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Energetic mode contributions in stochastic modal analysis: An application to mode classification

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Abstract

Complementary to classical modal analysis, usually performed on a test bench, identification of in-operation structures is more and more widely performed by the means of so-called "stochastic identification". In this framework, modal information should be usefully completed by a criterion indicating the importance of each mode in the total structure response. In this paper, a mechanics-based energetic criterion is proposed in three forms: kinetic energy, potential energy and dissipated power. In the framework of stochastic subspace analysis, a method for the identification of these three quantities is developed. Two simulation test cases with two then five degrees of freedom confirm the method accuracy. Next an experimental test case is presented and analysed. Its validation requires the development of an independent approach. Globally the comparison of both methods shows satisfactory correlation. Finally a real-life case is used to illustrate some practical difficulties of the method.

1. Introduction

1.1. Input–output and output-only analysis

From its beginnings, modal analysis has usually been performed by submitting a structure to a given excitation, then capturing its dynamic response by means of appropriate sensors. The analysis of crossed correlations between input and output signals provides the structure modal characteristics, generally presented in the mathematical form of transfer functions [1]. Then the structure behaviour in real-life conditions can be simulated by integrating excitations into the dynamic model.

This approach accurately provides the pure characteristics of the structure: modal frequencies, damping ratios and normal modes. However, the method fails when the excitations cannot be evaluated with sufficient accuracy. Indeed in many real-life situations, these excitations are unknown, largely scattered and ill-structured. Considering for example a train carriage, aerodynamic excitations are applied all over the body, while rolling excitations resulting from complex wheel–rail–ballast interactions continuously pass from the rail

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Nomenclature

$\mathbf{A}, \mathbf{A}_c$ transition matrices of the discrete and continuous state equations
$\mathbf{B}_c$ excitation matrix of the continuous state equations
$\mathbf{C}$ observation matrix of the discrete state equation
$\mathbf{C}_d, \mathbf{C}_v, \mathbf{C}_a$ selection matrices to extract displacements, velocities and accelerations from the state vector
$\mathbf{f}$ external force vector
$\mathbf{K}, \mathbf{k}, k_p$ stiffness matrix, modal stiffness matrix and its pth diagonal term
$\mathbf{M}, \mathbf{m}, m_p$ mass matrix, modal mass matrix and its pth diagonal term
N length of the available recorded signals
$P_d, P_d^{(p)}$ identified dissipated power and its pth modal contribution
$\mathbf{q}$ modal coordinate vector
$\mathbf{T}, \mathbf{T}_1, \mathbf{T}_2$ base change matrix from natural to identification space, its upper and lower halves
T_e, $t_{\max}$ sampling period and total record time
$\mathbf{u}, \mathbf{U}$ displacement vector in the state equation and in FE analysis
$W_k, W_k^{(p)}$ identified kinetic energy and its pth modal contribution
$W_p, W_p^{(p)}$ identified potential energy and its pth modal contribution
$\mathbf{x}, \mathbf{y}$ natural state vector and output vector of the state equation
$\boldsymbol{\gamma}$ damping matrix of a structure
$\boldsymbol{\Gamma}_\beta$ extended observability matrix of order β
$\bar{\boldsymbol{\delta}}, \underline{\boldsymbol{\delta}}$ selection matrices to extract the upper and lower halves from a matrix
$\boldsymbol{\Delta}$, $\boldsymbol{\Delta}_1$, $\boldsymbol{\Delta}_2$, $\boldsymbol{\Delta}'$ excitation matrix of the discrete state space equations, its upper and lower halves and its inverse
η_p, λ_p respectively pth damping ratio and complex pole
$\hat{\boldsymbol{\mu}}, \hat{\mu}_k$ eigenvalue matrix and its kth diagonal term
φ_p pth normal mode of a structure
$\boldsymbol{\Phi}$ normal mode matrix of a structure
$\boldsymbol{\Psi}, \boldsymbol{\psi}, \boldsymbol{\Psi}_i, \boldsymbol{\psi}_i$ normal mode matrices (from FE analysis), complete and reduced, and their ith column vectors
Ω_p pth modal angular frequencies

Sub and superscripts

$(\cdot)_k$ and $(\cdot)_k$ respectively, line k and column k of a matrix $(\cdot)$
$\hat{\cdot}$ estimate of the variable "$\cdot$" from identification procedure
$\underline{\cdot}$ value of the variable "$\cdot$" obtained from "direct method" (validation)

to the body throughout the axles. Excitations can sometimes be approximately estimated, as is the case for instance in aeronautics, by means of numerical simulation. Mathematical coupling of structure and environmental models can then be performed. However in some cases this is not possible, so that a different approach is needed.

The research on the subject has progressed rapidly during the last decades. Early papers considered the problem in the framework of parametric identification: autoregressive (AR), autoregressive-moving-average (ARMA). The first industrial applications were presented, applied for instance to the identification of an offshore platform [2]. At the end of the 1990s further developments—some with industrial applications [3]—resulted in the present "subspace stochastic identification" under various versions. Papers by Van Overshee and De Moor [4] (N4SID), Viberg [5] (IV-4SID), Verhaegen [6] (MOESP), Basseville et al. (Covariance-based approach) [7], and their co-authors provide a large overview on this evolution.

Contrasting with the classical input–output analysis, generally implemented on a test bench, these methods are basically output-only analyses, thus directly operating on the structure in-operation. Obviously the excitation is unknown, and moreover must be regarded as part of the system, with its own contribution to dynamics, possibly coupled to that of the structure. Consequently the stochastic nature of the excitation entails some statistical dispersion of the identified parameters [8,9]. The problem of sensor placement is also relevant to this situation, since a transfer matrix cannot be completely rebuilt in this case. For these reasons, on-bench analysis and stochastic analysis must be considered complementary, each one having its own advantages and drawbacks.

1.2. The need for mode classification

From the usual input–output procedure, modes appear with very different amplitudes. These differences are the direct consequences of the excitation levels but also of the transducer locations through the notion of participation factors. They can provide some mode classification only by reference to this specific information. However it should not be forgotten that in real-life conditions any identified mode, even hardly detectable, is liable to become a major one at some time, provided that the structure is excited under an appropriate frequency and a suitable force distribution. For this reason, up to now, the engineer has designed a structure by considering all the modes with the same importance.

The issue of mode classification appears differently when considering the structure embedded into its operational environment [10]. Vibration modes are excited at very different levels during the real life of the structure and the designer should be advised to consider the most excited modes more carefully than the small ones. This explains why engineers are becoming more and more concerned by possible in-operation modal analysis, as a complement of classical input–output analysis. From this point of view, quantifying the importance of modes, in the framework of in-operation modal analysis, is a worthy objective which could allow a reliable anticipation of structural fatigue, noise generation and user discomfort. This is the object of the present paper.

It should now be discussed which criterion should be taken for a useful mode classification. Obviously the answer depends on the ultimate purpose of the concerned analysis. For instance to prevent material rupture, a maximum local stress criterion would be appropriate, whereas from the acoustic point of view a global displacement criterion would be better. In the absence of precise information, an energetic criterion is proposed since it gathers information of both internal forces and displacements. This choice has no rational justification, but is chosen because of its universal value.

1.3. The energetic criterion

The energetic point of view being chosen, the question arises to define which energy definition should be considered. From elementary examples it can be shown that a pure "signal processing-based approach", from the observation of signal responses, can be dramatically misleading. The two-degrees-of-freedom (dof) example used as a simulation test case in Section 4.1 illustrates this danger. The sensors being placed near the node of one mode, this mode almost disappears from the output spectrum. The notion of "Power Spectral Density" (PSD) is by itself misleading since it does not represent any mechanical power carried by the signal. Only a mechanics-based approach can provide a meaningful energy definition.

To define a mechanical criterion, several definitions can be proposed. Assuming the small displacement hypothesis, the state space equation of a structure has the form:

$$\mathbf{M}\ddot{\mathbf{u}} + \boldsymbol{\gamma}\dot{\mathbf{u}} + \mathbf{K}\mathbf{u} = \mathbf{f}(t), \tag{1.1}$$

where $\mathbf{M}, \boldsymbol{\gamma}, \mathbf{K}$ are, respectively, the mass, damping and stiffness matrices of the structure, $\mathbf{u}$ and $\mathbf{f}$ the displacement and external force vectors.

At least three energetic criteria can be defined:

- dissipated power

$$P_d = \frac{1}{t_{\max}} \int_0^{t_{\max}} \mathbf{f}^{\mathrm{T}}(t)\dot{\mathbf{u}}(t)\,\mathrm{d}t, \tag{1.2}$$

- mean kinetic energy

$$W_k = \frac{1}{2t_{\max}} \int_0^{t_{\max}} \dot{\mathbf{u}}^{\mathrm{T}}(t)\mathbf{M}\dot{\mathbf{u}}(t)\,\mathrm{d}t, \tag{1.3}$$

- mean potential energy

$$W_p = \frac{1}{2t_{\max}} \int_0^{t_{\max}} \mathbf{u}^{\mathrm{T}}(t)\mathbf{K}\mathbf{u}(t)\,\mathrm{d}t. \tag{1.4}$$

The choice of one criterion depends once more on the practical purpose of the analysis, and none has universal value. For instance in terms of vibration absorption the dissipated power could be advised, but in terms of structure reliability the mean potential energy would be better. In the present paper, the three energetic contributions will be calculated, since they result essentially from the same mathematical development.

