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# Harmonic differential quadrature-finite differences coupled approaches for geometrically nonlinear static and dynamic analysis of rectangular plates on elastic foundation

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**Abstract**

The geometrically nonlinear static and dynamic analysis of thin rectangular plates resting on elastic foundation has been studied. Winkler–Pasternak foundation model is considered. Dynamic analogues Von Karman equations are used. The governing nonlinear partial differential equations of the plate are discretized in space and time domains using the harmonic differential quadrature (HDQ) and finite differences (FD) methods, respectively. The analysis provides for both clamped and simply supported plates with immovable inplane boundary conditions at the edges. Various types of dynamic loading, namely a step function, a sinusoidal pulse and an *N*-wave, are investigated and the results are presented graphically. The accuracy of the proposed HDQ–FD coupled methodology is demonstrated by the numerical examples.


## 1. Introduction

The practical importance of vibration analysis of plates with or without on elastic foundation has been increased in structural, aerospace, civil and mechanical engineering applications. Nonlinear static and dynamic analyses of plates of various shapes have been carried out by various researchers [1–4]. More detailed information can be found in a recent review paper by Sathyamorth [5]. Nath et al. [6] presented the finite differences methods for spatial discretization and Houbolt's time marching discretization to study the dynamic analysis of rectangular plates resting on elastic foundation. Dumir [7] and Dumir and Bhaskar [8] have investigated nonlinear static and dynamic analysis of rectangular plates on elastic foundation employing the orthogonal point collocation method. A few studies concerning to static and dynamic analysis of rectangular plates resting on elastic foundation have been carried out, namely by Cheung and Zienkiewicz [9] and Liew et al. [10] and Liu [11].

In structural mechanics, differential quadrature (DQ) methods are becoming popular as many important researchers demonstrated their successful applications of the method to the static, vibration and buckling analysis of various type beams, plates and shells. These applications include the work of Bert and his

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**Nomenclature**

| | |
|---|---|
| $a$, $b$ | dimensions of plate in $x$- and $y$-directions |
| $A_{ij}$ | weighting coefficients for first-order derivative |
| $B_{ij}$ | weighting coefficients for second-order derivative |
| $C$ | non-dimensional damping coefficients, Eq. (20) |
| $D$ | flexural rigidity of plate |
| $D_{ij}$ | weighting coefficients for fourth-order derivative |
| $E$ | modulus of elasticity |
| FD | finite differences |
| $G$ | non-dimensional Pasternak parameter, Eq. (20) |
| $G_f$ | shear parameter of Pasternak foundation |
| $h_k(x)$ | Lagrange interpolated functions |
| $h$, $h_f$ | thicknesses of the plate and foundation |
| HDQ | harmonic differential quadrature |
| $k_f$ | stiffness parameter of Winkler foundation |
| $K$ | non-dimensional Winkler parameter, Eq. (20) |
| $m_r$ | dimensionless mass ratio, Eq. (20) |
| $N$ | number of discrete points |
| $N_x$, $N_y$ | number of discrete points in the $x$- and $y$-directions |
| $P$ | non-dimensional load parameter |
| $\Delta t$ | time steps |
| $T$ | time |
| $U$, $V$, $W$ | displacement components |
| $W_c$ | central deflection |
| $x_i$ | discrete points in the variable domain |
| $\nu$ | Poisson's ratio |
| $\rho$, $\rho_f$ | mass density of the foundation and plate material |
| $\tau$ | non-dimensional time parameter, Eq. (20) |

co-workers [12–15], Liew et al. [16–18], Shu et al. [19,20], Striz et al. [21,22], Civalek et al. [23–27], and Wu and Liu [28]. Details on the development of the DQ methods and on its applications to the structural and fluid mechanics problems may be found in a well-known paper by Bert and Malik [14].

The practical importance of dynamic analysis of plates and shells on elastic foundation has been increased in structural, aerospace, biomechanics, civil and mechanical engineering applications. There are many situations such as seismic tests, nuclear explosions, earthquakes, etc. in which these structures are subjected to transient loads and large amplitudes of motion may occur. The objective of this study is to present an approximate numerical solution of the Von Karman–Donnel type governing equations for the geometrically nonlinear analysis of rectangular plates resting on Winkler–Pasternek elastic foundations under the various types of dynamic loading. To the author's knowledge, it is the first time the DQ method has been successfully applied to thin, isotropic rectangular plates resting on an elastic foundation problem for the geometrically nonlinear dynamic analysis.

## 2. Differential quadrature (DQ) method

In the DQ method, a partial derivative of a function with respect to a space variable at a discrete point is approximated as a weighted linear sum of the function values at all discrete points in the region of that variable. For simplicity, we consider a one-dimensional function $u(x)$ in the $[-1, 1]$ domain, and $N$ discrete points. Then the first derivatives at point $i$, at $x = x_i$ is given by

$$u_x(x_i) = \left.\frac{\partial u}{\partial x}\right|_{x=x_i} = \sum_{j=1}^{N} A_{ij} u(x_j), \quad i = 1, 2, \ldots, N, \tag{1}$$

where $x_j$ are the discrete points in the variable domain, $u(x_j)$ are the function values at these points and $A_{ij}$ are the weighting coefficients for the first-order derivative attached to these function values. Bellman et al. [29] suggested two methods to determine the weighting coefficients. The first one is to let Eq. (1) be exact for the test functions

$$u_k(x) = x^{k-1}, \quad k = 1, 2, \ldots, N, \tag{2}$$

which leads to a set of linear algebraic equations

$$(k-1)x_i^{k-2} = \sum_{j=1}^{N} A_{ij}\, x_j^{k-1}, \quad \text{for} \quad i = 1,2,\ldots,N \quad \text{and} \quad k = 1,2,\ldots,N, \tag{3}$$

which represents $N$ sets of $N$ linear algebraic equations. Another way to determine the weighting coefficients is to employ harmonic functions, named the harmonic differential quadrature (HDQ). HDQ has been proposed by Striz et al. [22]. Unlike the DQ that uses the polynomial functions, such as power functions, Lagrange interpolated, and Legendre polynomials as the test functions, HDQ uses harmonic or trigonometric functions as the test functions. Shu and Xue proposed an explicit means of obtaining the weighting coefficients for the HDQ [20]. When the $f(x)$ is approximated by a Fourier series expansion in the form

$$f(x) = c_0 + \sum_{k=1}^{N/2}\left(c_k \cos\frac{k\pi x}{L} + d_k \sin\frac{k\pi x}{L}\right) \tag{4}$$

