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On symmetry-breaking effects in propagation of waves in sandwich plates with and without heavy fluid loading

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Abstract

This paper addresses analysis of wave motions in an unbounded sandwich plate with and without heavy fluid loading in the plane problem formulation. The effects of coupling of otherwise independent upon each other in-phase and anti-phase (with respect to the transverse motion of skins) modes, which could be produced by uneven pre-stress of the skins, uneven properties of the acoustic media on the opposite sides of the plate, or by a difference in properties of the skin plies, are brought to light. These effects are called hereafter 'symmetry-breaking' for brevity. Although the suggested methodology is applicable to treat the above-mentioned cases in a general manner, this paper is concerned with the use of classic perturbation theory to study coupled in-phase and anti-phase modes in a plate of symmetric composition. The coupling perturbation parameter is introduced and is shown to be asymptotically small. The predictions of the perturbation theory are compared with the direct solutions in the cases of regular and singular perturbation and very good agreement is observed.

1. Introduction

This paper continues analysis of wave motions in an unbounded sandwich plate with heavy fluid loading in plane problem formulation performed in Refs. [1–4]. Sandwich plates and shells are widely used in many technical applications, e.g., naval architecture, aerospace or chemical industry, etc because this composition of a thin-walled structure conveniently combines the properties of high strength and low weight. In practice, it is often the case that a sandwich plate of symmetric composition is subjected to some symmetry-breaking loading. Most typically, the symmetry is broken by an uneven static pre-stress of skin plies or—which is particularly relevant for naval and aerospace engineering—by a difference in properties of the surrounding media. In these cases, the exact theory may be used straightforwardly (see, for example Refs. [5,6] in the case of no fluid loading), but it is relatively difficult to solve the transcendental dispersion equation and to identify the modes. It also presents serious difficulties

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to join the elementary theories, which are valid in refined symmetric case [1–4] to capture the symmetry-breaking effects, because they are derived independently based on some assumptions, which are not consistent with each other. The meaningful alternative to overcome these difficulties is offered by use of a perturbation theory, which is perfectly applicable because the above-mentioned symmetry-breaking effects are weak. This paper addresses exactly the issues of asymptotic analysis of perturbed dispersion curves for otherwise independent upon each other in-phase and anti-phase (with respect to the transverse motion of skins) modes. The theoretical background of such an analysis is given in classic publications [7,8].

In Section 2, propagation of waves in sandwich beams with heavy fluid loading is considered in the framework of a theory of elasticity for the core ply of a plate, standard Kirchhoff theory for skin plies and standard acoustics for the surrounding media. The system of governing equations derived in this section is valid for arbitrary composition of a sandwich plate (i.e., not necessarily symmetric) and for arbitrary fluid loading (i.e., different fluids on the different sides). Two classes of wave motions ('in-phase' and 'anti-phase' ones) are introduced for the case of symmetric composition of a sandwich plate in Section 3, and the coupling between them is studied. Section 4 addresses the weak coupling generated by uneven pre-stress of skin plies of a plate with heavy fluid loading on both sides, whereas one-sided heavy fluid loading is considered in Section 5. Finally, in Section 6 conclusions are presented.

2. The problem formulation

Consider an infinitely long sandwich plate consisting of two thin and relatively stiff plies (skins) and a soft core ply between them as is shown in Fig. 1a. It is loaded on both sides by acoustic media, in general, of different properties. Mechanical properties of skin and core plies of sandwich plates used, for example, in naval or aerospace structures are very different. Specifically, elastic and geometry parameters of skin plies considered individually are normally those of conventional thin plates, so that their dynamics are adequately described by a standard Kirchhoff theory. However, due to the interaction between skin and core plies, it is not sufficient to take into account only their flexural wave motions, and the longitudinal components of displacements also should be included to analysis of the wave propagation. The corresponding