In real-life conditions, let us remember that excitations $\mathbf{f}(t)$ are unknown, so that expression (1.2) cannot be directly calculated. Expressions (1.3) and (1.4) cannot be determined either since matrices $\mathbf{M}$ and $\mathbf{K}$ may also be unknown. The discussion hereafter will show that the three problems are closely related. Nevertheless mode identification is possible, using the so-called "stochastic identification method" [2–6], which does not require the knowledge of excitations. The environmental excitation is then considered part of an enlarged system including the structure and its environment, supposedly submitted to a white noise excitation. This excitation is an abstract concept which is introduced to represent the pure random part of the system.

Several methods of this type are available today in operational packages, but unfortunately do not provide any information on these excitations. Moreover a problem lies in the fact that the dynamic matrices are identified up to a similarity relationship, in the usual sense of system control theory [11]. From these matrices, estimates of the force, displacement and velocity sequences can be derived, but in an identification space which has no physical meaning. This indeterminacy forbids direct application of Eqs. (1.2)–(1.4). The dot-product involved in Eq. (1.2) cannot be performed without any knowledge of the "metric tensor" of the identification space. Mass and stiffness matrices involved in Eqs. (1.3) and (1.4) also depend on it. Ultimately the evaluation of the energetic criterion is reduced to the determination of this "metric tensor".

The present paper investigates the possibility to identify this "metric tensor" and therefore the modal energy contributions. As a result the object can only be reached by providing further information, i.e. information on the modal masses of the structure, and also some conditions on the number of observed responses.

The theoretical development will be based on the subspace stochastic identification method, in the version proposed by Van Overschee and De Moor [4], but could easily be adapted to other time-domain approaches, for instance the AR method [2,12].

2. Basic equations and notations

2.1. Discrete state space equations

This section recalls some basic equations and notation which will be useful for the analysis. From the dynamic Eq. (1.1), let us briefly recall the steps to obtain the so-called "discrete state space equation". Eq. (1.1) is first written in the form of a "continuous state space equation" [13]:

$$\dot{\mathbf{x}} = \mathbf{A}_c\mathbf{x} + \mathbf{B}_c\mathbf{f}, \tag{2.1}$$

where $\mathbf{x}$ stands for the generalised state vector

$$\mathbf{x} = \begin{bmatrix} \mathbf{u} \\ \dot{\mathbf{u}} \end{bmatrix} \tag{2.2}$$

and

$$\mathbf{A}_c = \begin{bmatrix} 0 & \mathbf{I} \\ -\mathbf{M}^{-1}\mathbf{K} & -\mathbf{M}^{-1}\boldsymbol{\gamma} \end{bmatrix}, \quad \mathbf{B}_c = \begin{bmatrix} 0 \\ \mathbf{M}^{-1} \end{bmatrix} \tag{2.3a, b}$$

for the transition and input matrices.

The discrete state space equation is obtained from time integration of Eq. (2.1), then comparing the responses at two moments kT_e and $(k+1)T_e$, T_e being the sampling period. Defining a "discrete transition matrix"

$$\mathbf{A} = \mathrm{e}^{T_e \mathbf{A}_c} \tag{2.4}$$

the discrete state space equation is finally obtained in the form [13]:

$$\mathbf{x}(k+1) = \mathbf{A}\mathbf{x}(k) + \int_0^{T_e} \mathrm{e}^{(T_e-\tau)\mathbf{A}_c} \mathbf{B}_c \mathbf{f}(kT_e+\tau)\,\mathrm{d}\tau \tag{2.5}$$

Generally, vector $\mathbf{f}$ is considered as nearly constant during the time interval $[kT_e,(k+1)T_e]$, for example equal to its value at time kT_e noted $\mathbf{f}(k)$. Moreover, state vector $\mathbf{x}$ is never completely observed, but generally either a part, or some linear combination $\mathbf{y}$ of the state variables is captured. Noting

$$\mathbf{\Delta} = \int_0^{T_e} \mathrm{e}^{(T_e-\tau)\mathbf{A}_c}\,\mathrm{d}\tau \tag{2.6}$$

and

$$\begin{cases} \mathbf{w}(k) = \mathbf{\Delta}\mathbf{B}_c\mathbf{f}(k) = \mathbf{\Delta}\begin{bmatrix} 0 \\ \mathbf{M}^{-1}\mathbf{f}(k) \end{bmatrix}, \\ \mathbf{v}(k) = \mathbf{L}\mathbf{f}(k) \end{cases} \tag{2.7a, b}$$

the dynamic equation takes the form:

$$\begin{cases} \mathbf{x}(k+1) = \mathbf{A}\mathbf{x}(k) + \mathbf{w}(k), \\ \mathbf{y}(k) = \mathbf{C}\mathbf{x}(k) + \mathbf{v}(k). \end{cases} \tag{2.8a, b}$$

2.2. The subspace method

In the framework of stochastic analysis, the excitation forces are unknown. Model (2.8) is then formally written, signals $\mathbf{w}$ and $\mathbf{v}$ being regarded as white noises.

Various methods have been proposed [4–6] for the identification of system (2.8). The main ideas are briefly summarised. The starting point is always the identification of the "extended observability matrix"

$$\mathbf{\Gamma}_\beta = \begin{bmatrix} \mathbf{C} \\ \mathbf{CA} \\ \cdots \\ \mathbf{CA}^{\beta-1} \end{bmatrix}, \tag{2.9}$$

where β is a chosen index. To reach this objective, the first step is the elimination of the noises from the dynamic equations. That is performed through the introduction of two response subspaces. One subspace is defined by the α response sequences starting, respectively, from time 0 to time $\alpha-1$. Let it be called "past response subspace". The other one is defined by the β response sequences starting from time α to time $\alpha+\beta-1$, and is called "future response subspace". Projecting the "future subspace" onto the "past subspace", noises asymptotically vanish since they are orthogonal to any independent signal. Let $\mathbf{Y}^{\S}$ be the matrix whose lines contain the β projected sequences. The second step performs simultaneously a space size reduction and the extraction of $\mathbf{\Gamma}_\beta$ from $\mathbf{Y}^{\S}$. That is performed from the singular value decomposition of $\mathbf{Y}^{\S}$

$$\mathbf{Y}^{\S} = \hat{\mathbf{Q}}\hat{\mathbf{S}}\hat{\mathbf{V}}^{\mathrm{T}} = [\hat{\mathbf{Q}}_s \quad \hat{\mathbf{Q}}_n]\begin{bmatrix} \hat{\mathbf{S}}_s & 0 \\ 0 & \hat{\mathbf{S}}_n \end{bmatrix}\begin{bmatrix} \hat{\mathbf{V}}_s^{\mathrm{T}} \\ \hat{\mathbf{V}}_n^{\mathrm{T}} \end{bmatrix}, \tag{2.10}$$

where the diagonal sub-matrix $\hat{\mathbf{S}}_s$ gathers the r most significant singular values of $\mathbf{Y}^{\S}$.

The extended observability matrix $\mathbf{\Gamma}_\beta$ is then estimated from matrix $\hat{\mathbf{Q}}_s$. Generally, the choice

$$\hat{\mathbf{\Gamma}}_\beta = \hat{\mathbf{Q}}_s \hat{\mathbf{S}}_s^{1/2} \tag{2.11}$$

is made, but actually $\mathbf{\Gamma}_\beta$ is still defined up to an arbitrary right multiplication by some square matrix $\mathbf{T}$. It is well known that this indetermination simply corresponds to a change of state space, and has no effect on the identified modal parameters.

From $\hat{\mathbf{Q}}_s$ matrices $\mathbf{A}$ and $\mathbf{C}$ can be estimated [4,5,14,15], considering the "shift structure" of $\mathbf{\Gamma}_\beta$, as illustrated by Eq. (2.9). Once more, the system $\{\mathbf{A}, \mathbf{C}\}$ is obtained up to a similarity relationship, in the usual sense of the system control theory.

An additional result was shown by Van Overschee and De Moor [4] who pointed out the physical meaning of the joint factor of $\mathbf{\Gamma}_\beta$ in the SVD, proving relation:

$$\mathbf{Y}^{\S} = \hat{\mathbf{\Gamma}}_\beta \hat{\mathbf{X}}_\alpha \tag{2.12}$$

where $\hat{\mathbf{X}}_\alpha$ is a sequence of "predictive states" $\{\hat{\mathbf{x}}(k); k > \alpha\}$—in the sense of Kalman filter prediction—estimated from moment α earlier data

$$\hat{\mathbf{X}}_\alpha = \begin{bmatrix} \hat{\mathbf{x}}(\alpha) & \hat{\mathbf{x}}(\alpha+1) & \dots & \hat{\mathbf{x}}(\alpha+N-1) \end{bmatrix}. \tag{2.13}$$

He also proved that unbiased estimates $\hat{\mathbf{A}}$ and $\hat{\mathbf{C}}$ of the dynamic matrices can be obtained from the least-squares solution of the equation

$$\begin{bmatrix} \hat{\mathbf{X}}_{\alpha+1} \\ \mathbf{Y}_\alpha \end{bmatrix} = \begin{bmatrix} \hat{\mathbf{A}} \\ \hat{\mathbf{C}} \end{bmatrix} \hat{\mathbf{X}}_\alpha, \tag{2.14}$$

where $\mathbf{Y}_\alpha$ is a sequence of vectors defined from $\mathbf{y}(k)$ as $\mathbf{X}_k$ was from $\mathbf{x}(k)$ in Eq. (2.13).

