Available online at www.sciencedirect.com
SCIENCE DIRECT®
Journal of Sound and Vibration 294 (2006) 927–943
JOURNAL OF SOUND AND VIBRATION
www.elsevier.com/locate/jsvi

The natural vibration of a conical shell with an annular end plate

Sen. Liang*, H.L. Chen

Institute of Vibration and Noise Control, School of Mechanical Engineering, Xi'an Jiaotong University, Xi'an 710049, PR China

Received 23 November 2004; received in revised form 27 June 2005; accepted 17 December 2005
Available online 9 March 2006

Abstract

In this paper, natural frequencies and mode shapes for a conical shell with an annular end plate or a round end plate are investigated in detail by combining the vibration theory with the transfer matrix method. The governing equations of vibration for this structure are expressed in terms of a matrix differential equation, and a novel recurrence formula method that can close in on analytical solution of transfer matrix is presented. Once the transfer matrix of single component has been determined, and the product of each component matrix and the joining matrix can obtain entire structure matrix. The frequency equations and mode shape functions are represented in terms of the elements of the structural transfer matrices. The 3D finite element numerical simulations have verified the present formulas of natural frequencies and mode shapes. The conclusions show that the transfer matrix method can accurately obtain natural vibration characteristics of the conical shell with an annular end plate.

1. Introduction

A conical shell with an annular end plate is a kind of useful structure in the aeronautical, aerospace and civil industries. Nowadays, the shell structure becomes larger and thinner and its vibration problem becomes more and more important than before. Therefore, the wide engineering applications of the shell structures have attracted considerably researchers' interest and many methods for investigating their dynamical characteristics have been promoted. Presently, many researchers concentrate their investigations on dynamic characteristics of a conical shell, a thin-walled round plate and a combination of cylindrical shell and round plates [1–12], but there are few research reports relative to this system of the conical shell with an annular end plate.

Transfer matrix method has been utilized in engineering applications for many years. Holzer originally employed the transfer matrix approach for an approximate solution of the differential equation governing the torsional vibration of rod in 1921 and the method is generally known as Holzer's method. Myklestad presented an approach quite analogous to Holzer's for the treatment of beam [13]. The relative equations were rearranged and simplified by Thomson to permit a systematic tabular computation and to expand this method to more general problems [14]. One of the earliest applications of transfer matrix method was the steady-state description of four terminal electrical networks, in which the method is commonly designated as a ‘‘four-pole

*Corresponding author. Tel.: +86 29 82663186; fax: +86 29 82660487.
E-mail address: liangsen98@mailst.xjtu.edu.cn (S. Liang).

0022-460X/$ - see front matter
doi:10.1016/j.jsv.2005.12.033

parameters". Rubin et al. [15–17] systematically applied four-pole parameters approach to acoustical, mechanical, and electromechanical vibrations. Murthy, Kalnins, Irie et al. [18–26] extended the applications of transfer matrix approach to a rotor blade, plate, symmetric shell and stiffened ring through a completely general treatment.

In this paper, natural frequencies and mode shapes of a conical shell with an annular end plate or a round plate are investigated in detail by combining the vibration theory with the transfer matrix method. The governing equations of vibration for a conical shell and an annular end plate are expressed in terms of matrix differential equations, and a new recurrence formula method that can close in on analytical solution of transfer matrix is presented. Once the transfer matrix of single component has been determined, the product of each component matrix and the joining matrix can form the matrix of entire structure. The frequency equations and mode shape functions are represented in terms of the matrix elements. 3D finite element analysis has validated the present formulas of natural frequencies and mode shapes. The conclusions show the transfer matrix obtained by present formulas can accurately reveal natural vibration characteristics of the conical shell with an annular end plate.

2. Theoretical analysis

In order to investigate the dynamic characteristics of the combined shell, the coefficient matrices of a conical shell and an annular end plate should be derived, respectively. Then the elements of transfer matrix can be obtained by employing recurrence formula method. The frequency equations and mode shape functions are represented in terms of the elements of the structural transfer matrices.

2.1. Coefficient matrix of the conical shell

Fig. 1 shows a combined structure of the conical shell with an annular end plate. α denotes the semi-vertex angle of the truncated conical shell, h expresses the thickness of conical shell and annular plate, and H is distance form the middle surface of annular plate to the large edge of conical shell. The radius of large edge for the conical shell is R, and outer radius and inner radius of the annular plate are R_1 and R_2. The surface co-ordinates (ς, θ, z) are taken in the middle surface as shown in Fig. 1. Based on the flÜgge theory [27], the equations of flexural vibration for a thin-walled conical shell can be written in a differential equation of the coefficient matrix $A(\varsigma)$ as

$$\mathrm{d}X(\varsigma)/\mathrm{d}\varsigma = A(\varsigma).X(\varsigma), \tag{1}$$

here state vector $X(\varsigma) = \{\bar{u}\ \bar{v}\ \bar{w}\ \bar{\phi}\ \overline{M_\varsigma}\ \overline{V_\varsigma}\ \overline{N_\varsigma}\ \overline{S_{\varsigma\theta}}\}$ is denoted by the dimensionless variables. $\bar{u}$, $\bar{v}$ and $\bar{w}$ are the displacements in circumferential, meridional and normal directions, respectively; $\bar{\phi}$, $\overline{N_\varsigma}$, $\overline{M_\varsigma}$, $\overline{V_\varsigma}$ and $\overline{S_{\varsigma\theta}}$ are the bending slope, the membrane force, the bending moment, the Kevlin–Kirchhoff shearing force and shear

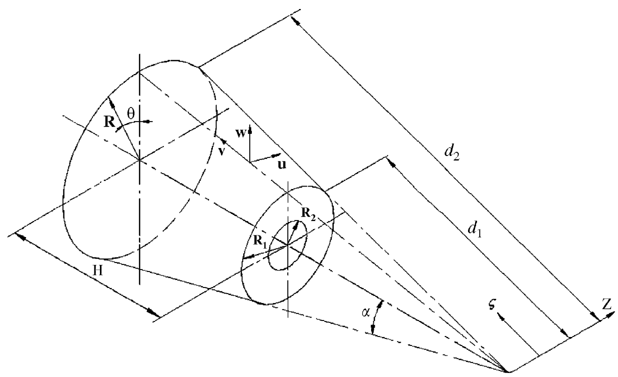

Fig. 1. The conical shell with an annular end plate.

resultant, respectively. They are defined as follows:

$$\{\overline{u}\ \overline{v}\ \overline{w}\} = \frac{1}{R}\{u\ v\ w\},\quad \bar{\phi} = \phi,\quad \overline{N_\varsigma} = N_\varsigma R^2/D,\ \overline{M_\varsigma} = M_\varsigma R/D,$$

$$\{\overline{V_\varsigma}\ \overline{S_{\varsigma\theta}}\} = \{V_s\ S_{\varsigma\theta}\}R^2/D,\quad D = Eh^3/[12(1-v^2)], \tag{2}$$

where D is flexural rigidity expressed by Young's modulus E, Poisson's ratio v and thickness h of the shell. In order to simplify the formulas, the following dimensionless parameters are introduced:

$$\varsigma = d/R,\quad (\varsigma_1 = d_1/R,\ \varsigma_2 = d_2/R,\ \varsigma_3 = R_1/R,\ \varsigma_4 = R_2/R). \tag{3}$$

The frequency parameter is described as

$$p^4 = \rho h R^4 \omega^2/D, \tag{4}$$

here ρ is the mass density, and ω is the frequency in the rad/s. d is the distance form vertex to arbitrary point in the middle surface of the conical shell. The non-zero elements of the coefficient matrix $A(\varsigma)$ in Eq. (1) can be derived as follows (a detailed derivation is shown in Appendix A):

$$A_{11} = \frac{1}{\varsigma},\quad A_{12} = \frac{n}{\varsigma\ \sin\ a},\quad A_{13} = \frac{nh^2}{6\varsigma^3 R^2 tg\alpha\ \sin\ \alpha},\quad A_{14} = -\frac{nh^2}{6\varsigma^2 R^2 tg\alpha\ \sin\ \alpha},$$

$$A_{18} = \frac{h^2}{6R^2(1-v)},\quad A_{21} = -\frac{vn}{\varsigma\ \sin\ \alpha},\quad A_{22} = -\frac{v}{\varsigma},\quad A_{23} = -\frac{v}{\varsigma tg\alpha},\quad A_{27} = \frac{h^2}{12R^2},$$

$$A_{34} = 1,\quad A_{43} = \frac{vn^2}{\varsigma^2 \sin^2\alpha},\quad A_{44} = -\frac{v}{\varsigma},\quad A_{45} = 1,\quad A_{53} = -\frac{(3+v)(1-v)n^2}{\varsigma^3 \sin^2\alpha},$$

$$A_{54} = \left(1 + v + \frac{2n^2}{\sin^2\alpha}\right)\frac{(1-v)}{\varsigma^2},\quad A_{55} = -\frac{1-v}{\varsigma},\quad A_{56} = 1,\quad A_{61} = -\frac{12(1-v^2)nR^2}{h^2\varsigma^2 tg\alpha\ \sin\ \alpha},$$

