b^2)/k_0^4 \\ 12k_0^4 & 4a \end{pmatrix}\frac{\partial^4}{\partial x^4}\begin{pmatrix} U \\ T \end{pmatrix} + \begin{pmatrix} 4(2a^2+b^2) & -8a(a^2+b^2)/k_0^4 \\ -8ak_0^4 & 4(2a^2+b^2) \end{pmatrix}\begin{pmatrix} U \\ T \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{B.5}$$

We seek solutions of the form

$$\begin{pmatrix} U \\ T \end{pmatrix} = \begin{pmatrix} A \\ B \end{pmatrix} \mathrm{e}^{\lambda x}. \tag{B.6}$$

The corresponding characteristic equation is

$$(\lambda^8 - 4a\lambda^4 + 4(2a^2+b^2))^2 - (a^2+b^2)(12\lambda^4 + 8a)^2 = 0 \tag{B.7}$$

or alternatively,

$$\left(\lambda^8 - (4a + 12\sqrt{a^2+b^2})\lambda^4 + 4(2a^2+b^2) - 8a\sqrt{a^2+b^2}\right) \times \left(\lambda^8 - (4a - 12\sqrt{a^2+b^2})\lambda^4 + 4(2a^2+b^2) + 8a\sqrt{a^2+b^2}\right) = 0. \tag{B.8}$$

Under the conditions $a>0$ and $|b/a| \ll 1$, the solutions take the simple form

$$\{\lambda_i\}_i = \left\{ \frac{\sqrt[4]{a}}{2}\frac{|b|}{a}, -\frac{\sqrt[4]{a}}{2}\frac{|b|}{a}, \mathrm{i}\frac{\sqrt[4]{a}}{2}\frac{|b|}{a}, -\mathrm{i}\frac{\sqrt[4]{a}}{2}\frac{|b|}{a}, \right.$$
$$2\sqrt[4]{a}, -2\sqrt[4]{a}, 2\mathrm{i}\sqrt[4]{a}, -2\mathrm{i}\sqrt[4]{a},$$
$$(1+\mathrm{i})\sqrt[4]{a}\sqrt[4]{1+\frac{|b|}{a}}, -(1+\mathrm{i})\sqrt[4]{a}\sqrt[4]{1+\frac{|b|}{a}}, (1-\mathrm{i})\sqrt[4]{a}\sqrt[4]{1+\frac{|b|}{a}}, -(1-\mathrm{i})\sqrt[4]{a}\sqrt[4]{1+\frac{|b|}{a}},$$
$$\left. (1+\mathrm{i})\sqrt[4]{a}\sqrt[4]{1-\frac{|b|}{a}}, -(1+\mathrm{i})\sqrt[4]{a}\sqrt[4]{1-\frac{|b|}{a}}, (1-\mathrm{i})\sqrt[4]{a}\sqrt[4]{1-\frac{|b|}{a}}, -(1-\mathrm{i})\sqrt[4]{a}\sqrt[4]{1-\frac{|b|}{a}} \right\}. \tag{B.9}$$

Appendix C. Derivation of Eq. (46)

Active power P is defined as [14]

$$P = P_V + P_M, \tag{C.1}$$

where P_V represents the active power transmitted due to the shearing force,

$$P_V = \frac{\omega E_0 I}{4}\left(\mathrm{i}\left(-\frac{\partial^3 u}{\partial x^3}\bar{u} + \overline{\frac{\partial^3 u}{\partial x^3}}u\right) + \eta\left(\frac{\partial^3 u}{\partial x^3}\bar{u} + \overline{\frac{\partial^3 u}{\partial x^3}}u\right)\right) \tag{C.2}$$

and P_M represents the active power transmitted due to the bending moment,

$$P_M = \frac{\omega E_0 I}{4}\left(\mathrm{i}\left(\frac{\partial^2 u}{\partial x^2}\overline{\frac{\partial u}{\partial x}} - \overline{\frac{\partial^2 u}{\partial x^2}}\frac{\partial u}{\partial x}\right) - \eta\left(\frac{\partial^2 u}{\partial x^2}\overline{\frac{\partial u}{\partial x}} + \overline{\frac{\partial^2 u}{\partial x^2}}\frac{\partial u}{\partial x}\right)\right). \tag{C.3}$$