The eigenmodes of the structure are finally obtained from the eigendecomposition of matrix $\hat{\mathbf{A}}$. If $\boldsymbol{\tau}$ is the eigenvector matrix, the matrix

$$[\hat{\boldsymbol{\mu}}] = \boldsymbol{\tau}^{-1} \hat{\mathbf{A}} \boldsymbol{\tau} \tag{2.15}$$

is diagonal. Eigenvalues $\hat{\mu}_p$, obtained in conjugate pairs, finally provide estimates of modal frequencies Ω_p and modal damping ratios η_p by the equations:

$$\hat{\lambda}_p = \frac{1}{T_e} \operatorname{Ln}(\hat{\mu}_p), \quad \hat{\lambda}_p = -\Omega_p \eta_p + j\Omega_p \sqrt{1-\eta_p^2}. \tag{2.16a, b}$$

This approach simultaneously allows an estimation of the system noises. Van Overschee and De Moor [4] proved that an unbiased estimate is given by

$$\left| \begin{aligned} \hat{\mathbf{w}}_k &= \hat{\mathbf{x}}(k+1) - \hat{\mathbf{A}}\,\hat{\mathbf{x}}(k), \\ \hat{\mathbf{v}}_k &= \mathbf{y}(k) - \hat{\mathbf{C}}\,\hat{\mathbf{x}}(k). \end{aligned} \right. \tag{2.17a, b}$$

All the elements of the state space model (2.8) are thus estimated. Several authors [8,9] showed that the accuracy of the identification directly depends on the length N of the analysed data records, with standard deviations decreasing like $1/\sqrt{N}$.

2.3. Relationship between natural, identification and modal spaces

As was pointed out above, the dynamic matrices are obtained up to a similarity relationship, defined by an arbitrary transformation matrix $\mathbf{T}$. The key point to evaluate mechanical energy meaningfully is the return to the natural space. Matrix $\mathbf{T}$ can be used to perform this operation.

At this point three spaces are involved in the formulation. From the identification procedure, all the information is obtained in the so-called "identification space" where transition matrix $\hat{\mathbf{A}}$ is defined. The

eigendecomposition of $\hat{\mathbf{A}}$ brings us into the "modal space" through transformation matrix $\boldsymbol{\tau}$, as defined by Eq. (2.15). Finally, three base changes can be defined:

- "identification space" towards "modal space":

$$\hat{\mathbf{x}} = \boldsymbol{\tau}\boldsymbol{\xi}, \tag{2.18}$$

  where ξ stands for the vector of modal coordinates.
- "natural space" towards "identification space"

$$\mathbf{x} = \mathbf{T}\hat{\mathbf{x}}. \tag{2.19}$$

- "natural space" towards "modal space"

$$\mathbf{x} = \boldsymbol{\theta}\boldsymbol{\xi} \quad \text{with } \boldsymbol{\theta} = \mathbf{T}\,\boldsymbol{\tau}. \tag{2.20a, b}$$

3. Return to the "natural space"

3.1. Space change matrix

In the "identification space", the relationship between discrete and continuous transition matrices is

$$\hat{\mathbf{A}}_c = \frac{1}{T_e}\,\mathrm{Ln}\,\hat{\mathbf{A}}. \tag{3.1}$$

In the "natural space", continuous matrix $\mathbf{A}_c$ has the special structure (2.3a). The problem is thus to find some matrix $\mathbf{T}$ which brings back matrix $\hat{\mathbf{A}}_c$ to this form [16]. First recall that both continuous matrix $\mathbf{A}_c$ and discrete matrix $\mathbf{A}$ are transformed from one state space to another one by the same transformation matrix $\mathbf{T}$. This can be expressed as

$$\hat{\mathbf{A}}_c = \mathbf{T}^{-1}\mathbf{A}_c\mathbf{T}, \quad \hat{\mathbf{C}} = \mathbf{C}\mathbf{T} \tag{3.2a, b}$$

which yields

$$\mathbf{A}_c\mathbf{T} = \mathbf{T}\hat{\mathbf{A}}_c \quad \text{or} \quad \begin{bmatrix} 0 & \mathbf{I} \\ -\mathbf{M}^{-1}\mathbf{K} & -\mathbf{M}^{-1}\boldsymbol{\gamma} \end{bmatrix} \begin{bmatrix} \mathbf{T}_1 \\ \mathbf{T}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{T}_1 \\ \mathbf{T}_2 \end{bmatrix} \hat{\mathbf{A}}_c, \tag{3.3a, b}$$

where matrix $\mathbf{T}$ is split into two $(n, 2n)$ blocks, n being the number of (dof) of the structure. Eq. (3.3b) can be displayed as

$$\begin{vmatrix} \mathbf{T}_2 = \mathbf{T}_1\hat{\mathbf{A}}_c, \\ -\mathbf{M}^{-1}\mathbf{K}\mathbf{T}_1 - \mathbf{M}^{-1}\boldsymbol{\gamma}\mathbf{T}_2 = \mathbf{T}_2\hat{\mathbf{A}}_c. \end{vmatrix} \tag{3.4a, b}$$

For the observation matrix $\mathbf{C}$ three cases must be considered.

(a) If only displacements are measured, $\mathbf{C}$ is a $(l, 2n)$ matrix of the form

$$\mathbf{C} = \mathbf{C}_d \left[\mathbf{I} \,\vdots\, 0 \right] \tag{3.5}$$

where l being the number of sensors and $\mathbf{C}_d$ a (l, n) selection matrix. Eq. (3.2b) results in

$$\hat{\mathbf{C}} = \mathbf{C}_d\mathbf{T}_1. \tag{3.6}$$

(b) If only velocities are measured, $\mathbf{C}$ is also a size $(l, 2n)$ selection matrix

$$\mathbf{C} = \mathbf{C}_v \left[0 \,\vdots\, \mathbf{I} \right] \tag{3.7}$$

and, from Eqs. (3.2b) and (3.4a), it results that:

$$\hat{\mathbf{C}} = \mathbf{C}_v \mathbf{T}_2 = \mathbf{C}_v \mathbf{T}_1 \hat{\mathbf{A}}_c. \tag{3.8}$$

(c) If only accelerations are measured, which is generally the case using piezo-electric accelerometers, Eqs. (2.8b), (2.1) then (2.3a) give

$$\mathbf{C} = \mathbf{C}_a \begin{bmatrix} -\mathbf{M}^{-1}\mathbf{K} & -\mathbf{M}^{-1}\boldsymbol{\gamma} \end{bmatrix} \tag{3.9}$$

thus, from Eqs. (3.2b), (3.9) then (3.4)

$$\hat{\mathbf{C}} = \mathbf{C}_a \mathbf{T}_2 \hat{\mathbf{A}}_c = \mathbf{C}_a \mathbf{T}_1 \hat{\mathbf{A}}_c^2. \tag{3.10}$$

Cases where a mixture of these signals is captured will not be considered since they do not correspond to usual situations.

From Eqs. (3.6), (3.8) or (3.10), matrix $\mathbf{T}_1$ can be extracted only if the selection matrix $\mathbf{C}_d$, $\mathbf{C}_v$ or $\mathbf{C}_a$ is an identity matrix. This restriction is important. It means that the number of sensors must be at least equal to half the size of the state space, which is theoretically the number of dof of the structure. However in the context of pure stochastic analysis, where no prior "modelling" of the structure is made, the notion of "degree of freedom" is unclear. Indeed in this case the latter cannot be distinguished from the notion of mode. Assuming that all the modes are captured, at least in a defined frequency range, the modal shapes are supposed to define a sufficient representation base of the structure behaviour in the operative framework. With this hypothesis, the number of dof is the same as the number m of the modes observed. Therefore two cases must be considered:

(1) The number l of sensors is equal to the number of modes: $l = m$. Matrix $\mathbf{T}$ is then completely determined:
(a) if only displacements are observed

$$\mathbf{T}_1 = \hat{\mathbf{C}}, \quad \mathbf{T}_2 = \hat{\mathbf{C}}\,\hat{\mathbf{A}}_c, \tag{3.11a, b}$$

(b) if only velocities are observed

$$\mathbf{T}_1 = \hat{\mathbf{C}}\,\hat{\mathbf{A}}_c^{-1}, \quad \mathbf{T}_2 = \hat{\mathbf{C}}, \tag{3.12a, b}$$

(c) if only accelerations are observed

$$\mathbf{T}_1 = \hat{\mathbf{C}}\,\hat{\mathbf{A}}_c^{-2}, \quad \mathbf{T}_2 = \hat{\mathbf{C}}\,\hat{\mathbf{A}}_c^{-1}. \tag{3.13a, b}$$

(2) The number l of sensors is inferior to the number of modes: $l < m$. Matrix $\mathbf{T}$ then can be split into three blocks:

$$\mathbf{T} = \begin{bmatrix} \mathbf{T}_{11} \\ \mathbf{T}_{12} \\ \mathbf{T}_2 \end{bmatrix},$$

where block $\mathbf{T}11$ has size $(l, 2n)$. In case (a) for instance, sub-matrices $\mathbf{T}_{11}$, then $\mathbf{T}_2$ can be extracted from Eq. (3.6): $\mathbf{T}_{11} = \hat{\mathbf{C}}; \quad \mathbf{T}_2 = \hat{\mathbf{C}}\,\hat{\mathbf{A}}_c^{-1}$, but sub-matrix $\mathbf{T}_{12}$ remains undetermined. Actually matrix $\mathbf{T}$ can be arbitrarily completed, for example by a block $\mathbf{T}_{12}$ of line vectors orthogonal to the line subspace of matrices $\mathbf{T}_{11}$ and $\mathbf{T}_2$. This arbitrary choice has some consequences on what follows. Exact return to the "nominal space" is not possible and therefore the natural metric tensor is not restored. There is a lack of information resulting from insufficient observation of the structure response. The relative energetic levels can be concerned, and possibly lead to a wrong mode classification.