$$A_{62} = -\frac{12(1-v^2)R^2}{h^2\varsigma^2 tg\alpha},\quad A_{63} = p^4 - \frac{12(1-v^2)R^2}{h^2\varsigma^2 tg^2\alpha} - \left[2 + \frac{(1+v)n^2}{\sin\alpha}\right]\frac{(1-v)n^2}{\varsigma^4 \sin^2\alpha},$$

$$A_{64} = \frac{(3+v)(1-v)n^2}{\varsigma^3 \sin^2\alpha},\quad A_{65} = \frac{vn^2}{\varsigma^2 \sin^2\alpha},\quad A_{66} = -\frac{1}{\varsigma},\quad A_{67} = -\frac{v}{\varsigma tg\alpha},$$

$$A_{71} = \frac{12(1-v^2)R^2}{h^2\varsigma^2\ \sin\ \alpha},\quad A_{72} = -p^4 + \frac{12(1-v^2)R^2}{h^2\varsigma^2},$$

$$A_{73} = \left[\frac{12(1+v)R^2}{h^2} - \frac{n^2}{\varsigma^2 \sin^2\alpha}\right]\frac{1-v}{\varsigma^2 tg\alpha},\quad A_{74} = \frac{(1-v)n^2}{\varsigma^3 \sin^2\alpha tg\alpha},\quad A_{77} = -\frac{1-v}{\varsigma},$$

$$A_{78} = -\frac{n}{\varsigma\ \sin\ \alpha},\quad A_{81} = -p^4 + \frac{12(1-v^2)n^2R^2}{\varsigma^2 h^2 \sin^2\alpha},\quad A_{82} = \frac{12(1-v^2)nR^2}{\varsigma^2 h^2 \sin^2\alpha},$$

$$A_{83} = \left\{\frac{12(1+v)R^2}{h^2} + \left[\frac{1}{n^2} + \frac{(1+v)n^2}{n^2 \sin^2\alpha}\right]\right\}\frac{(1-v)n}{\varsigma^2 tg\alpha\ \sin\ \alpha},\quad A_{84} = -\frac{(2+v)(1-v)n}{\varsigma^3 tg\alpha\ \sin\ \alpha},$$

$$A_{85} = -\frac{vn}{\varsigma^2 tg\alpha\ \sin\ \alpha},\quad A_{87} = \frac{vn}{\varsigma\ \sin\ \alpha},\quad A_{88} = -\frac{2}{\varsigma}. \tag{5}$$

here n is the circumferential wavenumber.

2.2. Coefficient matrix of the annular plate

For an annular end plate, the governing equations can be derived as a special case of a conical shell by taking the limiting values $(1/\varsigma\sin\alpha) \to (1/\varsigma)$, $(1/\varsigma tg\alpha) \to 0$. The matrix equation has the same expression as Eq. (1). In this case, the non-zero elements of the coefficient matrix $A(\varsigma)$ become:

$$A_{11} = \frac{1}{\varsigma},\quad A_{12} = \frac{n}{\varsigma},\quad A_{18} = \frac{h^2}{6R^2(1-v)},\quad A_{21} = -\frac{vn}{\varsigma},\quad A_{22} = -\frac{v}{\varsigma},\quad A_{27} = \frac{h^2}{12R^2},\quad A_{34} = 1,$$

$$A_{43} = \frac{vn^2}{\varsigma^2},\quad A_{44} = -\frac{v}{\varsigma},\quad A_{45} = 1,\quad A_{53} = -\frac{(3+v)(1-v)n^2}{\varsigma^3},\quad A_{54} = (1+v+2n^2)\frac{(1-v)}{\varsigma^2},$$

$$A_{55} = -\frac{1-v}{\varsigma},\quad A_{56} = 1,\quad A_{63} = p^4 - \left[2+(1+v)n^2\right]\frac{(1-v)n^2}{\varsigma^4},\quad A_{64} = \frac{(3+v)(1-v)n^2}{\varsigma^3},$$

$$A_{65} = \frac{vn^2}{\varsigma^2},\quad A_{66} = -\frac{1}{\varsigma},\quad A_{71} = \frac{12\left(1-v^2\right)R^2}{h^2\varsigma^2},\quad A_{72} = -p^4 + \frac{12\left(1-v^2\right)R^2}{h^2\varsigma^2},\quad A_{77} = -\frac{1-v}{\varsigma},$$

$$U_{78} = -\frac{n}{\varsigma},\quad A_{81} = -p^4 + \frac{12\left(1-v^2\right)n^2R^2}{\varsigma^2h^2},\quad A_{82} = \frac{12\left(1-v^2\right)nR^2}{\varsigma^2h^2},\quad A_{87} = \frac{vn}{\varsigma},\quad A_{88} = -\frac{2}{\varsigma}. \tag{6}$$

2.3. The joining matrix

In order to investigate the dynamic characteristics of the combined structure, the conical shell and the annular end plate should be connected together. Therefore, the following continuity conditions and equilibrium relations must be satisfied:

$$\left\{X(\varsigma_1+0)\right\}_c = [B]_{p\to c}\left\{X(\varsigma_3-0)\right\}_p, \tag{7}$$

here the subscripts c and p denote conical shell and annular plate, respectively. The joining matrix B can be given by

$$[B]_{p\to c} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \cos\beta & -\sin\beta & 0 & 0 & 0 & 0 & 0 \\ 0 & \sin\beta & \cos\beta & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cos\beta & -\sin\beta & 0 \\ 0 & 0 & 0 & 0 & 0 & \sin\beta & \cos\beta & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \tag{8}$$

here $\beta = (\pi/2 - \alpha)$.

2.4. The transfer matrix

It is difficult to obtain the analytical solution of the coupled equations set (1) for a conical shell or an annular end plate, so the transfer matrix approach is adopted here. In Refs. [18,19], the derivation is only involved in zero-initial value ($\varsigma_0 = 0$). In this paper, the initial value has been extended to more general problems (non-zero initial value). Based on the definition of transfer matrix, the following equation can be written as

$$\{X(\varsigma)\} = [T(\varsigma)]\{X(\varsigma_0)\}, \tag{9}$$

here ς_0 denotes initial value. Differentiating Eq. (9) with respect to ς yields

$$\frac{\mathrm{d}}{\mathrm{d}\varsigma}\{X(\varsigma)\} = \frac{\mathrm{d}}{\mathrm{d}\varsigma}[T(\varsigma)]\{X(\varsigma_0)\}. \tag{10}$$

According to Eq. (9), the initial value of state vector can be obtained obviously

$$\{X(\varsigma_0)\} = [T(\varsigma)]^{-1}\{X(\varsigma)\}. \tag{11}$$

Substituting Eq. (11) into Eq. (10), the following relation is given:

$$\frac{\mathrm{d}}{\mathrm{d}\varsigma}\{X(\varsigma)\} = \frac{\mathrm{d}}{\mathrm{d}\varsigma}[T(\varsigma)][T(\varsigma)]^{-1}\{X(\varsigma)\}. \tag{12}$$

By comparing Eq. (1) with Eq. (12), the following equation is derived:

$$\left([A(\varsigma)] - \frac{\mathrm{d}}{\mathrm{d}\varsigma}[T(\varsigma)][T(\varsigma)]^{-1}\right)\{X(\varsigma)\} = \{0\}. \tag{13}$$

For all values of ς, the state vector X cannot be zero. The following equation can be easily written:

$$[A(\varsigma)] = \frac{\mathrm{d}}{\mathrm{d}\varsigma}[T(\varsigma)][T(\varsigma)]^{-1}. \tag{14}$$

Then, post-multiplying $T(\varsigma)$ on both sides of Eq. (14) yields

$$\frac{\mathrm{d}}{\mathrm{d}\varsigma}[T(\varsigma)] = [A(\varsigma)][T(\varsigma)]. \tag{15}$$

Therefore, the transfer matrix is obtained by the solution of Eq. (15). If ς equals to ς_0 in Eq. (9), the initial condition will be

$$[T(\varsigma_0)] = [1]. \tag{16}$$

Eq. (16) provides the sufficient initial conditions in order to solve the differential equations set (15). Thus, the transfer matrix of entire structure can be obtained

$$X(\varsigma)_{\varsigma=\varsigma_2} = T_c B_{p\to c} T_p X(\varsigma)_{\varsigma=\varsigma_4}. \tag{17}$$

2.5. Solution of the transfer matrix

Presently, the transfer matrix $T(\varsigma)$ can be obtained by many ways. Kalnins [28] developed a multisegment, direct, numerical integration approach, Cohen [29] presented an iteration method using approximate eigenfunctions, Tottenham and Shimizu [30] used a matrix progression method, Sankar [31] showed an extend transfer matrix method, and Irie et al. [22–26] employed Runge–Kutta–Gill method. Most of them are approximate methods, and this paper demonstrates that the recurrence formula method that can close in on analytical solution solves transfer matrix.