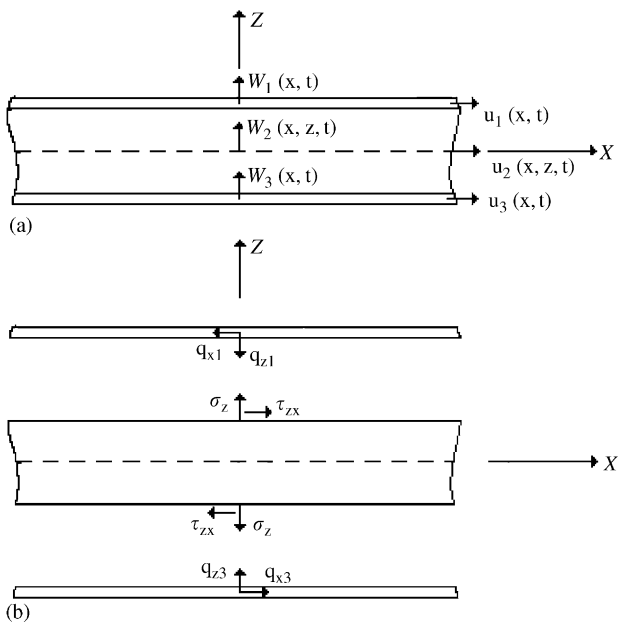

Fig. 1. Sandwich plate composition (a) displacements in plies (b) distributed forces.

governing equations are

$$D_1 w_1^{(4)} + \sigma_1 h_1 w_1'' + \rho_1 h_1 \ddot{w}_1 = q_{w1} + m_1' - p_+\left(x, t, \frac{h}{2} + h_1\right), \tag{1a}$$

$$E_1 h_1 u_1'' - \rho_1 h_1 \ddot{u}_1 = -q_{u1}, \tag{1b}$$

$$D_3 w_3^{(4)} + \sigma_3 h_3 w_3'' + \rho_3 h_3 \ddot{w}_3 = q_{w3} + m_3' + p_-\left(x, t, -\frac{h}{2} - h_3\right), \tag{1c}$$

$$E_3 h_3 u_3'' - \rho_3 h_3 \ddot{u}_3 = -q_{u3}. \tag{1d}$$

Here h_k, $k = 1$, 3 and h are thicknesses of skin and core plies, u_k (x,t) and w_k (x,t), $k = 1$, 3 are the longitudinal and the flexural displacements of the mid-surfaces of skin plies, positive if co-directed with the co-ordinate axes in Fig. 1a. Respectively, q_{wk} (x,t) and q_{uk} (x,t), $k = 1$, 3 are the distributed longitudinal and transverse forces acting on skin plies, see Fig. 1b. The distributed moments m_k (x,t), $k = 1$, 3 are also taken into account as well as static pre-stresses σ_k, $k = 1$, 3 acting in skin plies, $D_k = E_k h_k^3/12(1 - \nu^2)$, $k = 1$, 3 is the conventional formulation of bending stiffness, primes and dots denote derivatives on spatial and temporal co-ordinates x and t, respectively. The elastic properties of the material of each ply are specified by densities ρ_k, $k = 1$, 3, the Young moduli E_k, $k = 1$, 3, and Poisson's ratios $\nu_1 = \nu_3 = \nu$. The right-hand side of Eq. (1) are composed of stresses acting at the interfaces between skin and core plies. In general, they may also contain external driving forces and moments, but as far as propagation of free waves is concerned an external loading is omitted. An acoustic pressure in Eq. (1a,d) is defined as

$$p_\pm(x, t, z) = -\rho_\pm \dot{\varphi}_\pm(x, t, z), \tag{2}$$

and velocity potentials in an acoustic medium satisfy the wave equation

$$\Delta\varphi_\pm - \frac{1}{c_\pm^2}\ddot{\varphi}_\pm = 0. \tag{3}$$

The continuity conditions at the fluid–structure interfaces are formulated as

$$\begin{aligned} z = & -\frac{h}{2} - h_3 : \dot{w}_3(x, t) = \frac{\partial\varphi_-(x, t, z)}{\partial n_-} = \frac{\partial\varphi_-(x, t, z)}{\partial z}, \\ z = & \frac{h}{2} + h_1 : \dot{w}_1(x, t) = -\frac{\partial\varphi_+(x, t, z)}{\partial n_+} = \frac{\partial\varphi_+(x, t, z)}{\partial z}. \end{aligned} \tag{4}$$

In Eqs. (2–4), indices $\pm$ stand for the upper and the lower half-spaces, see Fig. 1. The properties of acoustic media in these half-spaces are specified by densities $\rho_\pm$ and sound speeds $c_\pm$, the outer unit normal vectors are designated as $n_\pm$.