3.2. Reconstruction of mass and stiffness matrices

Further calculation will show that knowledge of the mass matrix is compulsory to build the various energy expressions. This fact can be understood from simple consideration. The reduced dynamic equation

$$\ddot{\mathbf{u}} + \mathbf{M}^{-1}\boldsymbol{\gamma}\dot{\mathbf{u}} + \mathbf{M}^{-1}\mathbf{K}\mathbf{u} = \mathbf{M}^{-1}\mathbf{f} \tag{3.14}$$

clearly shows that in the absence of physical information on external forces $\mathbf{f}$, product $\mathbf{M}^{-1}\mathbf{f}$ can never be dissociated. Unfortunately, in the estimation of the dissipated energy (1.2), force vector $\mathbf{f}$ is required alone. Moreover in expression (1.3) of the kinetic energy, the mass matrix is explicitly concerned. The potential energy (1.4) also involves the stiffness matrix, which can be obtained from the mass matrix as soon as the modal frequencies Ω_p are known.

The consequences are important; it means that a realistic estimation of the vibratory energy requires the knowledge of the mass matrix or at least a mass matrix reduced to the observed dof.

Actually, since the transitions from "natural space" to "identification space", then to "modal space" are known, such a mass matrix can be restored if at least the modal masses are known. These modal masses can in no way be obtained from pure stochastic analysis. Their knowledge requires either some modelling effort, even rough modelling if the purpose is only mode classification, or some deterministic identification by means of a modal analysis in a laboratory.

Assuming these modal masses available, the reconstruction of a "physical" mass matrix follows a path going from eigenspace to identification space then to natural space, using the above matrices $\boldsymbol{\tau}$ and $\mathbf{T}$. However, this operation requires to know the real mode shapes. From stochastic identification, complex modes are obtained from which the normal modes can be derived after some algebraic manipulation. Another method is to exploit the low left quarter of "physical" matrix $\mathbf{A}_c$ theoretically equal to the product:

$$-\mathbf{L} = -\mathbf{M}^{-1}\mathbf{K} \tag{3.15}$$

as shown by Eq. (2.3a). The normal modes of the structure are the eigenvectors of this sub-matrix. Finally, "physical" mass matrix $\mathbf{M}$ is obtained by the equation

$$\mathbf{M} = \boldsymbol{\Phi}\mathbf{m}\boldsymbol{\Phi}^{-1}, \tag{3.16}$$

$\boldsymbol{\Phi}$ being the matrix of these normal modes, $\mathbf{m}$ the diagonal matrix of modal masses. By a similar equation, the "physical" stiffness matrix can be obtained:

$$\mathbf{K} = \boldsymbol{\Phi}\mathbf{k}\boldsymbol{\Phi}^{-1}, \tag{3.17}$$

$\mathbf{k}$ being the diagonal matrix of modal stiffnesses k_p directly obtained from modal masses m_p and modal frequencies Ω_p:

$$k_p = m_p\Omega_p^2. \tag{3.18}$$

4. Mode energetic contributions

4.1. Metric tensor of the identification space

In the "natural space", the dissipated energy defined by Eq. (1.2) can be approximately given by the discrete expression

$$W_d = T_e \sum_{k=0}^{N-1} \mathbf{f}^{\mathbf{T}}(k)\dot{\mathbf{u}}(k+1), \tag{4.1}$$

where N stands for the length of recorded signals available.

It was pointed out above that the dot-product involved in this equation cannot be applied in the identification space since the metric tensor of this space is unknown.

Defining the selection matrices

$$\bar{\boldsymbol{\delta}} = [\mathbf{I} \quad 0] \quad \text{and} \quad \underline{\boldsymbol{\delta}} = [0 \quad \mathbf{I}] \tag{4.2a, b}$$

leads to

$$\mathbf{u}(k) = \bar{\boldsymbol{\delta}}\mathbf{x}(k), \quad \dot{\mathbf{u}}(k) = \underline{\boldsymbol{\delta}}\,\mathbf{x}(k). \tag{4.3a, b}$$

Noting

$$\boldsymbol{\Delta}' = \boldsymbol{\Delta}^{-1} = \begin{bmatrix} \boldsymbol{\Delta}'_1 \\ \boldsymbol{\Delta}'_2 \end{bmatrix}. \tag{4.4}$$

Eq. (2.7a) results in

$$\mathbf{f}(k) = \mathbf{M}\boldsymbol{\Delta}'_2\mathbf{w}(k). \tag{4.5}$$

In terms of state variables, the dissipated power P_d is thus expressed as

$$P_d = \frac{1}{NT_e}\, W_d = \frac{1}{N}\sum_{k=0}^{N-1} \mathbf{w}^{\mathrm{T}}(k)\boldsymbol{\Delta}'^{\mathrm{T}}_2\mathbf{M}\,\underline{\boldsymbol{\delta}}\,\mathbf{x}(k+1). \tag{4.6}$$

It can be observed, from Eq. (2.6) that matrix $\boldsymbol{\Delta}$ can be approximated, up to the first order by a scalar matrix $T_e\mathbf{I}$, approximation which is sufficient for a simple mode classification. Expression (4.6) thus reduces to

$$P_d = \frac{1}{NT_e}\sum_{k=0}^{N-1} \mathbf{w}^{\mathrm{T}}(k)\underline{\boldsymbol{\delta}}^{\mathrm{T}}\mathbf{M}\,\underline{\boldsymbol{\delta}}\,\mathbf{x}(k+1). \tag{4.7}$$

Now the dissipated power can be expressed in terms of identified variables. All vectors are changed using matrix $\mathbf{T}$:

$$\mathbf{w}(k) = \mathbf{T}\hat{\mathbf{w}}(k), \quad \mathbf{x}(k) = \mathbf{T}\hat{\mathbf{x}}(k) \tag{4.8a, b}$$

which gives

$$P_d = \frac{1}{NT_e}\sum_{k=0}^{N-1} \hat{\mathbf{w}}^{\mathrm{T}}(k)\mathbf{T}^{\mathrm{T}}\underline{\boldsymbol{\delta}}^{\mathrm{T}}\mathbf{M}\,\underline{\boldsymbol{\delta}}\,\mathbf{T}\hat{\mathbf{x}}(k+1) = \frac{1}{NT_e}\sum_{k=0}^{N-1} \hat{\mathbf{w}}^{\mathrm{T}}(k)\mathbf{T}_2^{\mathrm{T}}\mathbf{M}\mathbf{T}_2\hat{\mathbf{x}}(k+1). \tag{4.9}$$

This result shows that in the identification space, the "metric tensor" is defined by

$$\mathbf{G} = \mathbf{T}_2^{\mathrm{T}}\mathbf{M}\mathbf{T}_2 \tag{4.10}$$

from which the dissipated energy can be calculated

$$P_d = \frac{1}{NT_e}\sum_{k=0}^{N-1} \hat{\mathbf{w}}^{\mathrm{T}}(k)\mathbf{G}\hat{\mathbf{x}}(k+1). \tag{4.11}$$

4.2. Kinetic and potential energy estimates

Kinetic and potential energy values are expressed by Eqs. (1.3) and (1.4). Always positive, they vary continuously according to the structure displacements. Their mean values are approximately given by

$$W_k = \frac{1}{2N}\sum_{k=0}^{N-1} \dot{\mathbf{u}}^{\mathrm{T}}(k)\mathbf{M}\dot{\mathbf{u}}(k) \tag{4.12}$$

for the kinetic energy and

$$W_p = \frac{1}{2N}\sum_{k=0}^{N-1} \mathbf{u}^{\mathrm{T}}(k)\mathbf{K}\mathbf{u}(k) \tag{4.13}$$

for the potential energy.