2.5.1. Recurrence formula

On the assumption that $A(\varsigma)$ is continuous in the range $[a, b]$, a and $b \geqslant 0$, and $T(\varsigma)$ is the only solution of Eq. (15) under the initial condition (16), the series $\{T_k(\varsigma)\}$ can be constructed firstly

$$T_0(\varsigma) = I.$$

$$T_1(\varsigma) = I + \int_{\varsigma_1}^{\varsigma} A(s).T_0(s)\,\mathrm{d}s,$$

$$\vdots$$

$$T_k(\varsigma) = I + \int_{\varsigma_1}^{\varsigma} A(s).T_{k-1}(s)\,\mathrm{d}s. \tag{18}$$

The difference between T_{k+1} and T_k is

$$T_{k+1}(\varsigma) - T_k(\varsigma) = \int_{\varsigma_1}^{\varsigma} A(s)[T_k(s) - T_{k-1}(s)]\,\mathrm{d}s. \tag{19}$$

Taking norm on both sides of Eq. (19) in normed linear space, the following equation is given

$$\begin{aligned}\|T_{k+1}(\varsigma) - T_k(\varsigma)\| &= \left\|\int_{\varsigma_1}^{\varsigma} A(s)[T_k(s) - T_{k-1}(s)]\,\mathrm{d}s\right\| \leqslant \int_{\varsigma_1}^{\varsigma} \|A(s)\|\ \|T_k(s) - T_{k-1}(s)\|\,\mathrm{d}s \\ &\leqslant M\int_{\varsigma_1}^{\varsigma} \|T_k(s) - T_{k-1}(s)\|\,\mathrm{d}s \\ &\leqslant M^2\int_{\varsigma_1}^{\varsigma}\mathrm{d}s\int_{\varsigma_1}^{\varsigma} \|T_{k-1}(s) - T_k(s)\|\,\mathrm{d}s \\ &\qquad\vdots \\ &\leqslant M^n\int_{\varsigma_1}^{\varsigma}\mathrm{d}s\int_{\varsigma_1}^{\varsigma}\mathrm{d}s\ldots\int_{\varsigma_1}^{\varsigma} \|T_1(s) - T_0(s)\|\,\mathrm{d}s,\end{aligned} \tag{20}$$

where

$$T_1(\varsigma) - T_0(\varsigma) = \int_{\varsigma_1}^{\varsigma} A(s).T_0(s)\,\mathrm{d}s = \int_{\varsigma_1}^{\varsigma} A(s)\,\mathrm{d}s. \tag{21}$$

Thus

$$\|T_1(\varsigma) - T_0(\varsigma)\| \leqslant \int_{\varsigma_1}^{\varsigma} \|A(s)\|\,\mathrm{d}s \leqslant M.t, \tag{22}$$

here $M = \max_{\varsigma_1 \leqslant \varsigma \leqslant b}\|A(\varsigma)\|$, and t is the length of integral region. By substituting Eq. (22) into Eq. (20), the following equation can be obtained

$$\|T_{k+1}(\varsigma) - T_k(\varsigma)\| \leqslant M^{k+1}\frac{t^{k+1}}{(k+1)!}. \tag{23}$$

Summing this Eq. (23) on both sides, the relationship is

$$\sum_{k=0}^{\infty}\|T_{k+1}(\varsigma) - T_k(\varsigma)\| \leqslant \sum_{k=0}^{\infty}\frac{(Mt)^{k+1}}{(k+1)!} \leq \mathrm{e}^{Mt}. \tag{24}$$

Obviously, the series $\sum_{k=0}^{\infty}(T_{k-1} - T_k)$ is consistent convergence and suppose its convergent result is $T(\varsigma) - I$. By reason of that $\sum_{k=0}^{k}(T_{k+1} - T_k) = T_{k+1} - I$, the following equation can be derived:

$$\lim_{k\to\infty} T_{k+1}(\varsigma) = T(\varsigma).$$

Taking the limits for recurrence Eq. (18) yields

$$T(\varsigma) = I + \int_{\varsigma_1}^{\varsigma} A(s).T(S)\,\mathrm{d}s. \tag{25}$$

Differentiating Eq. (25) with respect to ς gives

$$\frac{\mathrm{d}T(\varsigma)}{\mathrm{d}\varsigma} = A(\varsigma)T(\varsigma).$$

The result shows that the $T_k(\varsigma)$ in Eq. (18) will be the solution of matrix Eq. (15) when k becomes large enough.

2.5.2. The only solution of transfer matrix $T(\varsigma)$

On the assumption that $T(\varsigma)$ and $Y(\varsigma)$ are both the solutions of Eq. (15), the following equations can be written:

$$T(\varsigma) = I + \int_{\varsigma_1}^{\varsigma} A(s).T(s)\,\mathrm{d}s \quad \text{and} \quad Y(\varsigma) = I + \int_{\varsigma_1}^{\varsigma} A(s).Y(s)\,\mathrm{d}s. \tag{26}$$

The difference between $T(\varsigma)$ and $Y(\varsigma)$ is

$$T(\varsigma) - Y(\varsigma) = \int_{\varsigma_1}^{\varsigma} A(s)[T(s) - Y(s)]\,\mathrm{d}s. \tag{27}$$

Taking the norm on both sides of Eq. (27) gives

$$\|T(\varsigma) - Y(\varsigma)\| \leqslant \int_{\varsigma_1}^{\varsigma} \|A(s)\| \quad \|T(s) - Y(s)\|\,\mathrm{d}s. \tag{28}$$

Table 1
Eigenvalue equations of the conical shell with an annular end plate

$F_p - F_c$	$F_p - S_c$	$F_p - C_c$
$\begin{vmatrix} T_{51} & T_{52} & T_{53} & T_{54} \\ T_{61} & T_{62} & T_{63} & T_{64} \\ T_{71} & T_{72} & T_{73} & T_{74} \\ T_{81} & T_{82} & T_{83} & T_{84} \end{vmatrix} = 0$	$\begin{vmatrix} T_{11} & T_{12} & T_{13} & T_{14} \\ T_{31} & T_{32} & T_{33} & T_{34} \\ T_{51} & T_{52} & T_{53} & T_{54} \\ T_{71} & T_{72} & T_{73} & T_{74} \end{vmatrix} = 0$	$\begin{vmatrix} T_{11} & T_{12} & T_{13} & T_{14} \\ T_{21} & T_{22} & T_{23} & T_{24} \\ T_{31} & T_{32} & T_{33} & T_{34} \\ T_{41} & T_{42} & T_{43} & T_{44} \end{vmatrix} = 0$
$S_p - F_c$	$S_p - S_c$	$S_p - C_c$
$\begin{vmatrix} T_{52} & T_{54} & T_{56} & T_{58} \\ T_{62} & T_{64} & T_{66} & T_{68} \\ T_{72} & T_{74} & T_{76} & T_{78} \\ T_{82} & T_{84} & T_{86} & T_{88} \end{vmatrix} = 0$	$\begin{vmatrix} T_{12} & T_{14} & T_{16} & T_{18} \\ T_{32} & T_{34} & T_{36} & T_{38} \\ T_{52} & T_{54} & T_{56} & T_{58} \\ T_{72} & T_{74} & T_{76} & T_{78} \end{vmatrix} = 0$	$\begin{vmatrix} T_{12} & T_{14} & T_{16} & T_{18} \\ T_{22} & T_{24} & T_{26} & T_{28} \\ T_{32} & T_{34} & T_{36} & T_{38} \\ T_{42} & T_{44} & T_{46} & T_{48} \end{vmatrix} = 0$
$C_p - F_c$	$C_p - S_c$	$C_p - C_c$
$\begin{vmatrix} T_{55} & T_{56} & T_{57} & T_{58} \\ T_{65} & T_{66} & T_{67} & T_{68} \\ T_{75} & T_{76} & T_{77} & T_{78} \\ T_{85} & T_{86} & T_{87} & T_{88} \end{vmatrix} = 0$	$\begin{vmatrix} T_{15} & T_{16} & T_{17} & T_{18} \\ T_{35} & T_{36} & T_{37} & T_{38} \\ T_{55} & T_{56} & T_{57} & T_{58} \\ T_{75} & T_{76} & T_{77} & T_{78} \end{vmatrix} = 0$	$\begin{vmatrix} T_{15} & T_{16} & T_{17} & T_{18} \\ T_{25} & T_{26} & T_{27} & T_{28} \\ T_{35} & T_{36} & T_{37} & T_{38} \\ T_{45} & T_{46} & T_{47} & T_{48} \end{vmatrix} = 0$

Let $H = \max_{\varsigma_1 \leqslant \varsigma \leqslant b} \| T(\varsigma) - Y(\varsigma) \|$, derive as aforementioned

$$\| T(\varsigma) - Y(\varsigma) \| \leqslant H . \frac{(Mt)^{k+1}}{(k+1)!} . \tag{29}$$