Similarly, reactive power Q is defined by [14]

$$Q = Q_V + Q_M, \tag{C.4}$$

where Q_V represents the reactive power due to the shearing force,

$$Q_V = \frac{\omega E_0 I}{4}\left(-\left(\frac{\partial^3 u}{\partial x^3}\bar{u} + \overline{\frac{\partial^3 u}{\partial x^3}}u\right) + \mathrm{i}\eta\left(-\frac{\partial^3 u}{\partial x^3}\bar{u} + \overline{\frac{\partial^3 u}{\partial x^3}}u\right)\right) \tag{C.5}$$

and Q_M represents the reactive power due to the bending moment,

$$Q_M = \frac{\omega E_0 I}{4}\left(\left(\frac{\partial^2 u}{\partial x^2}\overline{\frac{\partial u}{\partial x}} + \overline{\frac{\partial^2 u}{\partial x^2}}\frac{\partial u}{\partial x}\right) + \mathrm{i}\eta\left(\frac{\partial^2 u}{\partial x^2}\overline{\frac{\partial u}{\partial x}} - \overline{\frac{\partial^2 u}{\partial x^2}}\frac{\partial u}{\partial x}\right)\right). \tag{C.6}$$

Combining Eqs. (C.2), (C.3), (C.5) and (C.6) results in

$$\frac{\omega E_0 I}{4}\left(\frac{\partial^3 u}{\partial x^3}\bar{u} + \overline{\frac{\partial^3 u}{\partial x^3}}u\right) = \frac{\eta P_V - Q_V}{1+\eta^2}, \tag{C.7}$$

$$\frac{\omega E_0 I}{4}\left(-\frac{\partial^3 u}{\partial x^3}\bar{u} + \overline{\frac{\partial^3 u}{\partial x^3}}u\right) = \frac{\eta Q_V + P_V}{\mathrm{i}(1+\eta^2)}, \tag{C.8}$$

$$\frac{\omega E_0 I}{4}\left(\frac{\partial^2 u}{\partial x^2}\overline{\frac{\partial u}{\partial x}} + \overline{\frac{\partial^2 u}{\partial x^2}}\frac{\partial u}{\partial x}\right) = -\frac{\eta P_M - Q_M}{1+\eta^2}, \tag{C.9}$$

$$\frac{\omega E_0 I}{4}\left(\frac{\partial^2 u}{\partial x^2}\overline{\frac{\partial u}{\partial x}} - \overline{\frac{\partial^2 u}{\partial x^2}}\frac{\partial u}{\partial x}\right) = \frac{\eta Q_M + P_M}{\mathrm{i}(1+\eta^2)}. \tag{C.10}$$

Using the definition of kinetic and potential energy densities (B.1) and (B.2) and Eq. (31), it can be shown that the following space derivatives of U and T can be written:

$$\begin{cases} \dfrac{\partial^3 U}{\partial x^3} = \dfrac{1}{\omega(1+\eta^2)}\big(3a(\eta P_V - Q_V) + 3b(\eta Q_V + P_V) - a(\eta P_M - Q_M) - b(\eta Q_M + P_M)\big) \\ \dfrac{\partial^3 T}{\partial x^3} = \dfrac{k_0^4}{\omega(1+\eta^2)}\big((\eta P_V - Q_V) - 3(\eta P_M - Q_M)\big) \\ \dfrac{\partial^7 U}{\partial x^7} = \dfrac{2}{\omega(1+\eta^2)}\big((14a^2 + 7b^2)(\eta P_V - Q_V) + 7ab(\eta Q_V + P_V) \\ -(18a^2 + 17b^2)(\eta P_M - Q_M) - ab(\eta Q_M + P_M)\big) \\ \dfrac{\partial^7 T}{\partial x^7} = \dfrac{2k_0^4}{\omega(1+\eta^2)}\big(18a(\eta P_V - Q_V) + 17b(\eta Q_V + P_V) - 14a(\eta P_M - Q_M) - 7b(\eta Q_M + P_M)\big) \end{cases} \tag{C.11}$$