From notations (4.3a, b) the following expressions hold

$$W_k = \frac{1}{2N}\sum_{k=0}^{N-1} \hat{\mathbf{x}}^{\mathrm{T}}(k)\mathbf{T}^{\mathrm{T}}\underline{\boldsymbol{\delta}}^{\mathrm{T}}\mathbf{M}\,\underline{\boldsymbol{\delta}}\,\mathbf{T}\hat{\mathbf{x}}(k), \tag{4.14}$$

$$W_p = \frac{1}{2N}\sum_{k=0}^{N-1} \hat{\mathbf{x}}^{\mathrm{T}}(k)\mathbf{T}^{\mathrm{T}}\bar{\boldsymbol{\delta}}^{\mathrm{T}}\mathbf{M}\bar{\boldsymbol{\delta}}\mathbf{T}\hat{\mathbf{x}}(k). \tag{4.15}$$

4.3. Modal contributions to energy expressions

The contribution of each mode to the different energy expressions results from the decomposition of the displacement in its modal components. From Eq. (2.17a)

$$\hat{\mathbf{x}}(k+1) = \hat{\mathbf{A}}\,\hat{\mathbf{x}}(k) + \hat{\mathbf{w}}(k) \tag{4.16}$$

is obtained and from Eq. (2.15)

$$\hat{\mathbf{x}}(k+1) = \boldsymbol{\tau}\hat{\boldsymbol{\mu}}\boldsymbol{\tau}^{-1}\hat{\mathbf{x}}(k) + \boldsymbol{\tau}\boldsymbol{\tau}^{-1}\hat{\mathbf{w}}(k). \tag{4.17}$$

This equation can be written in the form

$$\hat{\mathbf{x}}(k+1) = \sum_{p=1}^{m} \hat{\mathbf{x}}^{(p)}(k+1) \tag{4.18}$$

by defining

$$\hat{\mathbf{x}}^{(p)}(k+1) = \boldsymbol{\tau}_{.p}\boldsymbol{\tau}_{p.}^{-1}(\hat{\mu}_p\hat{\mathbf{x}}(k) + \hat{\mathbf{w}}(k)). \tag{4.19}$$

Likewise Eq. (4.19) is written as

$$\hat{\mathbf{x}}^{(p)}(k+1) = \boldsymbol{\tau}_{.p}\boldsymbol{\tau}_{p.}^{-1}\hat{\boldsymbol{\alpha}}_p(k) \tag{4.20}$$

if noting

$$\hat{\boldsymbol{\alpha}}_p(k) = \hat{\mu}_p\hat{\mathbf{x}}(k) + \hat{\mathbf{w}}(k). \tag{4.21}$$

The energetic contributions can now be obtained.

(1) Modal dissipated power
The expression of the dissipated power (4.11) can be expressed now as

$$P_d = \sum_{p=1}^{m} P_d^{(p)}, \tag{4.22}$$

where the component

$$P_d^{(p)} = \frac{1}{NT_e}\sum_{k=1}^{N} \hat{\mathbf{w}}^{\mathrm{T}}(k)\mathbf{G}\hat{\mathbf{x}}^{(p)}(k) \tag{4.23}$$

can be considered the energetic contribution of mode (p) to the total dissipated power.

(2) Modal kinetic energy
From Eqs. (4.12) and (4.20) the total kinetic energy is expressed as

$$P_k = \frac{1}{2}\left[\sum_{p=1}^{m}\left(\underline{\boldsymbol{\delta}}\,\mathbf{T}\boldsymbol{\tau}_{.p}\boldsymbol{\tau}_{p.}^{-1}\hat{\boldsymbol{\alpha}}_p(k)\right)^{\mathrm{T}}\right]\mathbf{M}\left[\sum_{q=1}^{m}\underline{\boldsymbol{\delta}}\,\mathbf{T}\boldsymbol{\tau}_{.q}\boldsymbol{\tau}_{q.}^{-1}\hat{\boldsymbol{\alpha}}_q(k)\right]. \tag{4.24}$$

In the brackets, products $\mathbf{T}\boldsymbol{\tau}_{.p}$ and $\mathbf{T}\boldsymbol{\tau}_{.q}$ restore the pth and qth columns of transformation matrix $\boldsymbol{\theta}$ defined by Eq. (2.20b):

$$P_k = \frac{1}{2}\left[\sum_{p=1}^{m}\left(\underline{\boldsymbol{\delta}}\,\boldsymbol{\theta}_{.p}\boldsymbol{\tau}_{p.}^{-1}\hat{\boldsymbol{\alpha}}_p(k)\right)^{\mathrm{T}}\right]\mathbf{M}\left[\sum_{q=1}^{m}\underline{\boldsymbol{\delta}}\,\boldsymbol{\theta}_{.q}\boldsymbol{\tau}_{q.}^{-1}\hat{\boldsymbol{\alpha}}_q(k)\right]. \tag{4.25}$$

A classical result from modal analysis theory allows a drastic simplification of this expression. Defining matrix

$$\mathbf{Q} = \begin{bmatrix} -\mathbf{K} & 0 \\ 0 & \mathbf{M} \end{bmatrix} \tag{4.26}$$

together with transformation matrix $\boldsymbol{\theta}$ of Eq. (2.20b), a classical result expresses that the product $\boldsymbol{\theta}^{\mathrm{T}}\mathbf{Q}\boldsymbol{\theta}$ is diagonal.

Introducing selection matrix $\underline{\boldsymbol{\delta}}$ defined by Eq. (4.2b), it results that the product $\boldsymbol{\theta}^{\mathrm{T}}\underline{\boldsymbol{\delta}}^{\mathrm{T}}\mathbf{M}\,\underline{\boldsymbol{\delta}}\,\boldsymbol{\theta}$ is diagonal. In Eq. (4.25), it means that any term including a product $\boldsymbol{\theta}_{.p}^{\mathrm{T}}\underline{\boldsymbol{\delta}}^{\mathrm{T}}\mathbf{M}\,\underline{\boldsymbol{\delta}}\,\boldsymbol{\theta}_{q.}$ is zero for $p \neq q$. Let $P_k^{(p)}$ be

$$P_k^{(p)} = \tfrac{1}{2}\hat{\boldsymbol{\alpha}}_p^{\mathrm{T}}(k)\boldsymbol{\tau}_{p.}^{-\mathrm{T}}\boldsymbol{\theta}_{.p}^{\mathrm{T}}\underline{\boldsymbol{\delta}}^{\mathrm{T}}\mathbf{M}\,\underline{\boldsymbol{\delta}}\,\boldsymbol{\theta}_{.p}\boldsymbol{\tau}_{p.}^{-1}\hat{\boldsymbol{\alpha}}_p(k) \tag{4.27}$$

the kinetic energy is finally expressed as

$$P_k = \sum_{p=1}^{m} P_k^{(p)}. \tag{4.28}$$

The term $P_k^{(p)}$ can be considered the modal contribution of mode p to the total kinetic energy.

(3) Modal potential energy

For potential energy a similar development leads to the expression:

$$P_p = \sum_{p=1}^{m} P_p^{(p)}, \tag{4.29}$$

where

$$P_p^{(p)} = \tfrac{1}{2}\hat{\boldsymbol{\alpha}}_p^{\mathrm{T}}(k)\boldsymbol{\tau}_{p.}^{-\mathrm{T}}\boldsymbol{\theta}_{.p}^{\mathrm{T}}\bar{\boldsymbol{\delta}}^{\mathrm{T}}\mathbf{K}\bar{\boldsymbol{\delta}}\boldsymbol{\theta}_{.p}\boldsymbol{\tau}_{p.}^{-1}\hat{\boldsymbol{\alpha}}_p(k) \tag{4.30}$$

represents the modal contribution of mode p to the total potential energy. In this equation, matrix $\mathbf{K}$ was obtained from the knowledge of modal masses by Eq. (3.18).

5. Validation of the method: simulated examples

5.1. A general test system

A n-dof system can be represented by using n independent mass-damper-spring oscillators (Fig. 1). Each elementary oscillator being given by its mass m_p, eigenfrequencies Ω_p and damping ratio η_p, the transition matrix of the subsystem is

$$\mathbf{A}_c^{(p)} = \begin{bmatrix} 0 & 1 \\ -\Omega_p^2 & -2\Omega_p\eta_p \end{bmatrix}. \tag{5.1}$$

The transition matrix of the whole system is obtained by assembling the elementary matrices into a block-diagonal matrix:

$$\mathbf{A}_c = \begin{bmatrix} \mathbf{A}_c^{(1)} & . & . & . \\ . & \mathbf{A}_c^{(2)} & . & . \\ . & . & \ldots & . \\ . & . & . & \mathbf{A}_c^{(m)} \end{bmatrix}. \tag{5.2}$$

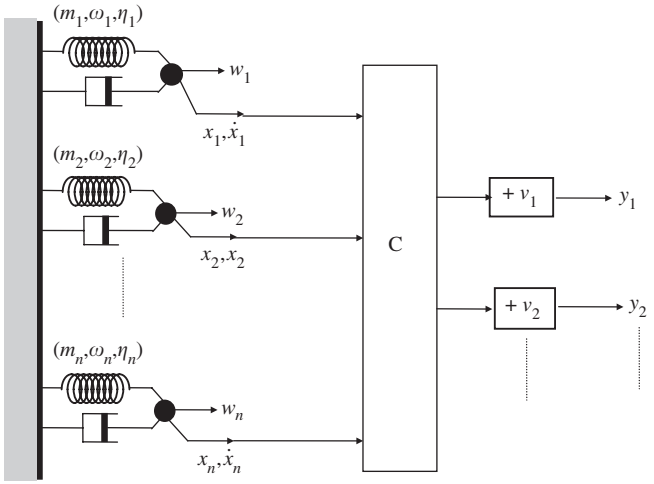

Fig. 1. An n-dof mechanical system.