and the Lagrange interpolated trigonometric polynomials are taken as

$$h_k(x) = \frac{\sin\frac{(x-x_0)\pi}{2}\cdots\sin\frac{(x-x_{k-1})\pi}{2}\sin\frac{(x-x_{k+1})\pi}{2}\cdots\sin\frac{(x-x_N)\pi}{2}}{\sin\frac{(x_k-x_0)\pi}{2}\cdots\sin\frac{(x_k-x_{k-1})\pi}{2}\sin\frac{(x_k-x_{k+1})\pi}{2}\cdots\sin\frac{(x_k-x_N)\pi}{2}} \tag{5}$$

for $k = 0, 1, 2, \ldots, N$. According to the HDQ, the weighting coefficients of the first-order derivatives $A_{ij}$ for $i \neq j$ can be obtained by using the following formula:

$$A_{ij} = \frac{(\pi/2)P(x_i)}{P(x_j)\ \sin[(x_i - x_j)/2]\pi}, \quad i,j = 1,2,3,\ldots,N, \tag{6}$$

where

$$P(x_i) = \prod_{j=1,\, j\neq i}^{N} \sin\left(\frac{x_i - x_j}{2}\pi\right), \quad \text{for}\ j = 1,2,3,\ldots,N. \tag{7}$$

The weighting coefficients of the second-order derivatives $B_{ij}$ for $i \neq j$ can be obtained using the following formula:

$$B_{ij} = A_{ij}\left[2A_{ii}^{(1)} - \pi\ \cot\left(\frac{x_i - x_j}{2}\right)\pi\right], \quad i,j = 1,2,3,\ldots,N. \tag{8}$$

The weighting coefficients of the first- and second-order derivatives $A_{ij}^{(p)}$ for $i = j$ are given as

$$A_{ii}^{(p)} = -\sum_{j=1,\, j\neq i}^{N} A_{ij}^{(p)}, \quad p = 1 \text{ or } 2 \quad \text{and for} \quad i = 1,2,\ldots,N. \tag{9}$$

The weighting coefficient of the third- and fourth-order derivatives can be computed easily from $A_{ij}$ and $B_{ij}$ by

$$C_{ij} = \sum_{k=1}^{N} A_{ik}B_{kj}, \quad D_{ij} = \sum_{k=1}^{N} B_{ik}B_{kj}. \tag{10,11}$$

Two different types of sampling grids are taken into consideration in this study. A natural, and often convenient, choice for sampling points is that of equally spaced grid (ES-G) points. These points are given by

$$\text{Type-I}: \ x_i = \frac{i-1}{N_x - 1} \quad \text{and} \quad y_i = \frac{i-1}{N_y - 1} \tag{12,13}$$

in the related directions. Sometimes, the DQ solutions deliver more accurate results with unequally spaced sampling points. Another choice that is found to be even better than the Chebyshev and Legendre polynomials

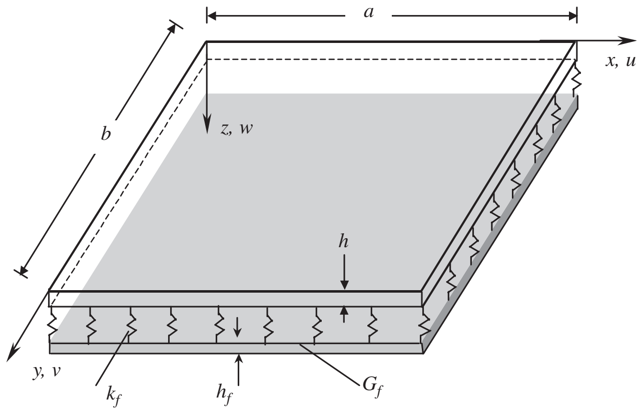


Fig. 1. Geometry and dimensions of rectangular plate on elastic foundation.

is the set of points proposed by Shu and Richards [19]. These points are given as

$$\text{Type-II}: \; x_i = \frac{1}{2}\left[1 - \cos\left(\frac{2i-1}{N_x - 1}\right)\pi\right] \quad \text{and} \quad y_i = \frac{1}{2}\left[1 - \cos\left(\frac{2i-1}{N_y - 1}\right)\pi\right] \tag{14,15}$$

in the $x$- and $y$- directions, respectively. These type grid points are known the Chebyshev–Gauss–Lobatto or non-equally spaced grid (NES-G) points.

## 3. Governing equations

We consider thin rectangular plates resting on Winkler–Pasternak elastic foundation of length $a$ in $x$- direction, width $b$ in the $y$-direction and thickness $h$ in the $z$-direction. The geometry of a typical rectangular plate resting on Winkler–Pasternak elastic foundation is shown in Fig. 1. The foundation is modelled in terms of Winkler parameter $k_f$ and shear parameter $G_f$ of the Pasternak model. More detailed information about the elastic and inelastic foundation models and the analysis of structures on elastic foundation can be found in the relevant literature [31–37]. Including the normal components of inertial forces and neglecting the damping of the foundation, the distributed reaction from the elastic foundation on the shell at any instant of time $t$ is given by

$$k_f w - G_f(w_{xx} + \beta^2 w_{yy}) + \rho_f h_f w_{tt}. \tag{16}$$

In this study, the plate–foundation interaction is considered and the in-plane and rotary inertia is neglected. The governing differential equations of motion in terms of non-dimensional displacements components $U$, $V$, and $W$ can be expressed as [6,27]

$$\begin{aligned} &U_{,XX} + \frac{\beta^2}{2}(1-\nu)U_{,YY} + \frac{\beta}{2}(1+\nu)V_{,XY} + W_{,X}\left[W_{,XX} + \frac{\beta^2}{2}(1-\nu)W_{,YY}\right] \\ &\quad + \frac{\beta^2}{2}(1+\nu)W_{,YY}W_{,XY} = 0, \end{aligned} \tag{17}$$

$$\begin{aligned} &\beta^2 V_{,YY} + \frac{1}{2}(1-\nu)V_{,XX} + \beta\frac{1}{2}(1+\nu)U_{,XY} + \beta W_{,Y}\left[\beta^2 W_{,YY} + \frac{1}{2}(1-\nu)W_{,XX}\right] \\ &\quad + \beta\frac{1}{2}(1+\nu)W_{,X}W_{,XY} = 0, \end{aligned} \tag{18}$$

$$W_{,XXXX} + 2\beta^2 W_{,XXYY} + \beta^4 W_{,YYYY} - 12(W_{,XX})\left[U_{,X} + \beta v V_{,Y} + \frac{1}{2}(W_{,X})^2 + \frac{1}{2}\beta^2 v(W_{,Y})^2\right]$$
$$- 12(\beta^2 W_{,YY})\left[\beta V_{,Y} + vU_{,X} + \frac{1}{2}v(W_{,X})^2 + \frac{1}{2}\beta^2(W_{,Y})^2\right]$$
$$- 12(1-v)\beta\left[\beta U_{,Y} + V_{,X} + \beta W_{,X} W_{,Y}\right] W_{,XY} - 12(1-v^2)P - KW - G(W_{,XX} + \beta^2 W_{,YY})$$
$$+ W_{,\tau\tau} + m_r W_{,\tau\tau} + CW_{,\tau} = 0. \quad (19)$$