As has been discussed in the Introduction, the core ply of a sandwich plate is much thicker and it is composed of material which is much softer than the skin plies. Thus, an elementary theory of plates is not applicable and the dynamics of a core ply should be described by the standard theory of elasto-dynamics, see for example Ref. [9]. In the plane problem formulation, Lamé equations are

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial z^2} - \frac{1}{c_1^2}\frac{\partial^2\phi}{\partial t^2} = 0, \tag{5a}$$

$$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial z^2} - \frac{1}{c_2^2}\frac{\partial^2\psi}{\partial t^2} = 0. \tag{5b}$$

Here $c_1^2 = E(1 - \nu)/\rho(1 + \nu)(1 - 2\nu)$ and $c_2^2 = E/2(1 + \nu)\rho$ are velocities of acoustic and shear waves in the material, respectively. The material density of the core ply, its Young's modulus and Poisson's ratio are denoted as ρ, E and ν, respectively. Potentials ϕ and ψ are introduced to formulate displacements as

$$u_2 = \frac{\partial\phi}{\partial x} - \frac{\partial\psi}{\partial z}, \quad w_2 = \frac{\partial\psi}{\partial x} + \frac{\partial\phi}{\partial z}. \tag{6}$$

Then stresses (see Fig. 1b) are defined as [9]

$$\sigma_x = \lambda\Delta\phi + 2\mu\left(\frac{\partial^2\phi}{\partial x^2} - \frac{\partial^2\psi}{\partial x\partial z}\right),$$
$$\sigma_x = \lambda\Delta\phi + 2\mu\left(\frac{\partial^2\phi}{\partial x^2} - \frac{\partial^2\psi}{\partial x\partial z}\right),$$
$$\tau_{xz} = \mu\left(2\frac{\partial^2\phi}{\partial x\partial z} + \frac{\partial^2\psi}{\partial x^2} - \frac{\partial^2\psi}{\partial z^2}\right) \tag{7}$$

In these equations, λ and μ are Lamé elastic moduli, defined as

$$\lambda = \frac{vE}{(1+v)(1-2v)}, \quad \mu = \frac{E}{2(1+v)}.$$

The system of differential equations (5) should be solved with the following compatibility conditions at the interfaces:

$$z = \frac{h}{2}: \quad w_2(x,z,t) = w_1(x,t), \quad u_2(x,z,t) = u_1(x,t) + \frac{h_1}{2}\frac{\partial w_1(x,t)}{\partial x}, \tag{8a}$$

$$z = -\frac{h}{2}: \quad w_2(x,z,t) = w_3(x,t), \quad u_2(x,z,t) = u_3(x,t) - \frac{h_3}{2}\frac{\partial w_3(x,t)}{\partial x}. \tag{8b}$$

Since the functions $u_k\ (x,t)$, $w_k\ (x,t)$, $k = 1$, 3 are defined for the mid-surfaces of skin plies, the continuity conditions at the interfaces for the longitudinal displacements are formulated with the components $\pm h_k/2\partial w_k(x,t)/\partial x$, $k = 1$, 3 (i.e., the angles of rotation due to the flexural motion) taken into account. It is consistent with the governing equations (1) for the skin plies, where distributed moments are also included. This formulation is valid as long as the elementary Kirchhoff theory is applicable to describe wave motion in the skins.

The interfacial distributed forces and moments are formulated as

$$q_{w3}(x,t) = \sigma_z\left(x, -\frac{h}{2}, t\right), \quad q_{u3}(x,t) = -\tau_{xz}\left(x, -\frac{h}{2}, t\right), \quad m_3(x,t) = \frac{h_3}{2}\tau_{xz}\left(x, -\frac{h}{2}, t\right), \tag{9a}$$

$$q_{w1}(x,t) = -\sigma_z\left(x, \frac{h}{2}, t\right), \quad q_{u1}(x,t) = \tau_{xz}\left(x, \frac{h}{2}, t\right), \quad m_1(x,t) = \frac{h_1}{2}\tau_{xz}\left(x, \frac{h}{2}, t\right). \tag{9b}$$

Hereafter the scaling is introduced as: $x = \bar{x}h$, $z = \bar{z}h$, $u_j = \bar{u}_j h$, $w_j = \bar{w}_j h$, $j = 1,2,3$.