Each subsystem is excited by independent noise w_p. At this point all subsystems are uncoupled. Any coupling can be obtained by combining the response signals of the n subsystems into l output signals through some arbitrary matrix $\mathbf{C}$. Such a system is by no way different from any mechanical system since matrix $\mathbf{C}$ can represent a possible assembly of rigid bars (see Section 5.2 for an example). Independent noise vector $\mathbf{v}$ is finally added to the output vector to simulate some measurement noises. All the noises are white Gaussian and are defined by their standard deviations $\boldsymbol{\sigma}(\mathbf{w})$ and $\boldsymbol{\sigma}(\mathbf{v})$.

The advantage of such a system is that each oscillator is associated with one mode whose normal mode shape is simply $\varphi_p = [0 \quad \ldots \quad 0 \quad 1 \quad 0 \quad \ldots \quad 0]^{\mathrm{T}}$ with a "1" at the pth place for the pth oscillator. N being the size of the signals available, the dissipated power, kinetic and potential energy mean values associated with the pth mode are then, respectively,

$$\underline{P}_d^{(p)} = \frac{1}{N}\sum_{k=0}^{N-1} f_P(k)\dot{u}_p(k+1), \tag{5.3}$$

$$\underline{W}_k^{(p)} = \frac{1}{2N}\sum_{k=0}^{N-1} m_p \dot{u}_p^2(k) \tag{5.4}$$

and

$$\underline{W}_p^{(p)} = \frac{1}{2N}\sum_{k=0}^{N-1} k_p u_p^2(k) = \frac{1}{2N}\sum_{k=0}^{N-1} m_p \Omega_p^2 u_p^2(k) \tag{5.5}$$

and can be calculated directly from the generation of the signal.

These values are supposedly unknown as well as the different variables in their expressions. Only the modal masses, which presently are the oscillator masses, and a set of output signals are known. From this data, the application of the identification procedure and calculation of the modal energies, from the formulas in Section 4.3, provide energy estimates $P_d^{(p)}$, $W_k^{(p)}$, $W_p^{(p)}$ which can be compared to their "direct values" $\underline{P}_d^{(p)}$, $\underline{W}_k^{(p)}$, $\underline{W}_p^{(p)}$.

5.2. The two-dof system

Let us consider the structure defined on Fig. 2. The structure is formed by a pair of independent mass-damper-spring subsystems joined by a zero-mass rigid bar. Clearly the two modes are defined by the independent oscillations of the two subsystems. All the characteristics are given in Table 1. The mode

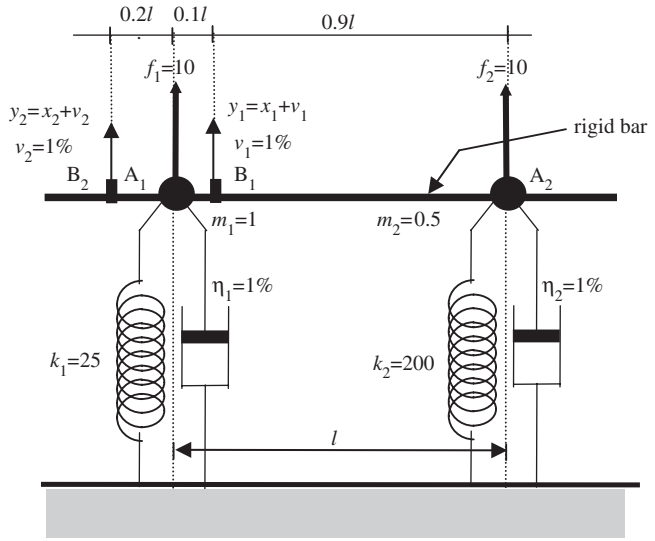

Fig. 2. A two-dof mechanical system.

frequencies and damping ratios are those of the oscillators, and the mode shapes have their nodes at point A_2 for the first mode and at A_1 for the second one.

To simulate a measurement noise, independent Gaussian white noises are added to the output signals, with a noise-to-signal ratio (NSR) of 1%. Let us assume that displacement sensors have been placed at points B_1 and B_2 near the second mode node (Fig. 2). As a consequence of this awkward sensor positioning the second mode (frequency $\Omega_2 = 20$) almost disappears from the transfer functions (Fig. 3). The table summarises the results obtained from the analysis of a series of 100 records of 2000 points, using the identification method. For all the modes, mean frequencies, damping ratios and energy values are presented. For comparison purposes, the mean values obtained from the "direct method", their mean ratios to the previous ones and the dispersions of these ratios are indicated. The results show that the modal characteristics of the system are accurately identified as well as the mechanical energy values, however with a greater dispersion of the dissipated power.

The results confirm that the kinetic and potential energy levels of the second oscillator are about 20% greater than those of the first oscillator, a result which could not be anticipated from the observation of the PSD (Fig. 3). The conclusion is worse when considering the dissipated power, which is four times greater for the second oscillator. Although being trivial this example clearly shows the danger of a pure signal processing-based classification.

As was highlighted in the above theory, modal masses must be provided as an input of the identification procedure. Of course in these simulations, exact masses were given. Table 1 presents the effect of providing wrong modal masses. A wrong classification is observed in this case. That result could have been anticipated since in this case energy levels are directly proportional to the given modal masses, because of the independence of the oscillators.

5.3. *The five-dof system*

Similarly, a five-dof system whose characteristics are given in Table 2 is tested. Mean results obtained from a series of 100 simulations of 2000 points are presented in the table. The value of the NSR is 1%. For all modes, the three energy values are obtained acurately, and no wrong classification occurs.

Once more, the dissipated power values are less coherent than the kinetic and potential energy ones. Moreover, a rapid degradation of this specific result is observed when increasing the NSRs. This result may be relevant regarding real-life applications, as will be discussed below.

Table 1
Two-dof system: energy estimates from identification and direct methods

2-dof system: statistics of 100 simulations (2000 points)			
Oscillator (mode): p	1	2	Noise to signal ratios (%)
Oscillator parameters			
Ω_p (rad/s)	5	20	
η_p (%)	1	2	
m_p	1	0.166	
σ (f_p)	10	10	
Observation matrix C (displacements)	0.9	0.1	1.0
	−1.2	0.2	1.0
Identified parameters			
Identified Ω_p (rad/s)	5.006	19.99	
Identified η_p (%)	1.72	2.17	
Kinetic energy			
Identified W_k	1.501	1.762	
$\underline{W_k}$ direct	1.503	1.817	
$r = W_k/\underline{W_k}$	0.997	0.966	
σ (r)	0.009	0.012	
Place (decreasing W_k)	2	1	
Potential energy			
Identified W_p	1.511	1.813	
$\underline{W_p}$ direct	1.498	1.804	
$r = W_p/\underline{W_p}$	1.009	1.006	
σ (r)	0.034	0.014	
Place (decreasing W_p)	2	1	
Dissipated power			
Identified P_d	0.973	4.43	
$\underline{P_d}$ direct	1.038	6.04	
$r = P_d/\underline{P_d}$	0.937	0.733	
σ (r)	0.346	0.151	
Place (decreasing P_d)	2	1	
Wrong modal masses given			
Wrong m_p	2	0.166	
Identified W_k	3.003	1.762	
Place (decreasing W_k)	1	2	

6. Experimental applications

6.1. The building frame model

Severe difficulties were encountered in defining an experimental validation of the method. A validation makes sense if some alternative method exists which provides accurate comparison values. In the present case such a method was not available and only an approximate approach, based on coarse hypotheses, could be set up. From this point of view, this method must be looked at as a different and independent approach, possibly less reliable than the identification method under test. For this reason this method will be called ''direct method'' and not ''validation method'' hereafter. However, the reader will not forget that this method is not an alternative to the ''identification method'', since it uses data which is not generally available during an in-operation analysis.

The structure tested is a steel-made reduced model of a two-floor building whose characteristics are given in Fig. 4. The structure being submitted to a white noise excitation onto the second floor level, the response is

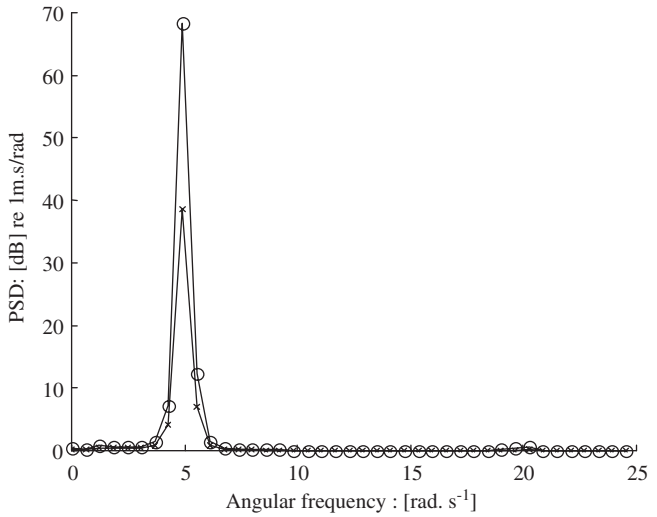

Fig. 3. Two-dof system: PSD of the two responses; × = channel 1, ○ = channel 2.