The non-dimensional quantities in the above equations are defined as

$$W = w/h, \quad X = x/a, \quad Y = y/b, \quad \beta = a/b, \quad U = ua/h^2, \quad V = va/h^2, \quad G = G_f a^2/D,$$
$$P = qa^4/Eh^4, \quad \tau = t\sqrt{D/\rho a^4 h}, \quad D = Eh^3/12(1-\upsilon^2), \quad m_r = h_f \rho_f / h\rho, \quad K = k_f a^4/D$$
$$C = c\sqrt{\gamma h a^4/D}, \quad (20)$$

where $u$, $v$ and $w$ are displacement componens in the $x$-, $y$-, and $z$-direction, respectively, $h$ and $h_f$ are the thickness of the plate and foundation, $E$ is Young's modulus, $v$ is Poisson's ratio, $\rho_f$ and $\rho$ are the mass density of the foundation and the material, $K$ is the stiffness parameters of Winkler foundation, $G$ is the shear modulus of Pasternak foundation, $D$ is the flexural rigidity, $a$ and $b$ are the sides of plate along $x$- and $y$-directions, $t$ is the time. Using the finite difference (central difference approach) method for the time-wise integration, the velocity and acceleration at time $i$, can be expressed as

$$\dot{u}_i = \frac{1}{2\Delta\tau}[u_{i+1} - u_{i-1}] \quad \text{and} \quad \ddot{u}_i = \frac{1}{(\Delta\tau)^2}[u_{i+1} - 2u_i + u_{i-1}]. \quad (21,22)$$

### 3.1. Boundary and initial conditions

In the present study the following two types of boundary conditions are considered. For all simply supported four edges and immovably constrained against in-plane translation (SSSS):

$$U = V = W = 0 \quad \text{and} \quad \left(\beta^2 \frac{\partial^2 W}{\partial Y^2} + v\frac{\partial^2 W}{\partial X^2}\right) = 0 \quad \text{at} \quad Y = 0 \quad \text{and} \quad Y = 1, \quad (23)$$

$$U = V = W = 0 \quad \text{and} \quad \left(\frac{\partial^2 W}{\partial X^2} + v\,\beta^2 \frac{\partial^2 W}{\partial Y^2}\right) = 0 \quad \text{at} \quad X = 0 \quad \text{and} \quad X = 1. \quad (24)$$

For all clamped four edges and immovably constrained against in-plane translation (CCCC):

$$U = V = W = 0 \quad \text{and} \quad \left(\frac{\partial W}{\partial Y}\right) = 0 \quad \text{at} \quad Y = 0 \quad \text{and} \quad Y = 1, \quad (25)$$

$$U = V = W = 0 \quad \text{and} \quad \left(\frac{\partial W}{\partial X}\right) = 0 \quad \text{at} \quad X = 0 \quad \text{and} \quad X = 1. \quad (26)$$

Furthermore, the zero initial conditions are assumed. These are given as

$$W = 0 \quad \text{and} \quad \left(\frac{\partial W}{\partial \tau}\right) = 0 \quad \text{at} \quad \tau = 0. \quad (27,28)$$

In this stage, the HDQ and FD methods are applied to discretize the derivatives for spatial and time domain in the governing equations, boundary and initial conditions. After spatial and time discretization, DQ form of the governing equations, boundary and initial conditions are given as [27]

$$\sum_{k=1}^{N_x} B_{ik} U_{kj} + \beta^2 d_1 \sum_{k=1}^{N_y} B_{jk} U_{ik} + \beta d_2 \sum_{m=1}^{N_y} A_{jm} \sum_{k=1}^{N_x} A_{ik} V_{km} + \sum_{k=1}^{N_x} A_{ik} W_{kj} \left[ \sum_{k=1}^{N_x} B_{ik} W_{kj} + \beta^2 d_1 \sum_{k=1}^{N_y} B_{jk} W_{ik} \right] + \beta^2 d_2 \sum_{k=1}^{N_y} A_{jk} W_{ik} \sum_{m=1}^{N_y} A_{jm} \sum_{k=1}^{N_x} A_{ik} W_{km} = 0, \tag{29}$$

$$\beta^2 \sum_{k=1}^{N_x} B_{jk} V_{ik} + d_1 \sum_{k=1}^{N_x} B_{ik} V_{kj} + \beta d_2 \sum_{m=1}^{N_y} A_{jm} \sum_{k=1}^{N_x} A_{ik} U_{km} + \beta \sum_{k=1}^{N_y} A_{jk} U_{ik} \left[ d_1 \sum_{k=1}^{N_x} B_{ik} W_{kj} + \beta^2 \sum_{k=1}^{N_y} B_{jk} W_{ik} \right] + \beta d_2 \sum_{k=1}^{N_y} A_{ik} W_{kj} \sum_{m=1}^{N_y} A_{jm} \sum_{k=1}^{N_x} A_{ik} W_{km} = 0, \tag{30}$$

$$\begin{aligned} &\sum_{k=1}^{N_x} D_{ik} W_{kj} + 2\beta^2 \sum_{m=1}^{N_y} B_{jm} \sum_{k=1}^{N_x} B_{ik} W_{km} + \beta^4 \sum_{k=1}^{N_y} D_{jk} W_{ik} \\ &- 12 \left( \sum_{k=1}^{N_x} B_{ik} W_{kj} \right) \left\{ \sum_{k=1}^{N_x} A_{ik} U_{kj} + \beta v \sum_{k=1}^{N_y} A_{jk} U_{ik} + \frac{1}{2} \left[ \left( \sum_{k=1}^{N_x} A_{ik}\ W_{kj} \right)^2 + \beta^2 v \left( \sum_{k=1}^{N_y} A_{jk}\ U_{ik} \right)^2 \right] \right\} \\ &- 12 \left( \beta^2 \sum_{k=1}^{N_y} B_{jk}\ W_{ik} \right) \left\{ \beta \sum_{k=1}^{N_y} A_{jk} V_{ik} + v \sum_{k=1}^{N_x} A_{ik} W_{kj} + \frac{1}{2} \left[ v \left( \sum_{k=1}^{N_x} A_{ik}\ W_{kj} \right)^2 + \beta^2 \left( \sum_{k=1}^{N_y} A_{jk}\ U_{ik} \right)^2 \right] \right\} \\ &- 12(1-v)\beta \left[ \beta \sum_{k=1}^{N_y} A_{jk} U_{ik} + \sum_{k=1}^{N_x} A_{ik} V_{kj} + \beta \sum_{k=1}^{N_x} A_{ik} W_{kj} \sum_{k=1}^{N_y} A_{jk} W_{ik} \right] \sum_{m=1}^{N_y} Ajm \sum_{k=1}^{N_x} A_{ik} W_{km} \\ &- 12(1-v^2)P - KW_{ik} - G \left( \sum_{k=1}^{N_x} B_{ik}\ W_{kj} + \beta^2 \sum_{k=1}^{N_y} B_{jk}\ W_{ik} \right) + \frac{1}{(\Delta\tau)^2} (W_{i+1} - 2W_i + W_{i-1}) \\ &+ m_r \left[ \frac{1}{(\Delta\tau)^2} (W_{i+1} - 2W_i + W_{i-1}) \right] + C \left[ \frac{1}{2\Delta\tau} (W_{i+1} - W_{i-1}) \right]. \end{aligned} \tag{31}$$