Propagation of a harmonic coupled acoustic and elastic wave in an infinitely long plate with heavy fluid loading is considered, so that

$$\bar{u}_j = U_j \exp(k\bar{x} - \mathrm{i}\omega t), \quad \bar{w}_j = W_j \exp(k\bar{x} - \mathrm{i}\omega t), \ \ j = 1,2,3,$$
$$\phi = \Phi(\bar{z})\exp(k\bar{x} - \mathrm{i}\omega t), \quad \psi = \Psi(\bar{z})\exp(k\bar{x} - \mathrm{i}\omega t). \tag{10}$$

Hereafter bars over non-dimensional variables are omitted, ω is a positive excitation frequency and k is, a priori, a complex wavenumber. Eqs. (10) are substituted into Eqs. (5), (7) and the problem in elasto-dynamics for the core ply is formulated as

$$\frac{\mathrm{d}^2\Phi}{\mathrm{d}z^2} + \left[k^2 + \left(\frac{\omega h}{c_1}\right)^2\right]\Phi = 0, \tag{11a}$$

$$\frac{\mathrm{d}^2\Psi}{\mathrm{d}z^2} + \left[k^2 + \left(\frac{\omega h}{c_2}\right)^2\right]\Psi = 0, \tag{11b}$$

$$z = \frac{1}{2}: \quad \frac{\mathrm{d}\Phi}{\mathrm{d}z} + k\Psi = h^2 W_1, \quad k\Phi - \frac{\mathrm{d}\Psi}{\mathrm{d}z} = h^2 U_1 + \frac{hh_1}{2}kW_1, \tag{11c}$$

$$z=-\frac{1}{2}:\quad \frac{\mathrm{d}\Phi}{\mathrm{d}z}+k\Psi=h^2W_3,\quad k\Phi-\frac{\mathrm{d}\Psi}{\mathrm{d}z}=h^2U_3-\frac{hh_3}{2}kW_3. \tag{11d}$$

Velocity potentials in the acoustic media are sought in the form

$$\varphi_\pm(x,t,z)=\varphi_\pm^{(0)}(z)\exp(kx-\mathrm{i}\omega t), \tag{12}$$

so the wave equation (3) is reduced to a 1-D Helmholtz equation

$$\frac{\mathrm{d}^2\varphi_\pm^{(0)}}{\mathrm{d}z^2}+\left[k^2+\left(\frac{\omega h}{c_\pm}\right)^2\right]\varphi_\pm^{(0)}=0. \tag{13}$$

The solution of this equation is sought as

$$\varphi_+^{(0)}(z)=A_{1+}\exp(\mathrm{i}\gamma_+z)+A_{1-}\exp(-\mathrm{i}\gamma_+z),\quad \gamma_+\equiv\sqrt{k^2+\left(\frac{\omega h}{c_+}\right)^2}.$$

$$\varphi_-^{(0)}(z)=A_{3+}\exp(\mathrm{i}\gamma_-z)+A_{3-}\exp(-\mathrm{i}\gamma_-z),\quad \gamma_-\equiv\sqrt{k^2+\left(\frac{\omega h}{c_-}\right)^2}.$$

The parameters $A_{1\pm}$, $A_{3\pm}$ should be selected to satisfy compatibility conditions (4) at fluid–structure interfaces and the radiation condition, which is formulated at infinity for the upper and the lower half-spaces occupied by the acoustic medium. Elementary algebra gives the following expressions for velocity potentials:

$$\varphi_+^{(0)}(z)=\frac{\omega h^2}{\gamma_+}W_1\exp(\mathrm{i}\gamma_+z),\quad z>\frac{1}{2},$$
$$\varphi_-^{(0)}(z)=\frac{\omega h^2}{\gamma_-}W_3\exp(-\mathrm{i}\gamma_-z),\quad z<-\frac{1}{2}.$$

Then the amplitudes of the contact acoustic pressure at the surfaces of skin plies are formulated via the amplitudes of displacements as

$$p_+=-\frac{\mathrm{i}\rho_+\omega^2h^2}{\sqrt{k^2+\left(\frac{\omega h}{c_+}\right)^2}}W_1,\quad z=\frac{1}{2}, \tag{14a}$$

$$p_-=\frac{i\rho_-\omega^2h^2}{\sqrt{k^2+\left(\frac{\omega h}{c_-}\right)^2}}W_3,\quad z=-\frac{1}{2}. \tag{14b}$$