Table 2
Five-dof system: energy estimates from identification and direct methods

5-dof system: statistics of 100 simulations (2000 points)						
Oscillator (mode): p	1	2	3	4	5	Noise to signal ratios (%)
Oscillator parameters						
Ω_p (rad/s)	5	10	15	20	30	
η_p (%)	1	1	0.5	5	3	
m_p	0.33	1	0.2	0.25	5	
σ (f_p)	10	15	5	20	25	
Observation matrix C (displacements)						
	20	10	5	2	0	1.0
	15	30	15	7	4	1.0
	2	5	10	5	2	1.0
	3	6	12	2.5	12	1.0
	1	2	4	7	15	1.0
Identified parameters						
Identified Ω_p (rad/s)	5.00	10.0	15.0	20.0	30.0	
Identified η_p (%)	1.79	1.40	0.8	4.8	2.93	
Kinetic energy						
Identified W_k	4.13	2.18	1.34	1.97	1.66	
$\underline{W_k}$ direct	4.14	2.20	1.34	2.07	1.74	
Place (decreasing W_k)	1	2	5	3	4	
Potential energy						
Identified W_p	4.16	2.21	1.35	2.03	1.71	
$\underline{W_p}$ direct	4.13	2.19	1.33	2.04	1.70	
Place (decreasing W_p)	1	2	5	3	4	
Dissipated power						
Identified P_d	2.83	1.99	1.05	13.54	9.75	
$\underline{P_d}$ direct	2.91	2.28	1.22	16.28	12.57	
Place (decreasing P_d)	3	4	5	1	2	

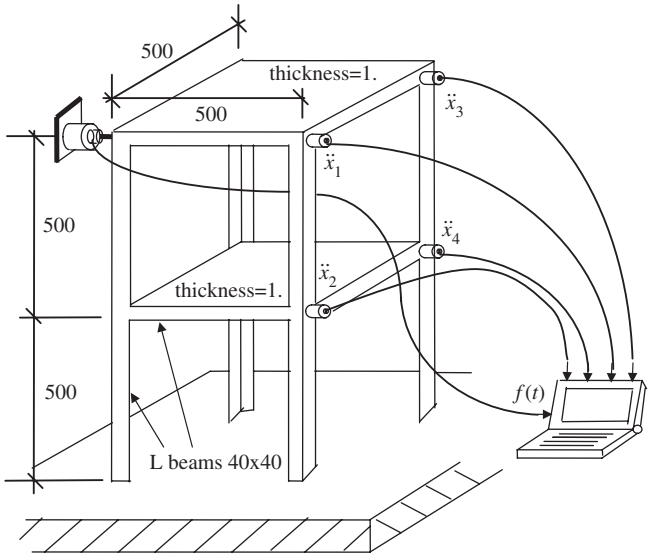

Fig. 4. The building frame reduced model (unit = mm).

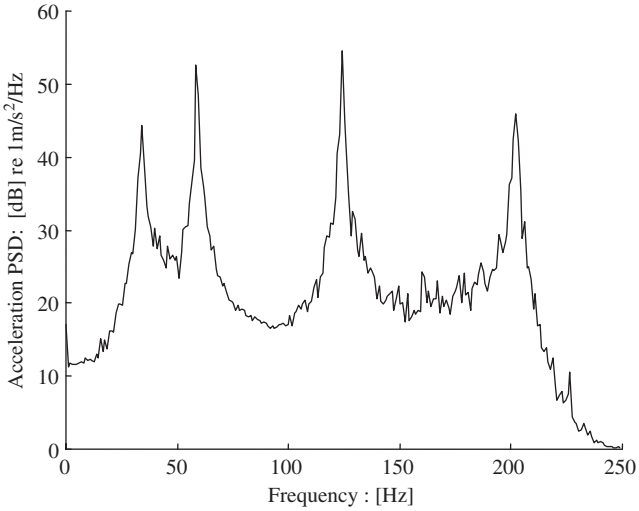

Fig. 5. Frame model: PSD of sensor 1 response.

captured from a set of four accelerometers installed at four structure node points 1–4, providing four 4096-point records. An accurate frequency domain analysis is also available using the PULSE system from B & K.

The PSD of channel 1 output, for example, averaged on eight 512-point records is shown on Fig. 5. The four channel outputs clearly reveal the existence of four main modes, i.e. in-phase floor translations, in-opposite floor translations, in-phase floor rotations and in-opposite floor rotations. Two translation modes in the orthogonal direction are not excited and does not appear.

The results from the identification procedure are summarised in Table 3.

As was explained above, modal masses must be given as an input of the identification program. These masses were obtained from finite element (FE) modelling, using the ABAQUS computer code. After fitting appropriate stiffness values for the column base rotations, this analysis confirmed the results obtained with both the PULSE system and with stochastic identification: modal frequencies and mode shapes, and provided

Table 3
Frame model: energy estimates from identification and direct methods

Frame model: identification and direct methods: (4096 points)				
Mode p	1	2	3	4
FE analysis				
F_p (Hz)	35.6	56.3	131.2	212.8
Identified parameters				
Identified F_p (Hz)	35.03	60.44	127.1	207.0
Identified η_p (%)	2.76	2.1	0.8	1.0
Kinetic energy				
Identified W_k ($\times 10^3$)	710	1534	866	85
Place (decreasing W_k)	3	1	2	4
$\underline{W_k}$ direct ($\times 10^3$)	886	1702	828	84
Place (decreasing $\underline{W_k}$)	2	1	3	4
Potential energy				
Identified W_p ($\times 10^3$)	703	1532	868	84
Place (decreasing W_p)	3	1	2	4
$\underline{W_k}$ direct ($\times 10^3$)	886	1702	828	84
Place (decreasing $\underline{W_p}$)	2	1	3	4
Dissipated power				
Identified P_d ($\times 10^3$)	4426	2869	899	586
Place (decreasing P_d)	1	2	3	4
$\underline{P_d}$ direct ($\times 10^3$)	5864	628	5	−109
Place (decreasing $\underline{P_d}$)	1	2	3	4

the modal masses which have been given in Table 3. Unfortunately the FE analysis did not allow a reliable simulation of the system behaviour under the real force program. Damping effects are not usually included into FE models, so that under continuous excitation the response infinitely increased, leading to an unrealistic structure behaviour. Consequently no energetic estimation could be obtained from this analysis.

6.2. The "direct method"

To obtain some energy estimates, a simplified method was developed. It is assumed that four modes only are excited by the external excitation. This is probably a coarse hypothesis but it is coherent with the fact that in the involved frequency range only these modes are identified. These modes thus set up a sufficient basis for the system displacements. Let $\{\mathbf{\Psi}_i; \quad i = 1-4\}$ be the set of four complete mode shapes resulting from the FE analysis (size $N_d = 150{,}000$ dof), $\mathbf{\Psi}$ the $(N_d \times 4)$ column matrix of these mode shapes. Let $\{\boldsymbol{\psi}_i; \quad i = 1-4\}$ be the set of four reduced mode shapes (size 4) extracted from modes $\mathbf{\Psi}_i$ at the four nodes 1–4, and $\boldsymbol{\psi}$ the matrix of these reduced mode shapes. From the previous hypothesis response $\mathbf{U}(t)$ of the structure is continuously a combination of modes $\mathbf{\Psi}_i$:

$$\mathbf{U}(t) = \mathbf{\Psi}\mathbf{q}(t) \tag{6.1}$$

$\mathbf{q}(t)$ being the modal coordinates vector. The same relation holds for the reduced coordinates

$$\mathbf{u}(t) = \boldsymbol{\psi}\mathbf{q}(t) \tag{6.2}$$

Therefore at any moment t the response $\mathbf{u}(t)$ can be projected onto the base $\boldsymbol{\psi}_i$:

$$\mathbf{q}(t) = \boldsymbol{\psi}^{-1}\mathbf{u}(t). \tag{6.3}$$

The same relation holds for accelerations:

$$\ddot{\mathbf{q}}(t) = \boldsymbol{\psi}^{-1}\ddot{\mathbf{u}}(t). \tag{6.4}$$

Another problem occurred due to the fact that sensors do not provide displacements but accelerations. Numerical integration is not possible since the highest frequencies are described by no more than three or four points per period. The difficulty was solved by having recourse to another approximate hypothesis. Essentially, a one-mode low-damped system has significant response only in the vicinity of its modal frequency Ω_i. It results that the accelerations can be derived from velocities approximately by multiplying them by Ω_i (and $\pi/2$ phase rotation), and from displacements by multiplying them by $-\Omega_i^2$. Therefore each one of the four modal acceleration sequences $\ddot{q}_i(t)$ can be divided by Ω_i to obtain (approximately) a modal velocity sequence $\dot{q}_i(t)$, and by $-\Omega_i^2$ to obtain a modal displacement sequence $q_i(t)$.