For SSSS boundary conditions:

$$U_{i1} = V_{i1} = W_{i1} = 0 \quad \text{and} \quad U_{iN} = V_{iN} = W_{iN} = 0, \tag{32}$$

$$U_{1j} = V_{1j} = W_{1j} = 0 \quad \text{and} \quad U_{Nj} = V_{Nj} = W_{Nj} = 0, \tag{33}$$

$$\sum_{k=1}^{N_y} B_{ik} W_{k1} + v\beta^2 \sum_{k=1}^{N_x} B_{jk} W_{j1} = \sum_{k=1}^{N_y} B_{ik} W_{kN} + v\beta^2 \sum_{k=1}^{N_x} B_{jk} W_{jN} = 0, \tag{34}$$

$$\beta^2 \sum_{k=1}^{N_y} B_{jk} W_{1k} + v \sum_{k=1}^{N_x} B_{ik} W_{1j} = \beta^2 \sum_{k=1}^{N_y} B_{jk} W_{iN} + v \sum_{k=1}^{N_x} B_{ik} W_{kN} = 0. \tag{35}$$

For CCCC boundary conditions:

$$U_{i1} = V_{i1} = W_{i1} = 0 \quad \text{and} \quad U_{iN} = V_{iN} = W_{iN} = 0, \tag{36}$$

$$U_{1j} = V_{1j} = W_{1j} = 0 \quad \text{and} \quad U_{Nj} = V_{Nj} = W_{Nj} = 0. \tag{37}$$

$$\sum_{k=1}^{N_y} A_{ik}\, W_{k1} = \sum_{k=1}^{N_y} A_{ik}\, W_{kN} = 0, \tag{38}$$

$$\sum_{k=1}^{N_y} A_{jk}\, W_{1k} = \sum_{k=1}^{N_y} A_{jk}\, W_{iN} = 0. \tag{39}$$

For initial conditions:

$$W_1 = 0 \quad \text{and} \quad \sum_{t=1}^{\tau_N} A_{jk} W_{1t} = 0, \tag{40}$$

where $A_{ij}$, $B_{ij}$ and $D_{ij}$ are the weighting coefficients for the first, second and fourth-order derivatives which can be determined as discussed in section two, $d_1 = (1 - v)/2$ and $d_2 = (1 + v)/2$. The set of nonlinear algebraic Eqs. (29)–(40) can be solved for $\{U\}$, $\{V\}$ and $\{W\}$ using nonlinear algorithms such as Newton–Raphson method [4,30].

## 4. Numerical applications

The title problem is analysed and some of HDQ–FD results are compared with results in the open literature [6–8] to show the applicability and efficiency of HDQ–FD coupled methodology. A uniform step load of infinite duration, sinusoidal loading of finite duration ($\tau = 0.16$), and N-shaped pulse load of finite ($\tau = 0.2$) duration (Fig. 2) have been considered.

First of all, to check whether the purposed formulation and programming are correct, a clamped immovable plate without an elastic foundation is analysed. The load–central displacement curve of clamped immovable rectangular ($b/a = 0.5$) and square plate ($b/a = 1$) under uniform distributed load is compared in Fig. 3 with the results of Dumir and Bhaskar [8]. The static response of clamped and simply supported immovable square plates on Winkler–Pasternak foundations is depicted in Figs. 4a and b for various values of the foundation parameters, $K$ and $G$. The obtained results agree excellently with those of Dumir and Bhaskar [8] solution.

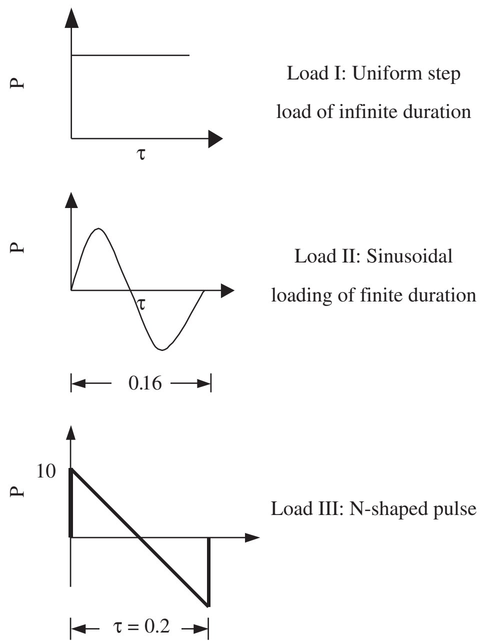


Fig. 2. Dynamic loads considered in numerical applications.

Fig. 5 shows the time–deflection curves of the clamped plate under the uniform step load of infinite duration ($P = 29.14$) for different damped coefficients, $C$. The results given by Nath et al. [6] are also plotted in this figure. The numerical solution of the HDQ method using non equally sampling grid (NES-G) points is equivalent to the Nath's results. The damping coefficient $C$ has been found to have significant influence on the dynamic response of the rectangular plates. From this curve given in Fig. 5, it may be concluded that decreasing the damping coefficient, $C$ will always result in increased deflection.