Finally, Eq. (1) are reduced to

$$\left[\frac{E_1h_1}{h}k^2+\rho_1hh_1\omega^2\right]U_1=-\hat{q}_{u1}, \tag{15a}$$

$$\left[\frac{E_3h_3}{h}k^2+\rho_3hh_3\omega^2\right]U_3=\hat{q}_{u3}, \tag{15b}$$

$$\left[\frac{E_1h_1^3}{12(1-\nu^2)h^3}k^4+\sigma_1\frac{h_1}{h}k^2-\rho_1hh_1\omega^2-\frac{\mathrm{i}\rho_+h^2}{\gamma_+}\right]W_1=\hat{q}_{w1}-k\hat{m}_1, \tag{15c}$$

$$\left[\frac{E_3h_3^3}{12(1-\nu^2)h^3}k^4+\sigma_3\frac{h_1}{h}k^2-\rho_3hh_1\omega^2-\frac{\mathrm{i}\rho_-h^2}{\gamma_-}\right]W_3=-\hat{q}_{w3}+k\hat{m}_3. \tag{15d}$$

The right-hand side of Eq. (15) are defined by formulas (9–10). These equations are valid for an arbitrarily composed sandwich plate with heavy fluid loading at its both sides.

3. Coupling of the in-phase and anti-phase modes

In the earlier paper [1], the symmetric composition of a sandwich plate ($h_1 = h_3$, $E_1 = E_3$, $\rho_1 = \rho_3$) has been considered and two uncoupled classes of linear wave motions (in-phase and anti-phase with respect to the transverse motion of skins) have been identified and analyzed separately. However, it is realistic to assume that even in the case of symmetric original composition of a sandwich plate one of skin plies may experience somewhat larger stress than its counterpart and that fluid loading is different (e.g., water and air) on opposite sides of the plate. In such a case, the otherwise independent upon each other in-phase and anti-phase wave motions become coupled due to these symmetry-breaking effects.

It is convenient to introduce the in-phase (W_+, U_-) and the anti-phase (W_-, U_+) components of displacements of skin plies as

$$\begin{aligned} W_+ &= \tfrac{W_1+W_3}{2}, \quad U_- = \tfrac{U_1-U_3}{2}, \\ W_- &= \tfrac{W_1-W_3}{2} \quad U_+ = \tfrac{U_1+U_3}{2}, \end{aligned} \tag{16}$$

while the elastic potentials are sought as

$$\begin{aligned} \Phi(z) &= A_1 \sinh(\gamma_1 z) + A_2 \cosh(\gamma_1 z), \quad \gamma_1^2 = -k^2 - \left(\tfrac{\omega h}{c_1}\right)^2, \\ \Psi(z) &= B_1 \cosh(\gamma_2 z) + B_2 \sinh(\gamma_2 z), \quad \gamma_2^2 = -k^2 - \left(\tfrac{\omega h}{c_2}\right)^2. \end{aligned} \tag{17}$$

Then the boundary conditions (8) are formulated as

$$\begin{aligned} &A_1\gamma_1 \cosh\left(\frac{\gamma_1}{2}\right) + A_2\gamma_1 \sinh\left(\frac{\gamma_1}{2}\right) + B_1 k \cosh\left(\frac{\gamma_2}{2}\right) + B_2 k \sinh\left(\frac{\gamma_2}{2}\right) = (W_+ + W_-)h^2, \\ &A_1\gamma_1 \cosh\left(\frac{\gamma_1}{2}\right) - A_2\gamma_1 \sinh\left(\frac{\gamma_1}{2}\right) + B_1 k \cosh\left(\frac{\gamma_2}{2}\right) - B_2 k \sinh\left(\frac{\gamma_2}{2}\right) = (W_+ - W_-)h^2, \\ &A_1 k \sinh\left(\frac{\gamma_1}{2}\right) + A_2 k \sinh\left(\frac{\gamma_1}{2}\right) - B_1\gamma_2 \sinh\left(\frac{\gamma_2}{2}\right) - B_2\gamma_2 \cosh\left(\frac{\gamma_2}{2}\right) \\ &= (U_+ + U_-)h^2 + \frac{hh_1}{2}k(W_+ + W_-), \\ &- A_1 k \sinh\left(\frac{\gamma_1}{2}\right) + A_2 k \sinh\left(\frac{\gamma_1}{2}\right) + B_1\gamma_2 \sinh\left(\frac{\gamma_2}{2}\right) - B_2\gamma_2 \cosh\left(\frac{\gamma_2}{2}\right) \\ &= (U_+ - U_-)h^2 + \frac{hh_1}{2}k(W_+ - W_-). \end{aligned} \tag{18}$$