Finally, the modal masses resulting from the FE analysis are used to build approximate values of energetic levels:

- kinetic energy: the usual definition

$$W_k(t) = \tfrac{1}{2}\dot{\mathbf{U}}^{\mathrm{T}}(t)\mathbf{M}\dot{\mathbf{U}}(t) \tag{6.5}$$

in modal space results in:

$$W_k^{(p)}(t) = \tfrac{1}{2}m_p\dot{q}_p^2(t) \tag{6.6}$$

for each mode p. A mean value of the kinetic energy is thus obtained from:

$$\underline{W_k^{(p)}} = \frac{1}{2N}\sum_{k=1}^{N} m_p\dot{q}_p^2(k). \tag{6.7}$$

- potential energy: the usual definition

$$W_p(t) = \tfrac{1}{2}\mathbf{U}^{\mathrm{T}}(t)\mathbf{K}\mathbf{U}(t) \tag{6.8}$$

in modal space results in

$$W_p^{(p)}(t) = \tfrac{1}{2}k_p q_p^2(t) \tag{6.9}$$

for each mode i. Using Eq. (3.18) and remembering the mean from which the velocity and displacement sequences were obtained, it results that from this approach $W_k^{(p)}(t) = W_p^{(p)}(t)$,
thus

$$\underline{W_k^{(p)}} = \underline{W_p^{(p)}}. \tag{6.10}$$

- dissipated power: excitation being applied onto node 1, a modal velocity at this node can be obtained at any moment t from

$$\dot{u}_1^{(p)}(t) = \dot{q}_p(t)\psi_{p1}. \tag{6.11}$$

Dissipated power results from the usual definition

$$\underline{P_d^{(p)}} = \frac{1}{N}\sum_{k=1}^{N} f(k)\dot{u}_1^{(p)}(k). \tag{6.12}$$

6.3. Comparison of the results of the identification and direct methods

Table 3 shows the results obtained from both the identification method and the "direct method". Kinetic and potential energy values are in good agreement, even considering the rough approximations which were made in the

elaboration of the "direct method". This is not the case for the dissipated power, where the "direct method" provides incoherent values, with a negative value for the fourth mode. On the other hand the identification method results in all positive values, increasing in accordance with the damping ratios, as expected. The failure of the correlated results in the case of dissipated power, especially the results from the "direct method", can be explained as follows. A detailed inspection of computational intermediate results have shown that during one period, dissipated energy is only a small fraction of the mean potential energy. The coarse hypotheses which were made to elaborate the "direct method" generate an error noise whose power is of the same order as the dissipated one. This hides the possibility to extract a satisfactory mode classification to compare with the identification method. On the other hand, the identification approach apparently provides reliable results since some mode filtering is achieved through coherent mathematical formulation. As a matter of fact a validation of the dissipated power obtained from the identification method is still missing.

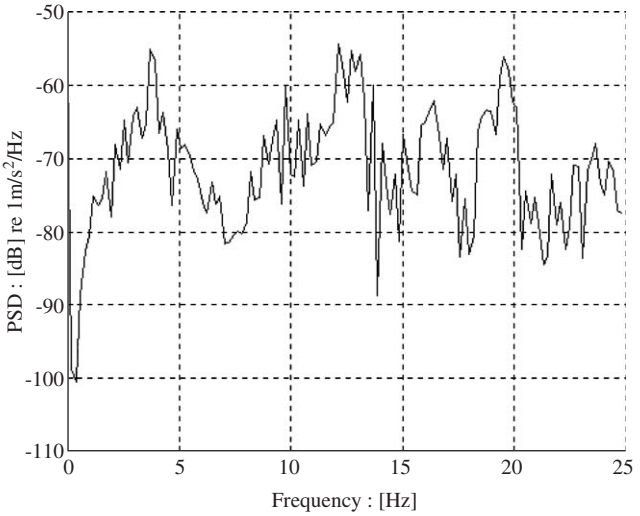

Fig. 6. Z24 Bridge: PSD of sensor 7 response.

Table 4
Z24 Bridge: energy estimates from identification method

Z24 Bridge: (4096 points)								
Mode	1	2	3	4	5	6	7	8
Assumed modal masses	1	1	1	1	1	1	1	1
Identified parameters								
Identified F_p (Hz)	3.74	4.89	9.64	10.9	12.1	13.3	17.2	19.8
Identified η_p (%)	0.81	1.16	0.86	0.33	0.29	0.76	0.5	0.54
Kinetic energy								
Identified W_k ($\times 10^{-4}$)	301	191	164	689	3330	1737	1950	3579
Place (decreasing W_k)	6	7	8	5	2	4	3	1
Potential energy								
Identified W_p ($\times 10^{-4}$)	298	197	99	480	2061	1417	809	1913
Place (decreasing W_p)	6	7	8	5	1	3	4	2
Dissipated power								
Identified P_d ($\times 10^{-8}$)	110	10	31	741	3762	676	6580	3453
Place (decreasing P_d)	6	8	7	4	2	5	1	3

6.4. Application to a real-life case

The method developed above was finally applied to a set of measurements stemming from the instrumentation of bridge Z24 in Switzerland, records which have been made public in order to be used as data sets for benchmarks. The records include nine signals which were truncated here up to 4096 points. A typical signal among them is illustrated in Fig. 6 by its PSD. The mass matrix of the structure being unavailable, the test cannot be considered a validation but is worthwhile appreciating the possible difficulties of a real-life case. The results are given in Table 4. Specific difficulties were encountered in applying the method in this case. In the subspace identification the space rank must be chosen a little higher than twice the expected number of modes. Moreover identified transition matrix $\hat{A}$ generally has a few real eigenvalues. The consequence is the likely presence of spurious modes. These modes could be simply ignored but the dimensional coherence of matrix calculation requires that they be kept for further calculation. The consequences of that are unknown.

In the absence of reliable information, the mass matrix was arbitrarily chosen as a unit matrix. The results show that the highest modes—above 10 Hz—dissipate more energy than others, independently of the peak height. Mean potential and kinetic energies lead to the same conclusion. Besides the fact that the group of highest frequencies probably carries much more energy than the other ones, in the absence of the modal mass information no more result can be drawn from this example.

7. Discussion

A method to evaluate the energetic mode contributions was developed under two restrictions which are worth discussing:

(1) The number of sensors must be at least equal to the number of modes. Strictly resulting from the present analysis, the conclusion is that if this condition is not fulfilled, the missing information can be arbitrarily completed, but a reliable energy estimation cannot be guaranteed. In the analysis of a complex body, such as a plane or a car body, this condition is linked to the issue of sensor locations. The problem is complex in itself and deserves proper investigation.
In the case where the above condition cannot be fulfilled, an original solution can be expected from a strategy of multiple analysis, using each time a different stationary excitation sequence. In practice this would mean that the structure is analysed in various operating conditions. The parameter over-determination resulting from this procedure could compensate for the lack of measurement sensors. At the present time the method has not been investigated but could be a basis for further developments.

(2) The mass matrix of the structure, reduced to the dof defined by the sensors, or at least the diagonal matrix of modal masses must be available. This requirement may appear as a serious restriction for practical applications. Possibly it may forbid any energy estimation from a pure in-operation analysis. However in general, modal mass estimates—or at least rough estimates if only mode classification is expected—may be available from a previous analysis or from the simplified modelling of the structure. For instance in the field of aeronautics, it can be assumed that the presence of environmental air does not essentially modify the modal masses, which are mainly attached to the structure masses. On the other hand, in the case of an offshore platform, the structure modal masses are modified because of the presence of added masses due to the water. However, in this case, from the knowledge of mode shapes, the elementary theory of fluid–structure coupling allows an approximate estimation of these masses.

Assuming that the above two conditions are fulfilled, it was shown that the identification method allows a mechanics-based classification of modes in terms of energy, in the framework of an in-operation analysis. In simulation the reconstruction of mode energetic contributions—mean kinetic energy, mean potential energy and dissipated power—was reached and verified accurately for a two- and a five-dof system.

To provide some appreciation of the method performance in practical applications, a reduced model of a two-floor building, submitted to white noise excitation, was studied. Modal masses were extracted from a FE analysis and given as an input to the identification program. A "direct approach" using coarse hypotheses was

proposed to compare with the identification method. The results of both methods showed satisfactory agreement except for the dissipated power, which was obtained with non-coherent values from the "direct approach". Clearly this was a consequence of the severe approximations made in the elaboration of the "direct method". On the other hand, from the identification method, dissipated power was obtained with apparently coherent values.

As the "direct method" above mentioned is only approximate, a more accurate validation would require that the experiment be made with perfect control of the excitation system. Assuming that these excitations are driven so that a pure combination of modes is involved, a direct evaluation of the injected energy for each mode would be possible. Such an excitation can be obtained by means of a closed-loop control with a feedback from the structure response to the applied forces, as it is practiced in aircraft testing. This solution can be a route for further research in this field.

Finally, the method was applied to a real-life test case involving nine transducers (nine output signals) stemming from an experiment on a bridge. As modal masses were not available, the purpose was restricted to appreciate the specific difficulties of such a complex case. The presence of real modes resulting from the identification process was remarked as a difficulty whose consequences on the results are unknown.

8. Conclusion

The purpose of this study was to provide a reliable method to classify the vibration modes of a structure from an energetic point of view, in the framework of an in-operation modal analysis. Such an evaluation would be useful to point out which modes are really activated during the structure real life, thus deserve the designer's special attention for.

The conclusion of the study can be summarised as follows: reliable values of mean modal kinetic energy, mean modal potential energy, can be reconstructed from the identification procedure provided that a sufficient number of sensors are used and that the set of modal masses is available. Then these values can be confidently used as energetic criteria for mode classification. Modal dissipated power can be reconstructed in the same way, but the reliability of the result could not be firmly determined except from pure simulation examples.

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