The effect of $K$ on the response of simply supported and clamped immovable rectangular plates resting on elastic foundation under the step load $P = 100$ is shown in Figs. 6a and b together with the results of Nath et al. [6]. The present results are in very good agreement with those of Nath et al. [6] for step load. For clamped support condition, the effect of $K$ on the response of rectangular plate under the $N$-wave and sinusoidal

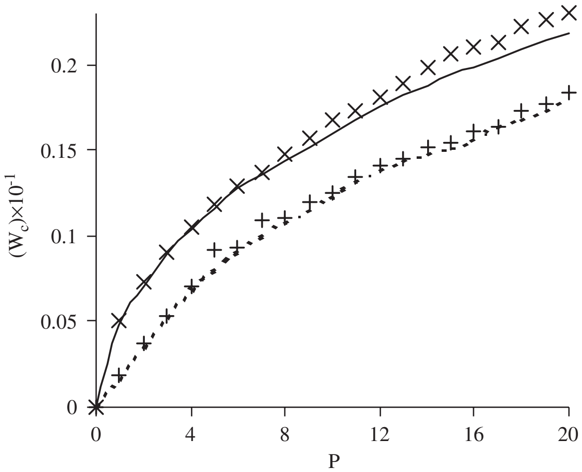


Fig. 3. Load versus central deflection response of clamped plate ($\nu = 0.3$) (—— Ref. 8 ($k = 0.5$); ······ Ref. 8 ($k = 1$); × HDQ ($k = 0.5$); + HDQ ($k = 1$)).

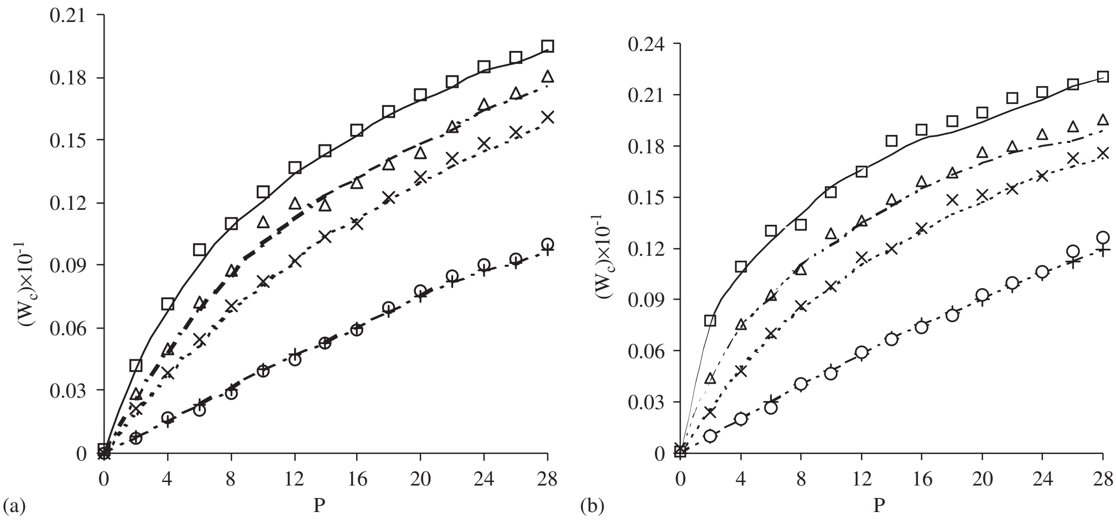


Fig. 4. Central deflection versus uniform load with different foundation parameters: (a) CCCC plate; (b) SSSS plate (–·–·– Ref. 8 ($K = 0.50; G = 0$); - - - - - Ref. 8 ($K = 100; G = 0$); – – + – – Ref. 8 ($K = 50; G = 50$); ——— Ref. 8 ($K = 0; G = 0$); □ HDQ ($K = 0; G = 0$); × HDQ ($K = 100; G = 0$); ○ HDQ ($K = 50; G = 50$); △ HDQ ($K = 50; G = 0$)).

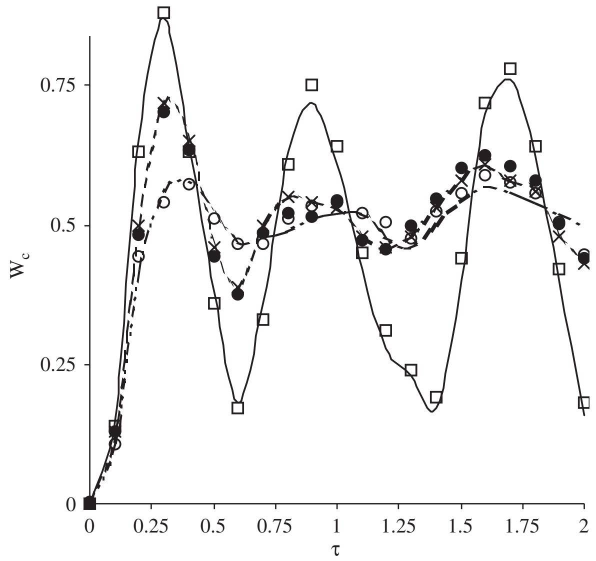


Fig. 5. Time–displacement curve of clamped plate for various damping coefficients ($\nu = 0.3$) (—— Ref. 6 ($C = 1.25$); -·-·- Ref. 6 ($C = 10$); -- × —- Ref. 6 ($C = 5$); □ HDQ–FD ($C = 1.25$); ○ HDQ–FD ($C = 10$); ● HDQ–FD ($C = 5$)).

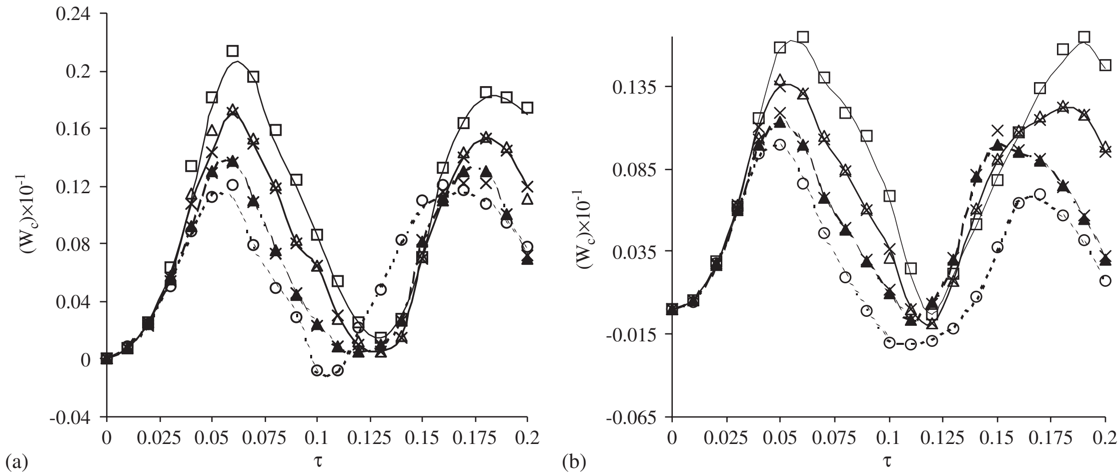


Fig. 6. Central deflection versus time for step load with different stiffness parameters of Winkler foundation ($P = 100; b/a = 1$): (a) CCCC plate; (b) SSSS plate (□ HDQ–FD ($K = 0$); △ HDQ–FD ($K = 800$); × HDQ–FD ($K = 1600$); ○ HDQ–FD ($K = 3200$); —— Ref. 6 ($K = 0$); —×— Ref. 6 ($K = 800$); -- △ -- Ref. 6 ($K = 1600$); ----- Ref. 6 ($K = 3200$)).