The solution of this system is readily available as

$$A_1 = \frac{h^2 k U_- \cosh\left(\frac{\gamma_2}{2}\right) + \frac{hh_1}{2}k^2 W_+ \cosh\left(\frac{\gamma_2}{2}\right) + h^2\gamma_2 W_+ \sinh\left(\frac{\gamma_2}{2}\right)}{k^2 \sinh\left(\frac{\gamma_1}{2}\right)\cosh\left(\frac{\gamma_2}{2}\right) + \gamma_1\gamma_2 \cosh\left(\frac{\gamma_1}{2}\right)\sinh\left(\frac{\gamma_2}{2}\right)}, \tag{19a}$$

$$B_1 = \frac{-h^2\gamma_1 U_- \cosh\left(\frac{\gamma_1}{2}\right) - \frac{hh_1}{2}\gamma_1 k W_+ \cosh\left(\frac{\gamma_1}{2}\right) + h^2 k W_+ \sinh\left(\frac{\gamma_1}{2}\right)}{k^2 \sinh\left(\frac{\gamma_1}{2}\right)\cosh\left(\frac{\gamma_2}{2}\right) + \gamma_1\gamma_2 \cosh\left(\frac{\gamma_1}{2}\right)\sinh\left(\frac{\gamma_2}{2}\right)}, \tag{19b}$$

$$A_2 = \frac{h^2 k U_+ \sinh\left(\frac{\gamma_2}{2}\right) + \frac{hh_1}{2}k^2 W_- \sinh\left(\frac{\gamma_2}{2}\right) + h^2\gamma_2 W_- \cosh\left(\frac{\gamma_2}{2}\right)}{k^2 \cosh\left(\frac{\gamma_1}{2}\right)\sinh\left(\frac{\gamma_2}{2}\right) + \gamma_1\gamma_2 \sinh\left(\frac{\gamma_1}{2}\right)\cosh\left(\frac{\gamma_2}{2}\right)}, \tag{19c}$$

$$B_2 = \frac{-h^2\gamma_1 U_+ \sinh\left(\frac{\gamma_1}{2}\right) - \frac{hh_1}{2}\gamma_1 k W_- \sinh\left(\frac{\gamma_1}{2}\right) + h^2 k W_- \cosh\left(\frac{\gamma_1}{2}\right)}{k^2 \cosh\left(\frac{\gamma_1}{2}\right)\sinh\left(\frac{\gamma_2}{2}\right) + \gamma_1\gamma_2 \sinh\left(\frac{\gamma_1}{2}\right)\cosh\left(\frac{\gamma_2}{2}\right)}. \tag{19d}$$

Substitution of in-phase (W_+, U_-) and anti-phase (W_-, U_+) components of displacements to equations for skin plies give

$$\left[\frac{E_1h_1}{h}k^2+\frac{E_3h_1}{h}k^2+\rho_1hh_1\omega^2+\rho_3hh_1\omega^2\right]U_+ + \left[\frac{E_1h_1}{h}k^2-\frac{E_3h_1}{h}k^2+\rho_1hh_1\omega^2-\rho_3hh_1\omega^2\right]U_- = -q_{u1}-q_{u3}, \tag{20a}$$

$$\left[\frac{E_1h_1}{h}k^2-\frac{E_3h_1}{h}k^2+\rho_1hh_1\omega^2-\rho_3hh_1\omega^2\right]U_+ + \left[\frac{E_1h_1}{h}k^2+\frac{E_3h_1}{h}k^2+\rho_1hh_1\omega^2+\rho_3hh_1\omega^2\right]U_- = -q_{u1}+q_{u3}, \tag{20b}$$