Table 2
Eigenvalue employed FEM and present method

		$m = 0$	$m = 1$	$m = 2$
$n = 0$	FEM	3.4170	6.7660	10.145
	Transfer matrix	3.4170	6.7660	10.155
	Error/%	0	0	0.00
$n = 1$	FEM	4.9232	8.3550	11.760
	Transfer matrix	4.9282	8.3639	11.772
	Error/%	0.10	0.11	0.10
$n = 2$	FEM	6.3191	9.8670	13.320
	Transfer matrix	6.3263	9.8810	13.342
	Error/%	0.11	0.14	0.16

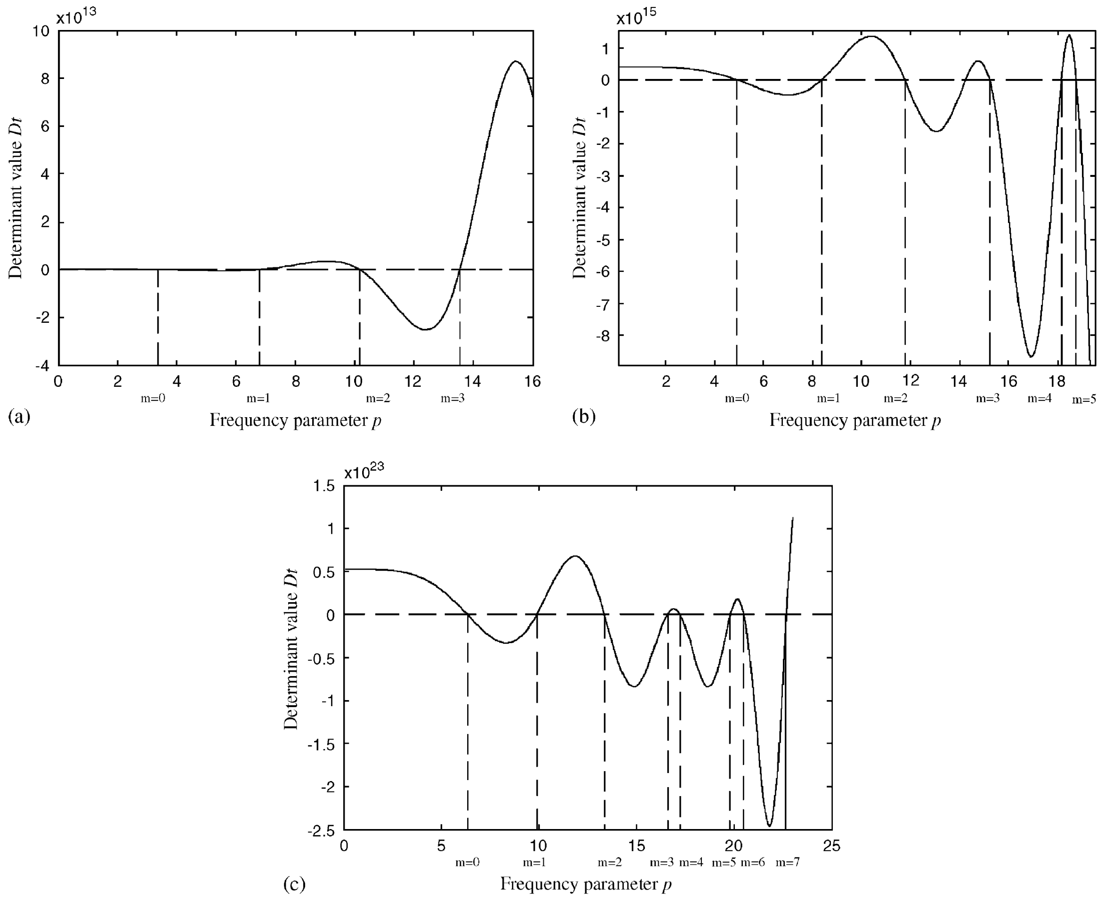

Fig. 2. Determinant value vs. frequency parameter p for $n = 0, 1, 2$: (a) determinant value Dt vs. frequency parameter p for $n = 0$; (b) determinant value Dt vs. frequency parameter p for $n = 1$; (c) determinant value Dt vs. frequency parameter p for $n = 2$.

Furthermore,

$$\lim_{k\to\infty}\frac{(Mt)^{k+1}}{(k+1)!}=0. \tag{30}$$

Substituting Eq. (30) into Eq. (29) yields

$$\left\|T(\varsigma)-Y(\varsigma)\right\|=0.$$

Therefore, $T(\varsigma)$ is the one and only solution of Eq. (15).

2.6. The eigenvalue equation

The elements of transfer matrix $T(\varsigma)$ can be determined by employing recurrence formula (18). The eigenvalue equations (or frequency equations) corresponding to the different boundary conditions can be obtained. Generally, both the inner edge of annular plate and the large edge of conical shell may be one of following three restriction conditions, i.e. free (F), simply supported (S) and clamped (C) boundary conditions:

At a free edge, $\bar{M}_\varsigma=\bar{V}_\varsigma=\bar{N}_\varsigma=\bar{S}_{\varsigma\theta}=0$;
at a clamped edge, $\bar{u}=\bar{v}=\bar{w}=\bar{\phi}=0$;
at a simply supported edge, $\bar{u}=\bar{w}=\bar{M}_\varsigma=\bar{N}_\varsigma=0$.

Substituting the boundary conditions of annular plate and conical shell into Eq. (9), the eigenvalue equation can be derived. Table 1 shows the eigenvalue equations of the structure under all nine combinations. The natural frequencies of the system are determined by calculating the eigenvalues of these equations in Table 1.

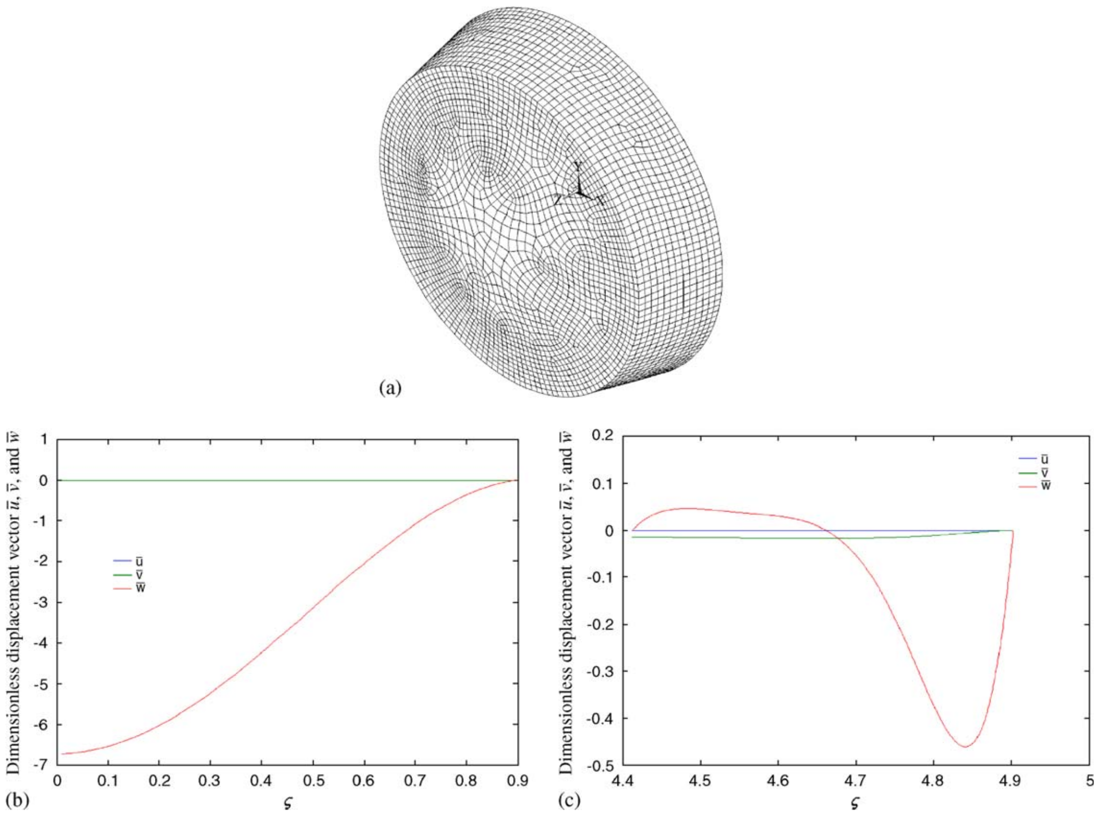

Fig. 3. Eigenvalue and mode shape analyzed by FEM and present method: (a) mode shape relative to eigenvalue $p=3.4170$ by FEM; (b) mode shape of round plate corresponding to $p_{00}=3.4170$ by present method; (c) mode shape of conical shell corresponding to $p_{00}=3.4170$ by present method.