loading ($P = 200$) are shown in Figs. 7 and 8. These figures show that the deflections will decrease with increase in foundation parameter for rectangular plates, and also the response to a step load is higher than the response to a sinusoidal load. Besides that, the response to a sinusoidal load is higher than the response to a $N$-wave load. Figs. 9a and b show the time–deflection curves of the simply supported and clamped plates for various values of shear parameter ($G$) of Pasternak foundation. In these figures a uniform step load of infinite duration ($P = 100$) has been considered. Figs. 10a and b show the variation of the deflection of the CCCC and SSSS immovable plates with the shear parameters for different values $G$ (0, 50, 100, 150, 200, 250). For these figures, a sinusoidal loading ($P = 200$) of finite duration has been considered. The present results are in very good agreement with those of Nath et al. [6]. The shear parameter $G$ of the foundation has been found to have a significant influence on the response of the plates. It is interesting to note that increase in shear modulus

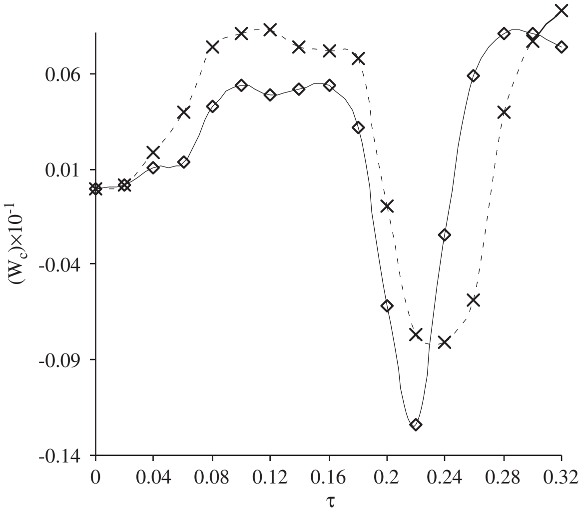


Fig. 7. Central deflection versus time for $N$-wave load with varying stiffness parameters of Winkler foundation (CCCC plate; $P = 200$; $b/a = 1$; $G = 0$; $v = 0.3$) (– – × – – HDQ–FD ($K = 400$); — ◇ — HDQ–FD ($K = 2400$)).

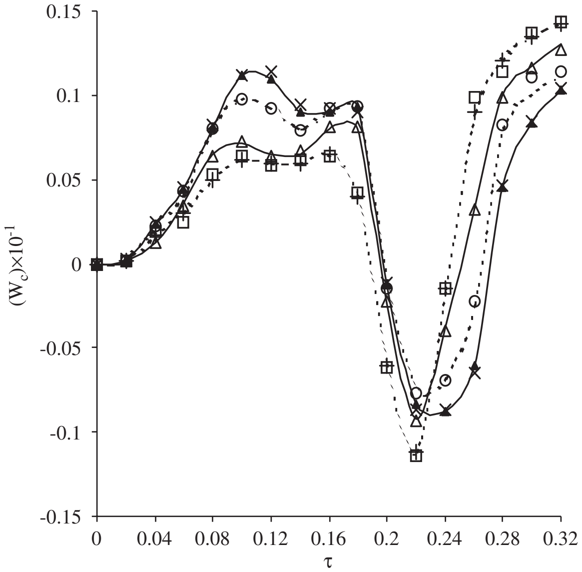


Fig. 8. Central deflection versus time for sinusoidal load with varying stiffness parameters of Winkler foundation (CCCC plate; $P = 200$; $b/a = 1$; $G = 0$; $v = 0.3$) ( × HDQ–FD ($K = 400$); ○ HDQ–FD ($K = 800$); △ HDQ–FD ($K = 1600$); □ HDQ–FD ($K = 2400$); —▲— Ref. 6 ($K = 400$); ----- Ref. 6 ($K = 800$); —— Ref. 6 ($K = 1600$); – – + – – Ref. 6 ($K = 2400$)).

parameter $G$ of the shear layer of a Pasternak foundation causes decrease in the deflections of the rectangular plates. It is shown that the response to a step load is higher than the response to a sinusoidal load. It is also interesting to note, however, that the response to a simply supported is higher than the response to a clamped supported. In Fig. 11, three different types of loading are considered for simply supported immovable square plate using $G = 50$. It is shown that the response to a step load is higher than the response to a sinusoidal load and $N$-wave load. Figs. 12a and b show the variation of the deflection of the simply supported immovable rectangular plates with the non-dimensional time parameters for different values $m_r$. For these figures,

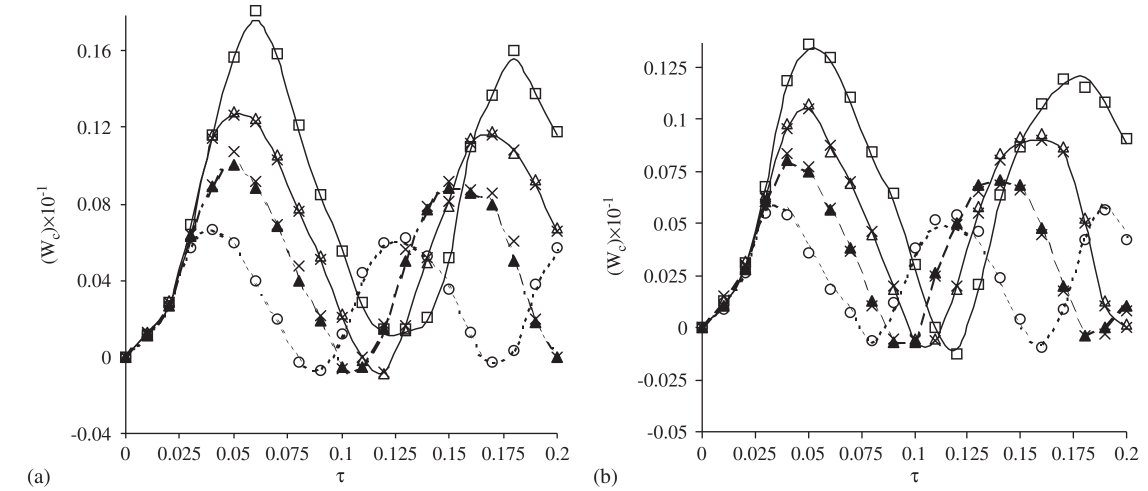


Fig. 9. Central deflection versus time for step load with varying shear modulus of foundation, $G$ ($P = 200$; $K = 800$; $b/a = 1$; $v = 0.3$): (a) SSSS plate; (b) CCCC plate (—— Ref. 6 ($G = 0$); — × — Ref. 6 ($G = 50$); —▲— Ref. 6 ($G = 100$); - - - - - Ref. 6 ($G = 200$); □ HDQ–FD ($G = 0$); △ HDQ–FD ($G = 50$); × HDQ–FD ($G = 100$); ○ HDQ–FD ($G = 200$)).