$$\begin{aligned}&\left[\frac{E_1h_1^3}{12(1-\nu^2)h^3}k^4+\frac{E_3h_1^3}{12(1-\nu^2)h^3}k^4+\sigma_1\frac{h_1}{h}k^2+\sigma_3\frac{h_1}{h}k^2 -\rho_1hh_1\omega^2-\rho_3hh_1\omega^2-\frac{\mathrm{i}\rho_+h^2}{\gamma_+}-\frac{\mathrm{i}\rho_-h^2}{\gamma_-}\right]W_+\\&+\left[\frac{E_1h_1^3}{12(1-\nu^2)h^3}k^4-\frac{E_3h_1^3}{12(1-\nu^2)h^3}k^4+\sigma_1\frac{h_1}{h}k^2-\sigma_3\frac{h_1}{h}k^2 -\rho_1hh_1\omega^2+\rho_3hh_1\omega^2-\frac{\mathrm{i}\rho_+h^2}{\gamma_+}+\frac{\mathrm{i}\rho_-h^2}{\gamma_-}\right]W_- = q_{w1}+q_{w3}-m_1'-m_3',\end{aligned} \tag{20c}$$

$$\begin{aligned}&\left[\frac{E_1h_1^3}{12(1-\nu^2)h^3}k^4-\frac{E_3h_1^3}{12(1-\nu^2)h^3}k^4+\sigma_1\frac{h_1}{h}k^2-\sigma_3\frac{h_1}{h}k^2-\rho_1hh_1\omega^2+\rho_3hh_1\omega^2-\frac{\mathrm{i}\rho_+h^2}{\gamma_+}+\frac{\mathrm{i}\rho_-h^2}{\gamma_-}\right]W_+\\&+\left[\frac{E_1h_1^3}{12(1-\nu^2)h^3}k^4+\frac{E_3h_1^3}{12(1-\nu^2)h^3}k^4+\sigma_1\frac{h_1}{h}k^2+\sigma_3\frac{h_1}{h}k^2-\rho_1hh_1\omega^2-\rho_3hh_1\omega^2-\frac{\mathrm{i}\rho_+h^2}{\gamma_+}-\frac{\mathrm{i}\rho_-h^2}{\gamma_-}\right]\\&\times W_- = q_{w1}-q_{w3}-m_1'+m_3'.\end{aligned} \tag{20d}$$

The interfacial force and moment resultants are

$$-q_{u1}-q_{u3} = -\tau_{zx}\left(\frac{1}{2}\right)+\tau_{zx}\left(-\frac{1}{2}\right), \tag{21a}$$

$$-q_{u1}+q_{u3} = -\tau_{zx}\left(\frac{1}{2}\right)-\tau_{zx}\left(-\frac{1}{2}\right), \tag{21b}$$

$$q_{w1}+q_{w3}-m_1'-m_3' = -\sigma_z\left(\frac{1}{2}\right)+\sigma_z\left(-\frac{1}{2}\right)+\frac{h_1}{2h}\frac{\mathrm{d}\tau_{zx}}{\mathrm{d}x}\bigg|_{z=-\frac{1}{2}}+\frac{h_1}{2h}\frac{\mathrm{d}\tau_{zx}}{\mathrm{d}x}\bigg|_{z=1/2}, \tag{21c}$$

$$q_{w1}-q_{w3}-m_1'+m_3' = -\sigma_z\left(\frac{1}{2}\right)-\sigma_z\left(-\frac{1}{2}\right)+\frac{h_1}{2h}\frac{\mathrm{d}\tau_{zx}}{\mathrm{d}x}\bigg|_{z=-1/2}-\frac{h_1}{2h}\frac{\mathrm{d}\tau_{zx}}{\mathrm{d}x}\bigg|_{z=1/2}, \tag{21d}$$

The stresses in Eq. (21) are formulated as

$$\begin{aligned}\sigma_z(z) = {}&\lambda\left[A_1\left(\gamma_1^2+k^2\right)\sinh\left(\gamma_1 z\right)+A_2\left(\gamma_1^2+k^2\right)\cosh\left(\gamma_1 z\right)\right]\\&+2\mu\left[A_1k^2\left(\gamma_1^2+\right)\sinh\left(\gamma_1 z\right)+A_2k^2\left(\gamma_1^2+\right)\cosh\left(\gamma_1 z\right)-B_1k\gamma_2\sinh\left(\gamma_2 z\right)-B_2k\gamma_2\cosh\left(\gamma_2 z\right)\right],\end{aligned} \tag{22a}$$

$$\tau_{zx}(z) = \mu\left[2A_1k\gamma_1\cosh\left(\gamma_1 z\right)+2A_2k\gamma_1\sinh\left(\gamma_1 z\right)+B_1\left(k^2-\gamma_2^2\right)\cosh\left(\gamma_2 z\right)+B_2\left(k^2-\gamma_2^2\right)\sinh\left(\gamma_2 z\right)\right]. \tag{22b}$$