2.7. The mode shape function

Once the eigenvalue is determined, mode shape can be derived. Here, taken the boundary condition $F_p - C_c$ for example, the transfer matrix of entire system can be written as

$$\begin{Bmatrix} \bar{u} \\ \bar{v} \\ \bar{w} \\ \bar{\phi} \\ \bar{M}_\varsigma \\ \bar{V}_\varsigma \\ \bar{N}_\varsigma \\ \bar{S}_{\varsigma\theta} \end{Bmatrix}_{\varsigma=\varsigma_2} = \left[T_{ij}\right] \begin{Bmatrix} \bar{u} \\ \bar{v} \\ \bar{w} \\ \bar{\phi} \\ \bar{M}_\varsigma \\ \bar{V}_\varsigma \\ \bar{N}_\varsigma \\ \bar{S}_{\varsigma\theta} \end{Bmatrix}_{\varsigma=\varsigma_4} \qquad (i,j = 1,2,3,4,5,6,7,8). \tag{31}$$

By substituting these boundary conditions into Eq. (31), extracting first, second, and third rows of this equation, and assigning arbitrarily $\bar{\phi}(\varsigma_4) = 1$, the following equations set yields:

$$\begin{bmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{bmatrix} \begin{Bmatrix} \bar{u} \\ \bar{v} \\ \bar{w} \end{Bmatrix}_{\varsigma=\varsigma_4} + \begin{Bmatrix} T_{14} \\ T_{24} \\ T_{34} \end{Bmatrix} = 0. \tag{32}$$

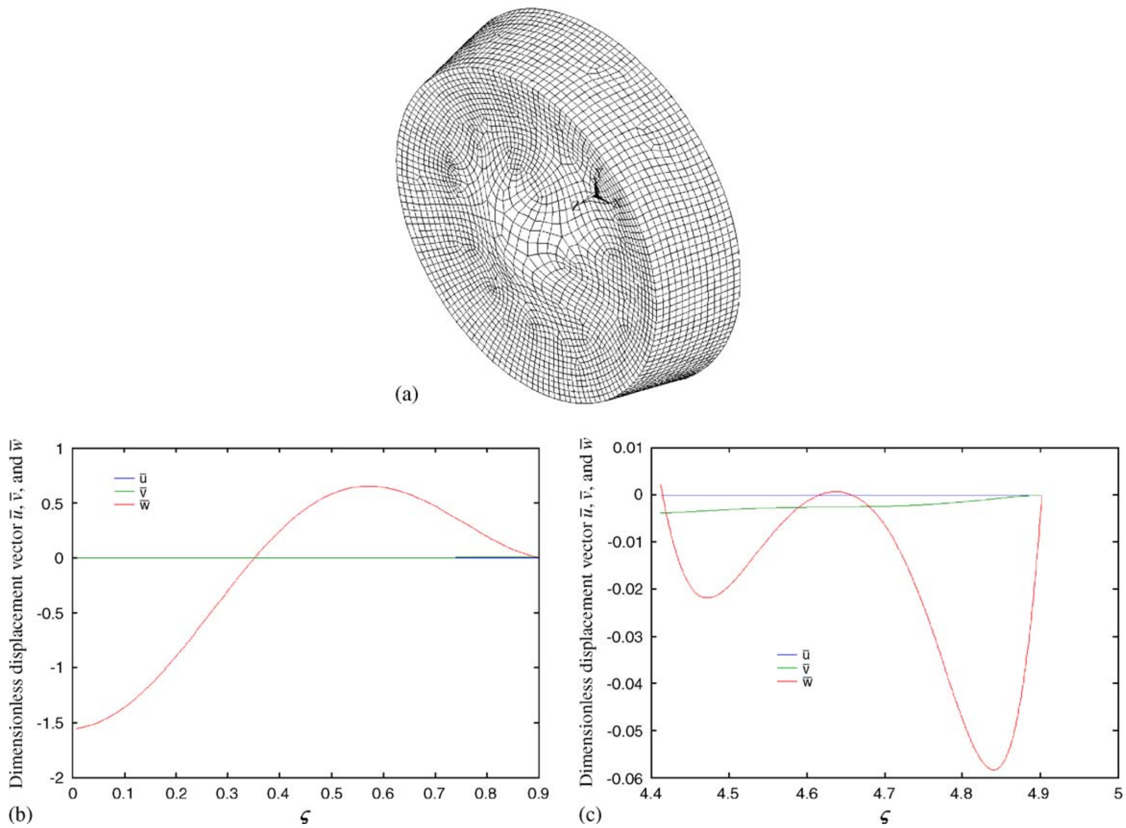

Fig. 4. Eigenvalue and mode shape analyzed by FEM and present method: (a) mode shape relative to eigenvalue $p = 6.7660$ by FEM; (b) mode shape of round plate corresponding to $p_{01} = 6.7660$ by present method; (c) mode shape of conical shell corresponding to $p_{01} = 6.7660$ by present method.

Solution of the displacement vector for the inner edge of annular plate can be written as

$$\begin{Bmatrix} \bar{u} \\ \bar{v} \\ \bar{w} \end{Bmatrix}_{\varsigma=\varsigma_4} = -\begin{bmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{bmatrix}^{-1} \begin{Bmatrix} T_{14} \\ T_{24} \\ T_{34} \end{Bmatrix} = \begin{Bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{Bmatrix}. \tag{33}$$

The displacement vector at any point in the middle surface of annular plate can be obtained as

$$\begin{Bmatrix} \bar{u} \\ \bar{v} \\ \bar{w} \end{Bmatrix}_{\varsigma_4\leqslant\varsigma\leqslant\varsigma_3} = \begin{bmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{bmatrix}_{\varsigma_4\leqslant\varsigma\leqslant\varsigma_3} \begin{Bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{Bmatrix} + \begin{Bmatrix} T_{14} \\ T_{24} \\ T_{34} \end{Bmatrix}_{\varsigma_4\leqslant\varsigma\leqslant\varsigma_3}. \tag{34}$$

With the same method, the displacement vector at any point in the middle surface of the conical shell can be derived as

$$\begin{Bmatrix} \bar{u} \\ \bar{v} \\ \bar{w} \end{Bmatrix}_{\varsigma_1\leqslant\varsigma\leqslant\varsigma_2} = \begin{bmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{bmatrix}_{\varsigma_1\leqslant\varsigma\leqslant\varsigma_2} \begin{Bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{Bmatrix} + \begin{Bmatrix} T_{14} \\ T_{24} \\ T_{34} \end{Bmatrix}_{\varsigma_1\leqslant\varsigma\leqslant\varsigma_2}. \tag{35}$$

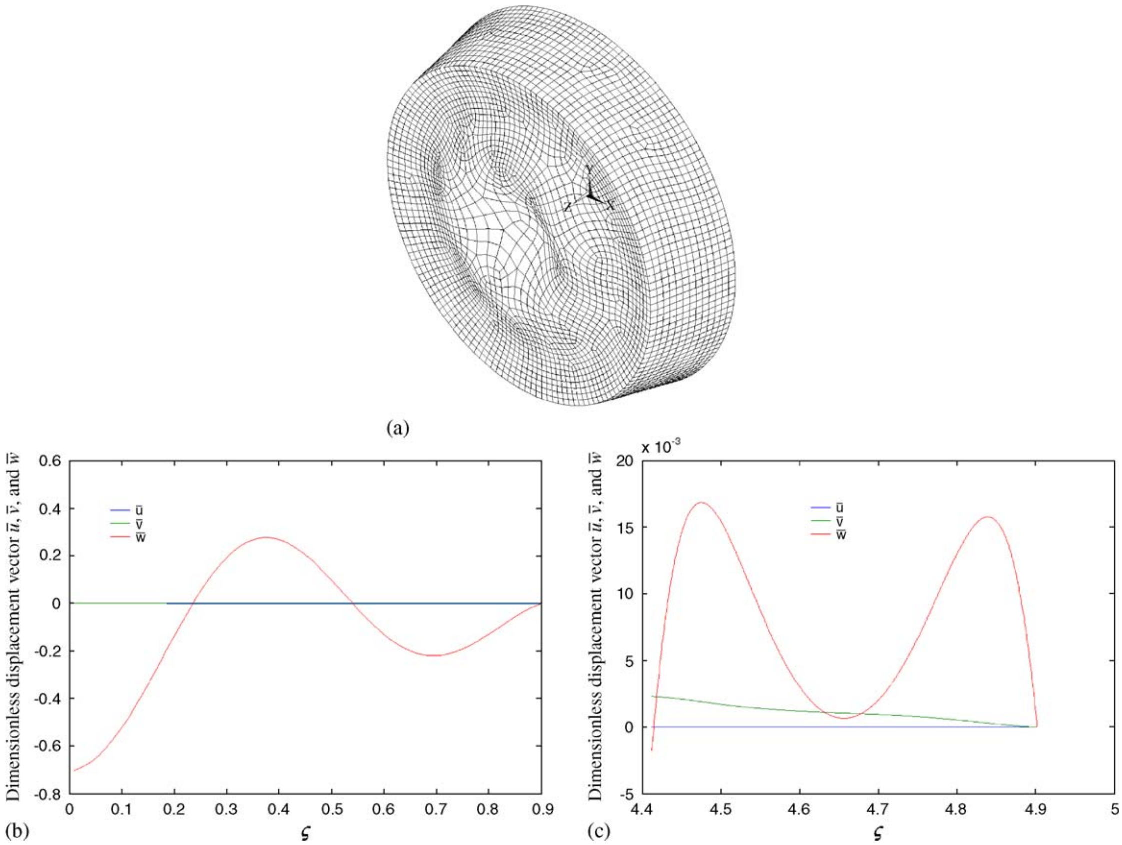

Fig. 5. Eigenvalue and mode shape analyzed by FEM and present method: (a) mode shape corresponding to eigenvalue $p = 10.145$ by FEM; (b) mode shape of round plate corresponding to $p_{02} = 10.155$ by present method; (c) mode shape of conical shell corresponding to $p_{02} = 10.155$ by present method.