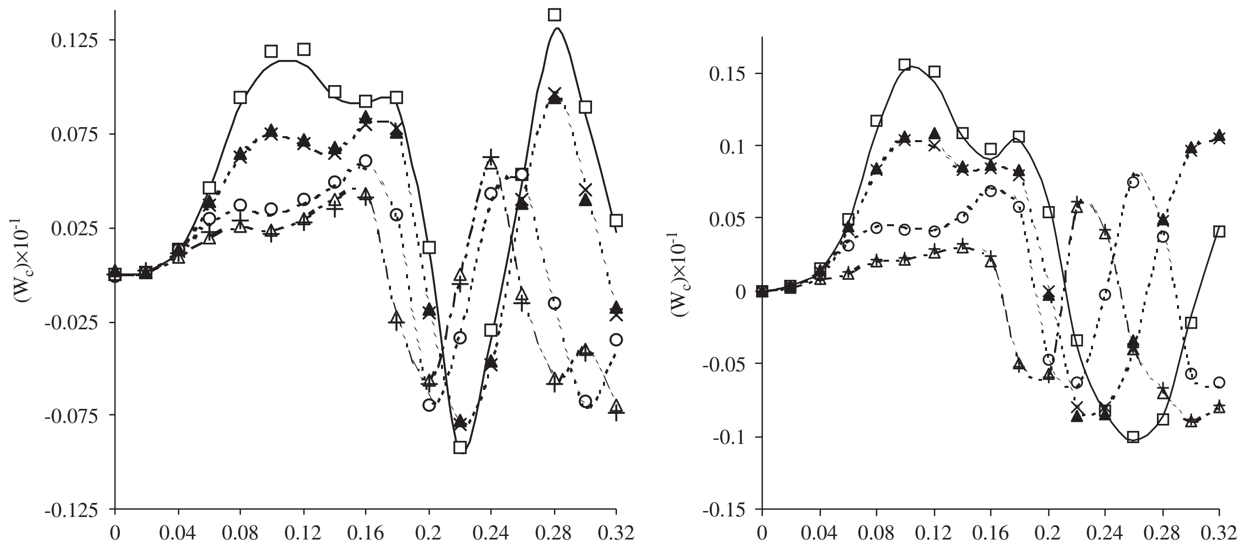


Fig. 10. Central deflection versus time for sinusoidal load of period 0.16 with different shear modulus of foundation, $G$ ($P = 200$; $K = 400$; $b/a = 1$; $v = 0.3$): (a) CCCC plate; (b) SSSS plate (—— Ref. 6 ($G = 0$); -- × -- Ref. 6 ($G = 50$); - - - - - Ref. 6 ($G = 150$); -- △ -- Ref. 6 ($G = 250$); □ HDQ–FD ($G = 0$); ▲ HDQ–FD ($G = 50$); ○ HDQ–FD ($G = 150$); + HDQ–FD ($G = 250$)).

a uniform step load of finite duration ($P = 100$) and a sinusoidal loading of finite duration ($P = 200$) have been considered. The results show clearly that the amplitude of the response has not been changed with increasing $m_r$. It is interesting to note, however, that the response to a step load is higher than the response to a sinusoidal load.

The number of sampling points in both $x$- and $y$-directions are taken to be 7, 9, 11, and 13. It is observed that $N_x = N_y = 11$ is sufficient to obtain accurate results. The percentage errors of the displacements between

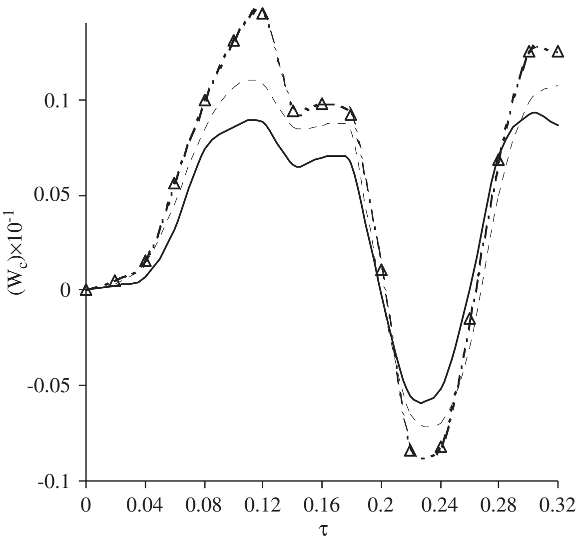


Fig. 11. Central deflection versus time for three different types of load (SSSS plate; $P = 200$; $K = 400$; $G = 50$; $b/a = 1$; $\nu = 0.3$) (—— HDQ–FD-Type-III loading; - - - - - HDQ–FD-Type-II loading; –- △ -– HDQ–FD-Type-I loading).