The inversion of (C.11) gives:

$$\begin{cases} (\eta P_V - Q_V) = -\dfrac{\omega(1+\eta^2)}{8k_0^4 b^2}\left(-3k_0^4\dfrac{\partial^7 U}{\partial x^7} - 3a\dfrac{\partial^7 T}{\partial x^7} + 48ak_0^4\dfrac{\partial^3 U}{\partial x^3} + (48a^2 + 34b^2)\dfrac{\partial^3 T}{\partial x^3}\right) \\ (\eta Q_V + P_V) = \dfrac{\omega(1+\eta^2)}{8k_0^4 b^3}\left(-4ak_0^4\dfrac{\partial^7 U}{\partial x^7} - (4a^2 + b^2)\dfrac{\partial^7 T}{\partial x^7}\right. \\ \qquad \left. +(64a^2k_0^4 + 14b^2k_0^4)\dfrac{\partial^3 U}{\partial x^3} + (64a^3 + 50ab^2)\dfrac{\partial^3 T}{\partial x^3}\right) \\ (\eta P_M - Q_M) = -\dfrac{\omega(1+\eta^2)}{8k_0^4 b^2}\left(-k_0^4\dfrac{\partial^7 U}{\partial x^7} - a\dfrac{\partial^7 T}{\partial x^7} + 16ak_0^4\dfrac{\partial^3 U}{\partial x^3} + (16a^2 + 14b^2)\dfrac{\partial^3 T}{\partial x^3}\right) \\ (\eta Q_M + P_M) = \dfrac{\omega(1+\eta^2)}{8k_0^4 b^3}\left(-4ak_0^4\dfrac{\partial^7 U}{\partial x^7} - (4a^2 + 3b^2)\dfrac{\partial^7 T}{\partial x^7}\right. \\ \qquad \left. +(64a^2k_0^4 + 34b^2k_0^4)\dfrac{\partial^3 U}{\partial x^3} + (64a^3 + 62ab^2)\dfrac{\partial^3 T}{\partial x^3}\right) \end{cases} \tag{C.12}$$

Finally, the active and reactive powers are expressed according to Eqs. (C.1)–(C.6):

$$\begin{aligned} P = \frac{\omega}{2k_0^4 b^3}\Bigg(&-k_0^4(2a - b\eta)\frac{\partial^7 U}{\partial x^7} - (2a^2 + b^2 - ab\eta)\frac{\partial^7 T}{\partial x^7} \\ &+4k_0^4(8a^2 + 3b^2 - 4ab\eta)\frac{\partial^3 U}{\partial x^3} + 4(8a^3 + 7ab^2 - 4a^2b\eta - 3b^3\eta)\frac{\partial^3 T}{\partial x^3}\Bigg) \end{aligned} \tag{C.13}$$

and

$$\begin{aligned} Q = \frac{\omega}{2k_0^4 b^3}\Bigg(&-k_0^4(2a\eta + b)\frac{\partial^7 U}{\partial x^7} - (2a^2\eta + b^2\eta + ab)\frac{\partial^7 T}{\partial x^7} \\ &+4k_0^4(8a^2\eta + 3b^2\eta + 4ab)\frac{\partial^3 U}{\partial x^3} + 4(8a^3\eta + 7ab^2\eta + 4a^2b + 3b^3)\frac{\partial^3 T}{\partial x^3}\Bigg). \end{aligned} \tag{C.14}$$