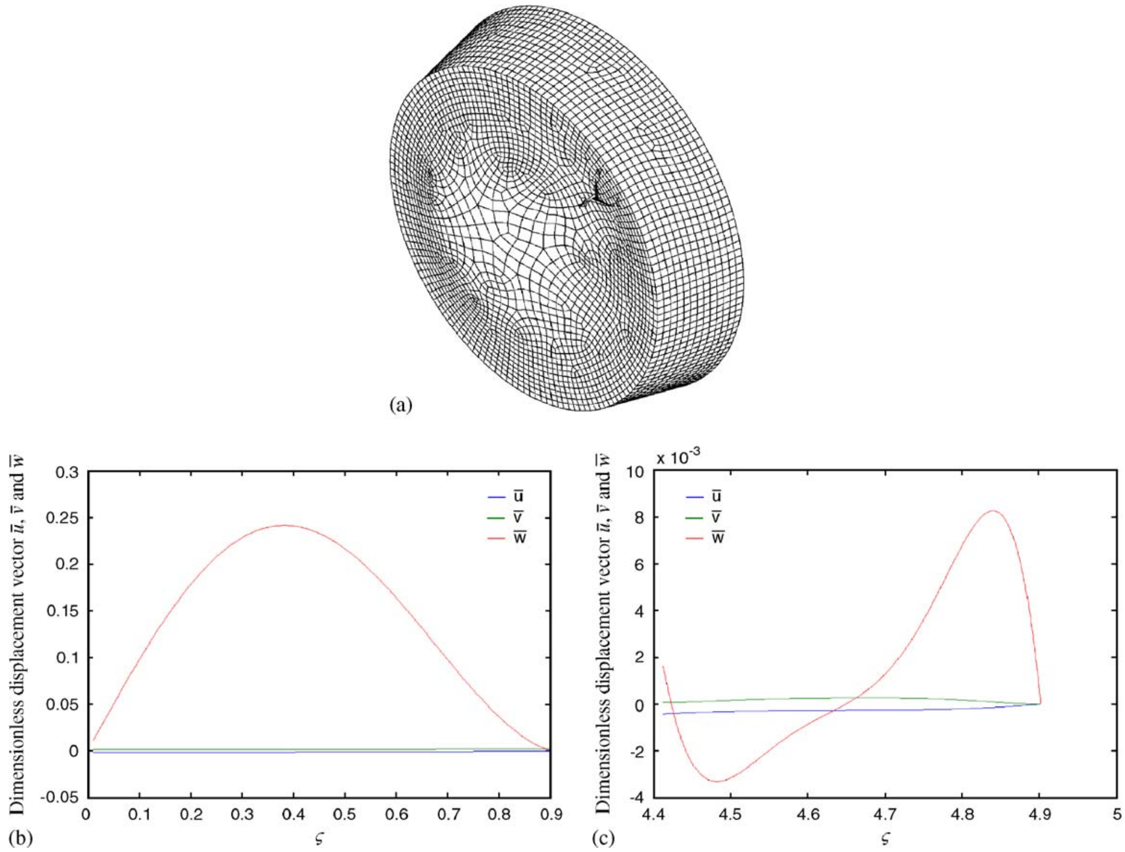

Fig. 6. Eigenvalue and mode shape analyzed by FEM and present method: (a) mode shape corresponding to eigenvalue $p = 4.9232$ by FEM; (b) mode shape of round plate corresponding to $p_{10} = 4.9282$ by present method; (c) mode shape of conical shell corresponding to $p_{10} = 4.9282$ by present method.

Thus, the mode shape (displacement vector) can be calculated from the Eqs. (34) and (35). Using the same method, the other elements in state vector will also be obtained easily.

3. Numerical analysis

A conical shell with a round end plate (made from aluminum) is taken for example, geometric parameters of the combined shell are $R = 200\,\text{mm}$, $h = 2\,\text{mm}$, $R_1 = 180\,\text{mm}$, $R_2 = 0$ and $H = 96\,\text{mm}$, and material parameters are $E = 68.97\,\text{GPa}$ and $\mu = 0.3$. The boundary conditions are $F_p - C_c$. When $\varsigma_4 = 0$, the annular plate will become a round plate. Therefore, the natural frequencies and their mode shapes can be obtained by taking an extremely small value for ς_4, in this paper $\varsigma_4 = 0.01$. In order to verify the formulas presented by this paper, finite element method (FEM) is also adopted here. The analytical software is ANSYS 7.0, and the element type employed here is shell 63. By using Lanczos method, the natural vibration characteristics can be acquired easily.

The nature frequencies of round plates and truncated conical shells can be calculated with the present formulas, respectively. By comparing these results with those in Refs. [23,32], k (this paper $k = 9$) in recurrence Eq. (18) can be determined. Once the transfer matrix of single component has been obtained, the product of each component matrix and the joining matrix can form the entire structure matrix. The solution of Eq. (18) can utilize the software Mathematica. Some curves of the determinant value Dt with frequency parameter p are provided partially. Figs. 2a–c show the curves for $n = 0$, 1 and 2, respectively. Eigenvalue (or natural frequency parameter) is expressed by p_{nm}. Here m is the axial mode number and n denotes the

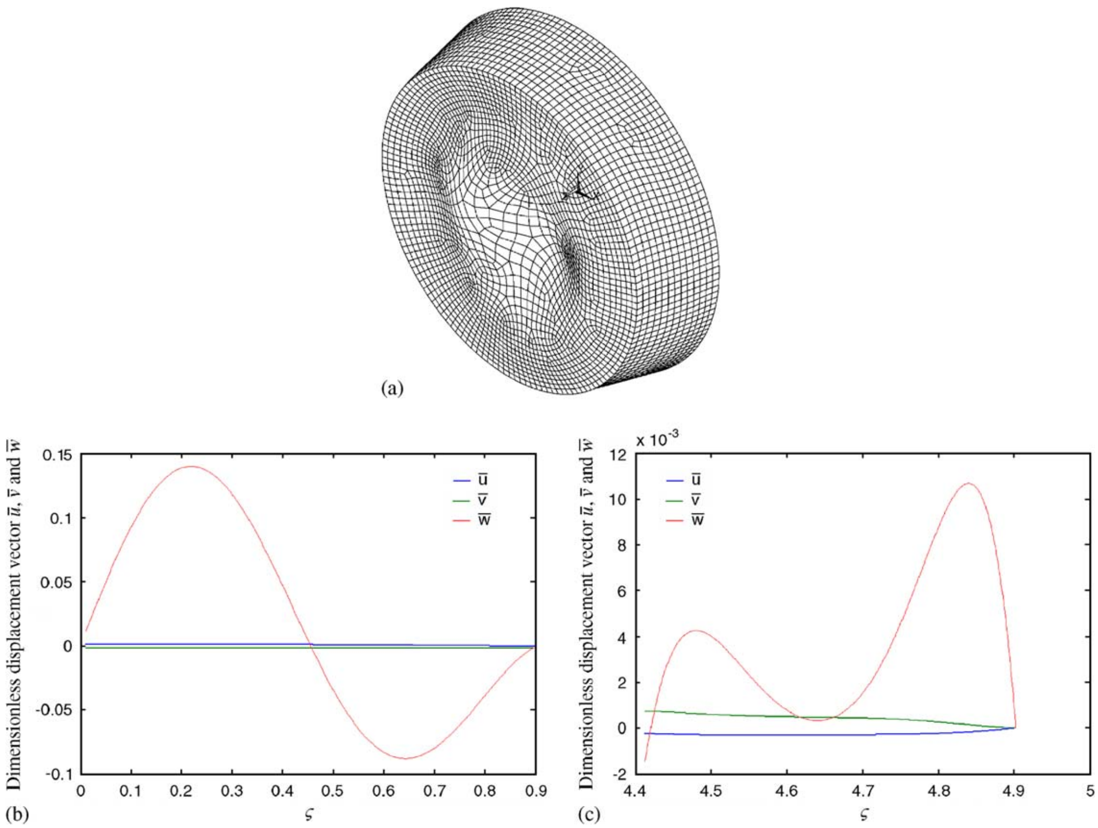

Fig. 7. Eigenvalue and mode shape analyzed by FEM and present method: (a) mode shape corresponding to eigenvalue $p = 8.3550$ by FEM; (b) mode shape of round plate corresponding to $p_{11} = 8.3639$ by present method; (c) mode shape of conical shell corresponding to $p_{12} = 8.3639$ by present method.

circumferential wave number as above mentioned. The mode shapes corresponding to eigenvalues can be calculated by Eq. (34) and (35). Figs. 3a–8a are some mode shapes obtained by FEM, and Figs. 3b–8b and Figs. 3c–8c are some mode shapes acquired by present formulas. A complete mode shape of the combined shell is comprised of mode shapes of an end plate and a conical shell. In other words, under the same eigenvalue, combining panel b with panel c forms a complete mode shape of the entire structure. In order to compare mode shapes obtained by present formulas with those by 3D FEM, panels b and c denote the dimensionless mode shapes of a meridian, $\bar{u}$, $\bar{v}$ and $\bar{w}$ must be transformed into u, v and w by Eq. (A.7a). $n = 0$, i.e. circumferential wavenumber is 0, and mode shapes is circular symmetry. $n = 1$, i.e. circumferential wavenumber is 1, and mode shape is axial symmetry. To check the accuracy of eigenvalue, comparison is also made in Table 2.