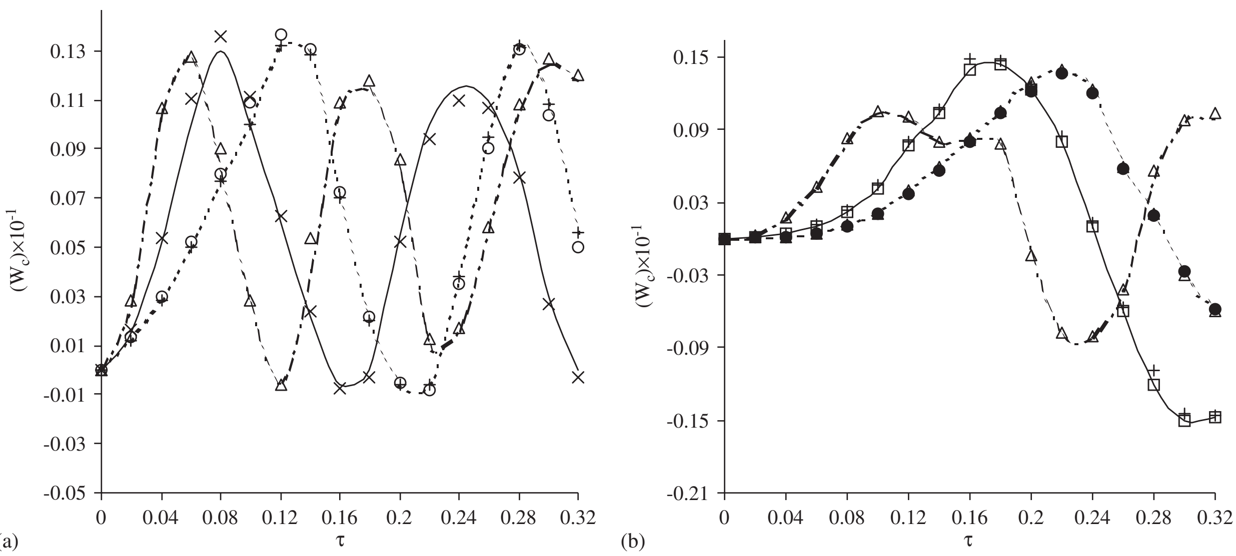


Fig. 12. Central deflection versus time for step load with varying mass ratio $m_r$ of SSSS plate: (a) $P = 100$; $K = 800$; $G = 50$; $b/a = 1$; $\nu = 0.3$; for step load; (b) $P = 200$; $K = 400$; $G = 50$; $b/a = 1$; $\tau = 0.1$; $\nu = 0.3$; for sinusoidal load (△ HDQ–FD ($m_r = 0$); × HDQ–FD ($m_r = 1$); ○ HDQ–FD ($m_r = 3$); + HDQ–FD ($m_r = 4$); ● HDQ–FD ($m_r = 8$); –––––Ref. 6 ($m_r = 0$); —— Ref. 6 ($m_r = 1$); - - + - - Ref. 6 ($m_r = 3$); - - ○ - - Ref. 6 ($m_r = 4$); –- △ -– Ref. 6 ($m_r = 8$)).

the HDQ solution and the references data of Nath et al. [6] are displayed in Fig. 13a. These error value is obtained for dynamic analysis of simply supported plates under the infinite duration ($P = 200$) dynamic step load for $K = 800$. The results of this analysis had been given in above as Fig. 11 for different Pasternak ($G$) parameter. For this error analysis Pasternak parameter is taken as 50. For the displacements of HDQ method provide acceptable results with a maximum discrepancy of 3.95% for ES-G points using nine grid points.

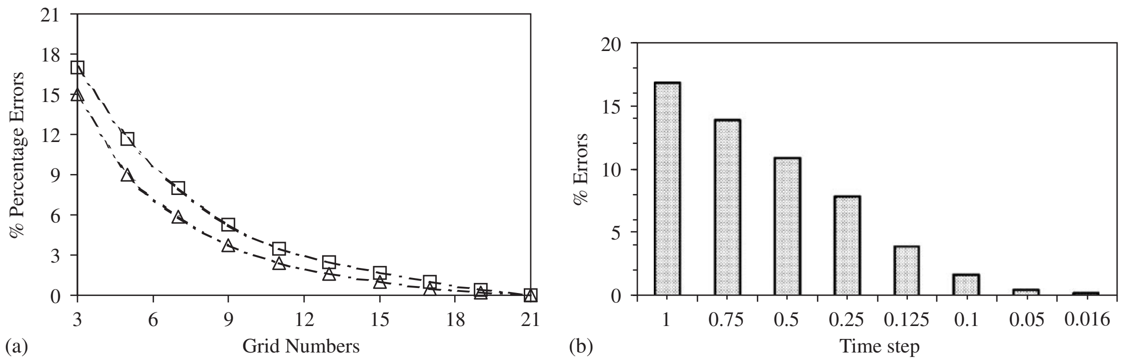


Fig. 13. Variation of % errors: (a) variation of errors with grid numbers for HDQ; (b) variation of errors with time steps for FD (- - □ - - HDQ(ES-G); –- △ -– HDQ(ES-G)).

If the grid numbers in each directon are taken as 13, the percentage error is reduced the value 1.02%. The percentage error is defined by

$$\%\text{Error} = \left| \left( \frac{\text{References value} - \text{HDQ value}}{\text{References value}} \right) \times 100 \right|.$$

For small value of $N$, the HDQ solutions with the stretched Chebyshev–Gauss–Lobatto grids or non equally sampling (NES-G) points is much more accurate than those with the conventional equally spaced sampling grid (ES-G) points. This means that the equally spaced grid (ES-G) points are not reliable in the HDQ solution of dynamic problems. The displacements of HDQ/FD coupled methodolgy provide acceptable results with a maximum discrepancy of 4.01% for $\Delta t = 0.2$ using NES-G points. If the time steps are taken as $\Delta t = 0.1$, the percentage error is reduced the value 1.13%. It was found that as the time intervals or step sizes increases, the percentage error also increases. This is clearly shown in Fig. 13b.

## 5. Conclusions

The geometrically nonlinear dynamic analysis of rectangular plates on Winkler–Pasternak elastic foundation has been presented using the HDQ method. Typical results obtained by HDQ–FD coupled methodolgy are compared with the avaliable results for various foundation parameters. The following conclusions can be obtained from the study:

1. It is appeared that the shear parameter $G$ of the Pasternak foundation and stiffness parameter $K$ of the Winkler foundation have been found to have a significant influence on the dynamic response of the plates.
2. The effect of Winkler parameter $K$, on the diplacements is greater than the Pasternak parameter, $G$.
3. Increase in shear modulus parameter $G$ of the shear layer of a Pasternak foundation causes decrease in the deflections of the plate.
4. The parameter $K$ of the Winkler foundation has been found to have significant influence on the static response of the rectangular plates. From the load–deflection curves, it may be concluded that increasing the foundation parameter, $K$ will always result in decreased deflection.
5. Increasing value of $m_r$ has a small effect on the amplitude of the response.
6. The step load of infinite duration has bigger effect on the dynamic response of the rectangular plates on elastic foundation compared with the other dynamic loads which considered in this study.
7. The damping coefficient $C$ has been found to have significant influence on the dynamic response of the plate. It may be concluded that decreasing the damping coefficient, $C$ will always result in increased deflection.
8. The response to a simply supported plate is higher than the response to a clamped supported.

9. The results obtained with non-equally sampling grid (NES-G) points are more accurate than the values calculated by equally sampling grid (ES-G) points.

As a conclusion, the HDQ–FD methodolgy is a simple, efficient, and accurate method for the linear and nonlinear analysis of immovably clamped and simply supported rectangular plates resting on elastic foundation.

## Acknowledgements

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