These two equations are written in matrix form

$$\begin{pmatrix} P \\ Q \end{pmatrix} = \frac{\omega}{2k_0^4 b^3}\left(-a\begin{pmatrix} k_0^4(2-(b/a)\eta) & a(2+(b/a)^2-(b/a)\eta) \\ k_0^4(2\eta+(b/a)) & a(2\eta+(b/a)^2\eta+(b/a)) \end{pmatrix}\frac{\partial^7}{\partial x^7}\begin{pmatrix} U \\ T \end{pmatrix} + 4a^2 \right.$$
$$\left. \times \begin{pmatrix} k_0^4(8+3(b/a)^2-4(b/a)\eta) & a(8+7(b/a)^2-4(b/a)\eta-3(b/a)^3\eta) \\ k_0^4(8\eta+3(b/a)^2\eta+4(b/a)) & a(8\eta+7(b/a)^2\eta+4(b/a)+3(b/a)^3) \end{pmatrix}\frac{\partial^3}{\partial x^3}\begin{pmatrix} U \\ T \end{pmatrix}\right). \quad \text{(C.15)}$$

References

[1] R.H. Lyon, R.G. Dejong, *Theory and Application of Statistical Energy Analysis*, second ed, MIT Press, Cambridge MA, 1975.

[2] C. Lesueur, *Rayonnement Acoustique des Structures*, Éditions Eyrolles, Paris, 1988.

[3] D.J. Nefske, S.H. Sung, Power flow finite element analysis of dynamic systems: basic theory and applications to beams, *Transactions of the ASME* 111 (1989) 94–100.

[4] A. Le Bot, L. Jezequel, In: *Fourth International Congress on Intensity Techniques, Senlis, France*, Energy methods applied to transverse vibrations of beams, 1993, pp. 371–378.

[5] O.M. Bouthier, R.J. Bernhard, Simple models of energy flow in vibrating membranes, *Journal of Sound and Vibration* 182 (1) (1995) 129–147.

[6] O.M. Bouthier, R.J. Bernhard, Simple models of energys of transversaly vibrating plates, *Journal of Sound and Vibration* 182 (1) (1995) 149–164.

[7] M. Djimadoum, J.-L. Guyader, Vibratory prediction with an equation of diffusion, *Acta Acustica* 3 (1995) 11–24.

[8] J.C. Wohlever, R.J. Bernhard, Mechanical energy flow models of rods and beams, *Journal of Sound and Vibration* 153 (1) (1992) 1–19.

[9] J.-M. Mencik, Formulation de la Réponse Dynamique d'une Structure Maîtresse Couplée à un Système Annexe et Formulation Locale du Comportement Énergétique des Structures vibrantes, PhD Thesis, Sherbrooke University, Canada; INSA de Lyon France, 2002.

[10] M.N. Ichchou, A. Le Bot, L. Jezequel, Energy models of one-dimensional multi-propagative systems, *Journal of Sound and Vibration* 201 (5) (1997) 535–554.

[11] R.S. Langley, On the vibrational conductivity approach to high frequency dynamics for two-dimensional structural components, *Journal of Sound and Vibration* 182 (4) (1995) 637–657.

[12] J.-M. Mencik, A. Berry, A theoretical formulation of the dynamical response of a master structure coupled with elastic continuous fuzzy subsystems with discrete attachments, *Journal of Sound and Vibration* 280 (3–5) (2005) 1031–1050.

[13] S. Timoshenko, G.H. Young, W. Weaver Jr., *Vibration Problems in engineering*, fourth ed, Wiley, New York, 1974.

[14] Y. Lase, M.N. Ichchou, L. Jezequel, Energy flow analysis of bars and beams: theoretical formulations, *Journal of Sound and Vibration* 192 (1) (1996) 281–305.

[15] A.G. Troshin, V.I. Popkov, A.V. Popov, The measurement of vibrational power flux in rod structures, *International Congress on Intensity Techniques, Senlis, France*, (1990) 265–272.

[16] Yu.I. Bobrovnitskii, On the energy flow in evanescent waves, *Journal of Sound and Vibration* 152 (1) (1992) 175–176.