Table 2 and Figs. 2–8 indicate that the results analyzed by present approach and FEM are in good agreement with each other, which demonstrates that the method using present study is valid and transfer matrix obtained by recurrence formula method can accurately calculate natural frequency and mode shape.

4. Conclusions

With the vibration theory and transfer matrix method combined, natural frequencies and mode shapes of the conical shell with an annular end plate are investigated in detail. The governing equations of vibration for this system are expressed in terms of matrix differential equations, and a novel recurrence formula method that can close in on analytical solution of transfer matrix is presented. Once the transfer matrix of single component has been determined, the product of each component matrix and the joining matrix can form the

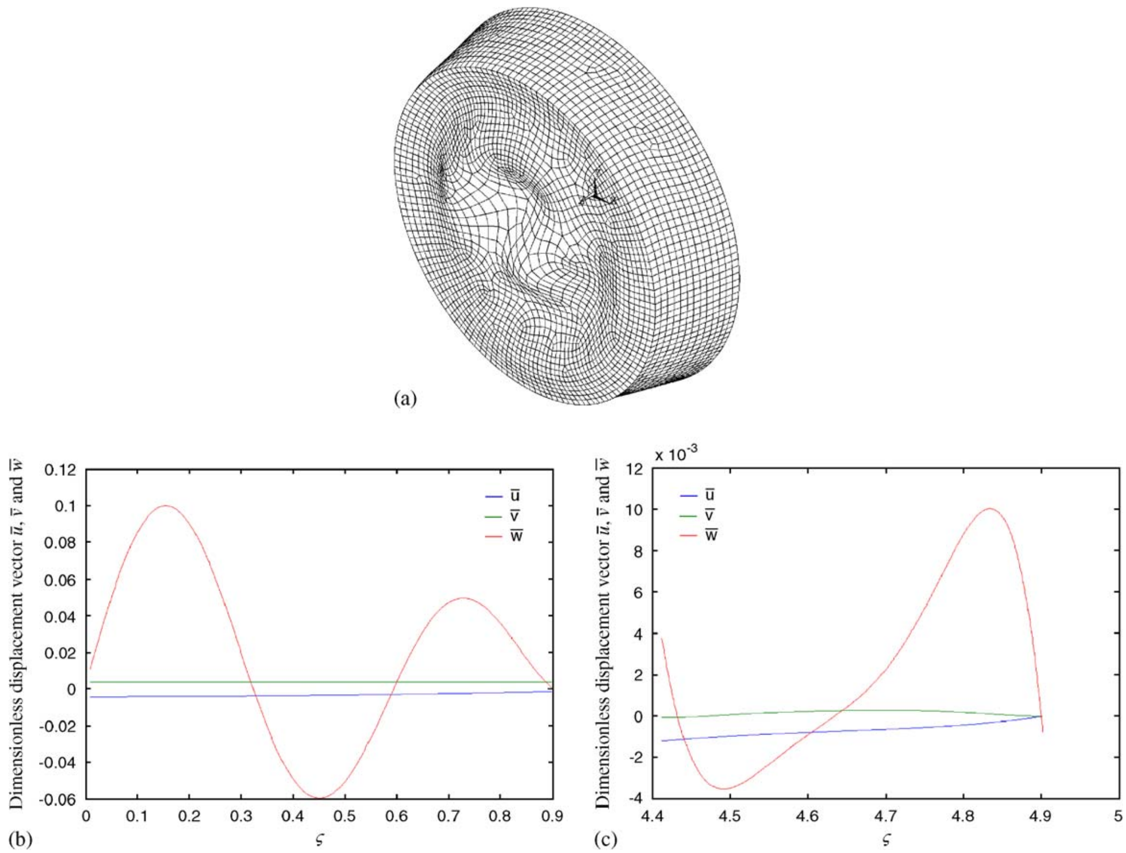

Fig. 8. Eigenvalue and mode shape analyzed by FEM and present method: (a) mode shape corresponding to eigenvalue $p = 11.760$ by FEM; (b) mode shape of round plate corresponding to $p_{12} = 11.772$ by present method; (c) mode shape of conical shell corresponding to $p_{12} = 11.772$ by present method.

matrix of entire structure, and the frequency equations and mode shape functions are represented in terms of the elements of the structural matrices. The 3D finite element numerical simulation has validated the present formulas of natural frequencies and mode shapes. The conclusions show the transfer matrix obtained by present method can accurately reveal dynamic characteristic of the conical shell with an annular end plate.

Acknowledgments

The works was supported by the research Grant no. 10076012 from the National Natural Science Foundation, and the research Grant no. 20010698011 from Doctoral Science Foundation of Ministry of Education of the People's Republic of China.

Appendix A

According to the flÜgge theory, the governing equation of flexural vibration for the conical shell are written as

$$\frac{1}{\varsigma}\frac{\partial(\varsigma N_{\varsigma\theta})}{\partial\varsigma}+\frac{1}{\varsigma\sin\alpha}\frac{\partial N_\theta}{\partial\theta}-\frac{Q_\theta}{\varsigma tg\alpha}+\rho h\omega^2 u=0,$$
$$\frac{1}{\varsigma}\frac{\partial(\varsigma N_{\varsigma})}{\partial\varsigma}+\frac{1}{\varsigma\sin\alpha}\frac{\partial N_{\theta\varsigma}}{\partial\theta}-\frac{N_\theta}{\varsigma}+\rho h\omega^2 v=0,$$

$$-\frac{N_\theta}{\varsigma tg\alpha}-\frac{1}{\varsigma\sin\alpha}\frac{\partial Q_\theta}{\partial\theta}-\frac{1}{\varsigma}\frac{\partial(\varsigma Q_\varsigma)}{\partial\varsigma}+\rho h\omega^2 w=0. \tag{A.1}$$

The Kevlin–Kirchhoff shearing force and shear resultant is

$$V_\varsigma=Q_\varsigma+\frac{1}{\varsigma\sin\alpha}\frac{\partial M_{\varsigma\theta}}{\partial\theta},\quad S_{\varsigma\theta}=N_{\varsigma\theta}-\frac{M_{\varsigma\theta}}{\varsigma tg\alpha}. \tag{A.2}$$

The components of the shearing force are written as

$$\begin{aligned}Q_\varsigma&=\frac{1}{\varsigma}\frac{\partial(\varsigma M_\varsigma)}{\partial\varsigma}+\frac{1}{\varsigma\sin\alpha}\frac{\partial M_{\theta\varsigma}}{\partial\theta}-\frac{M_\theta}{\varsigma},\\ Q_\theta&=\frac{1}{\varsigma}\frac{\partial(\varsigma M_{\varsigma\theta})}{\partial\varsigma}+\frac{1}{\varsigma\sin\alpha}\frac{\partial M_\theta}{\partial\theta}+\frac{M_{\theta\varsigma}}{\varsigma}.\end{aligned} \tag{A.3}$$

The components of membrane force are given by

$$\begin{aligned}N_\varsigma&=\frac{12D}{h^2}\left\{\frac{\partial v}{\partial\varsigma}+\frac{v}{\varsigma}\left(\frac{1}{\sin\alpha}\frac{\partial u}{\partial\theta}+v+\frac{w}{tg\alpha}\right)\right\},\\ N_\theta&=\frac{12D}{h^2}\left\{v\frac{\partial v}{\partial\varsigma}+\frac{1}{\varsigma}\left(\frac{1}{\sin\alpha}\frac{\partial u}{\partial\theta}+v+\frac{w}{tg\alpha}\right)\right\},\\ N_{\varsigma\theta}&=\frac{6(1-v)D}{h^2}\left\{\frac{\partial u}{\partial\varsigma}+\frac{1}{\varsigma}\left(\frac{1}{\sin\alpha}\frac{\partial v}{\partial\theta}-u\right)\right\}.\end{aligned} \tag{A.4}$$

The bending moment can be written as

$$\begin{aligned}M_\varsigma&=D\left\{\frac{\partial\phi}{\partial\varsigma}+\frac{v}{\varsigma}\left(\frac{1}{\varsigma\sin^2\alpha}\frac{\partial^2 w}{\partial\theta^2}+\phi\right)\right\},\\ M_\theta&=D\left\{v\frac{\partial\phi}{\partial\varsigma}+\frac{1}{\varsigma}\left(\frac{1}{\varsigma\sin^2\alpha}\frac{\partial^2 w}{\partial\theta^2}+\phi\right)\right\},\\ M_{\varsigma\theta}&=\frac{(1-v)D}{\varsigma\sin\alpha}\left(\frac{\partial\phi}{\partial\theta}-\frac{1}{\varsigma}\frac{\partial w}{\partial\theta}\right).\end{aligned} \tag{A.5}$$

The slope of the displacement w can be expressed by

$$\phi=\frac{\partial w}{\partial\varsigma}. \tag{A.6}$$

For the steady vibration of shell, let us take

$$\{u\quad v\quad w\}=R\{\bar{u}\sin n\theta\quad\bar{v}\cos n\theta\quad\overline{w}\cos n\theta\}, \tag{A.7a}$$

$$\phi=\overline{\phi}\cos n\theta, \tag{A.7b}$$

$$\{M_\varsigma\quad M_\theta\quad M_{\varsigma\theta}\}=\frac{D}{R}\{\overline{M_s}\cos n\theta\quad\overline{M_\theta}\cos n\theta\quad\overline{M_{\varsigma\theta}}\sin n\theta\}, \tag{A.7c}$$

$$\{N_\varsigma\quad N_\theta\quad N_{\varsigma\theta}\}=\frac{D}{R^2}\{\overline{N_s}\cos n\theta\quad\overline{N_\theta}\cos n\theta\quad\overline{N_{\varsigma\theta}}\sin n\theta\}, \tag{A.7d}$$

$$\{Q_\varsigma\quad Q_\theta\}=\frac{D}{R^2}\{\overline{Q_\varsigma}\cos n\theta\quad\overline{Q_\theta}\sin n\theta\}, \tag{A.7e}$$

$$\{V_\varsigma\quad S_{\varsigma\theta}\}=\frac{D}{R^2}\{\overline{V_\varsigma}\cos n\theta\quad\overline{S_{\varsigma\theta}}\sin n\theta\}. \tag{A.7f}$$

By eliminating M_θ $M_{\varsigma\theta}$ Q_ς Q_θ N_θ $N_{\varsigma\theta}$ from the equations (A.1–A.7), the matrix differential equation yields as follows:

$$\frac{\mathrm{d}}{\mathrm{d}\varsigma}\begin{Bmatrix} \bar{u} \\ \bar{v} \\ \bar{w} \\ \bar{\phi} \\ \bar{M}_\varsigma \\ \bar{V}_\varsigma \\ \bar{N}_\varsigma \\ \bar{S}_{\varsigma\theta} \end{Bmatrix} = \left[A_{ij}\right]\begin{Bmatrix} \bar{u} \\ \bar{v} \\ \bar{w} \\ \bar{\phi} \\ \bar{M}_\varsigma \\ \bar{V}_\varsigma \\ \bar{N}_\varsigma \\ \bar{S}_{\varsigma\theta} \end{Bmatrix} \quad (i,j = 1,2,3,4,5,6,7,8). \tag{A.8}$$

References

[1] K.Y. Lam, H. Li, T.Y. Ng, C.F. Chua, Generalized differential Quadrature method for the free vibration of truncated conical panels, *Journal of sound and Vibration* 251 (2002) 329–348.

[2] X.X. Hu, T. Sakiyama, H. Masuda, C. Morita, Vibration analysis of rotating twisted and open conical shells, *International Journal of Solids and Structures* 39 (2002) 6121–6134.

[3] X.X. Hu, T. Sakiyama, H. Masuda, C. Morita, Vibration analysis of twisted conical shells with tapered thickness, *International Journal of Engineering Science* 40 (2002) 1579–1598.

[4] A.M.I. Sweedan, A.A. EI Damatty, Experimental and analytical evaluation of the dynamic characteristics of conical shells, *Thin-Walled Structures* 40 (2002) 465–486.

[5] T. Irie, G. Yamada, Y. Muramoto, Natural frequencies of in-plane vibration of annular plates, *Journal of Sound and Vibration* 97 (1984) 171–175.

[6] T. Irie, G. Yamada, S. Aomura, Free vibration of a mindlin annular plate of varying thickness, *Journal of Sound and Vibration* 66 (1979) 187–197.

[7] T. Irie, G. Yamada, Y. Kaneko, Vibration and stability of non-uniform annular plate subjected to a follower force, *Journal of Sound and Vibration* 73 (1980) 261–269.

[8] A. Selmane, Natural frequencies of transverse vibrations of non-uniform circular and annular plates, *Journal of Sound and Vibration* 220 (1999) 225–249.

[9] H.N. Arafat, A.H. Nayfeh, W. Faris, Natural frequencies of heated annular and circular plates, *International Journal of Solids and Structures* 41 (2004) 3031–3051.

[10] L. Cheng, Y.Y. Li, L.H. Yam, Vibration analysis of annular-like plates, *Journal of Sound and Vibration* 262 (2003) 1153–1170.

[11] A. Joseph Stanley, N. Ganesan, Frequency response of shell-plate combinations, *Computers and Structures* 59 (1995) 1083–1094.

[12] D.T. Huang, W. Soedel, On the free vibrations of multiple plates welded to a cylindrical shell with special attention to mode pairs, *Journal of Sound and Vibration* 166 (1993) 315–339.

[13] N.O. Myklestad, New method of calculating natural modes of coupled bending-torsion vibration of beams, *Transactions of the American Society of Mechanical Engineers* 67 (1945) 61–67.

[14] W.T. Thomson, Matrix solution of vibration of non-uniform beams, *Journal of Applied Mechanics* 17 (1950) 337–339.

[15] J.C. Snowdon, Mechanical four-pole parameters and their application, *Journal of Sound and Vibration* 15 (1971) 307–323.

[16] S. Rubin, Review of mechanical immittance and transmission matrix concepts, *Journal of the Acoustical Society of America* 41 (1967) 1171–1179.

[17] E.C. Pestel, F.A. Leckie, *Matrix methods in Elastomechanics*, McGraw-Hill Book Company, New York, 1963.

[18] V.R. Murthy, N.C. Nigam, Dynamic characteristics of stiffened rings by transfer matrix approach, *Journal of Sound and Vibration* 39 (1975) 237–245.

[19] V.R. Murthy, Dynamic characteristics of rotor blades, *Journal of Sound and Vibration* 49 (1976) 483–500.

[20] T. Irie, G. Yamada, Y. Kaneko, Free vibration of non-circular cylindrical shell with longitudinal interior partitions, *Journal of Sound and vibration* 96 (1984) 133–142.

[21] A. Kalnins, Free vibration of rotationally symmetric shells, *The Journal of the Acoustical Society of America* 36 (1964) 1335–1365.

[22] T. Irie, G. Yamada, Y. Kaneko, Free vibration of a conical shell with variable thickness, *Journal of Sound and vibration* 82 (1982) 83–94.

[23] T. Irie, G. Yamada, K. Tanaka, Natural frequencies of truncated conical shells, *Journal of Sound and Vibration* 92 (1984) 447–453.

[24] T. Irie, G. Yamada, K. Tanaka, Free vibration of a thin-walled beam-shell of arc cross-section, *Journal of sound and vibration* 94 (1984) 563–572.

[25] T. Irie, G. Yamada, Y. Muramoto, Free vibration of joined conical-cylindrical shells, *Journal of Sound and Vibration* 95 (1984) 31–39.

[26] G. Yamada, T. Irie, T. Tamiya, Free vibration of a circular cylindrical double-shell system closed by end plates, *Journal of Sound and vibration* 108 (1986) 297–304.

[27] W. flÜgge, *Stresses in Shells*, Springer, New York, 1973.
[28] A. Kalnins, Free vibration of rotationally symmetric shells, *Journal of the Acoustical Society of America* 36 (1964) 1365–1964.
[29] G.A. Cohen, Computer analysis of asymmetric free vibrations of ring-stiffened orthotropic shell of revolution, *American Institute of Aeronautics and Astronautics Journal* 3 (1965) 2305–2312.
[30] H. Tottenham, K. Shimizu, Analysis of the free vibration of cantilever cylindrical thin elastic shells by the matric progression method, *International Journal of Mechanical Science* 14 (1972) 293–310.
[31] S. Sankar, Extend transfer matrix method for free vibration of shells of revolution, *Shock and Vibration Bulletin* 47 (1977) 121–133.
[32] Ni. Zhenhua, *Vibration mechanics*, Xi'an Jiaotong University, Xi'an, 1990.