Stresses (22) are substituted into formulas (21) and these interfacial forces and moments are substituted into Eq. (20) to yield the system of homogeneous linear equations with respect to W_+, W_-, U_+, U_-. It is appropriate to write these equations as follows

$$\begin{aligned} &z_{11}W_+ + z_{12}U_- + z_cW_- = 0,\\ &z_{21}W_+ + z_{22}U_- = 0,\\ &z_cW_+ + z_{33}W_- + z_{34}U_+ = 0,\\ &z_{43}W_- + z_{44}U_+ = 0. \end{aligned} \tag{23}$$

The coefficients z_{ij}, $i, j = 1, 2$ are very cumbersome and therefore they are not presented here in an explicit form. When the determinant of system (23) is set to zero, a dispersion equation is obtained. This equation is valid for determining wavenumbers as a function of excitation frequency for arbitrary parameters of sandwich plate composition and fluid loading. Its numerical solution is obtained via a code written in Mathematica [10] by use of the algorithm suggested in Ref. [1]. Although this algorithm is capable to yield the dispersion curves for any combination of parameters, it usage is less convenient and much more time-consuming than in the cases considered in this reference (see similar observations reported in Ref. [6]). Therefore it has appeared to be practical to seek an easier way for analysis of dispersion curves.

In this paper, a plate of symmetric composition is considered and the coupling between in-phase and anti-phase components is produced either by different pre-stress of identical skin plies or by a difference in properties of the acoustic media on opposite sides of the plate. Then it is possible to determine wavenumbers by use of perturbation theory. It is convenient to present the dispersion equation in the form

$$(z_{11}z_{22} - z_{12}z_{21})(z_{33}z_{44} - z_{34}z_{43}) - z_c^2 z_{22}z_{44} = 0. \tag{24}$$

The parameter z_c quantifies the coupling between in-phase and anti-phase modes. The wavenumbers of these modes are determined by the equations

$$F_1(k,\omega) \equiv z_{11}z_{22} - z_{12}z_{21} = 0, \tag{25a}$$

$$F_2(k,\omega) \equiv z_{33}z_{44} - z_{34}z_{43} = 0. \tag{25b}$$

These equations are solved in Ref. [1].

The coupling coefficient is

$$z_c = \frac{\sigma_1 - \sigma_3}{E_1}\frac{h_1}{h}k^2 - \frac{\mathrm{i}\rho_+\omega^2h^2}{E_1\gamma_+} + \frac{\mathrm{i}\rho_-\omega^2h^2}{E_1\gamma_-}. \tag{26}$$

The symmetry-breaking effects may be analysed in the framework of a perturbation theory as long as the parameter z_c remains small.

4. The effect of static pre-stress

If the properties of the fluids on the both sides of the plate are identical, then the perturbation parameter reduces to $z_c = (\sigma_1 - \sigma_3)/E_1\left(h_1/h\right)k^2$. It is formulated as a product of the small parameter h_1/h, which in naval structures is of order 10^{-1}, and another small parameter $(\sigma_1 - \sigma_3)/E_1$, which is of order 10^{-2}–10^{-4}. Thus, $\alpha \equiv (\sigma_1 - \sigma_3)/E_1 h_1/h$ is very small indeed and it is possible to apply the perturbation theory in solving the dispersion equation

$$F_1F_2 - \alpha^2 z_{22}z_{44}k^4 = 0. \tag{27}$$

Before addressing the case of an uneven pre-stress, it is relevant to assess the influence of symmetric pre-stress on the location of the dispersion curves plotted hereafter in $\Omega \equiv \omega h/c_{\mathrm{skin}}$, $\mathrm{K} \equiv k_{\mathrm{dim}}h$, $\left(c_{\mathrm{skin}} = \sqrt{E_1/\rho_1(1-\nu^2)}\right)$. In Fig. 2, four sets of dispersion curves are presented for a sandwich plate of the following parameters: $h_1/h = 0.1$, $E/E_1 = 0.01$, $\rho/\rho_1 = 0.1$, no fluid loading. The curves designated as 1 s and 1a display wavenumbers of in-phase and anti-phase modes in a plate without any pre-stress of both skins. The curves designated as 2 s and 2a display wavenumbers of in-phase and anti-phase modes in a plate with a

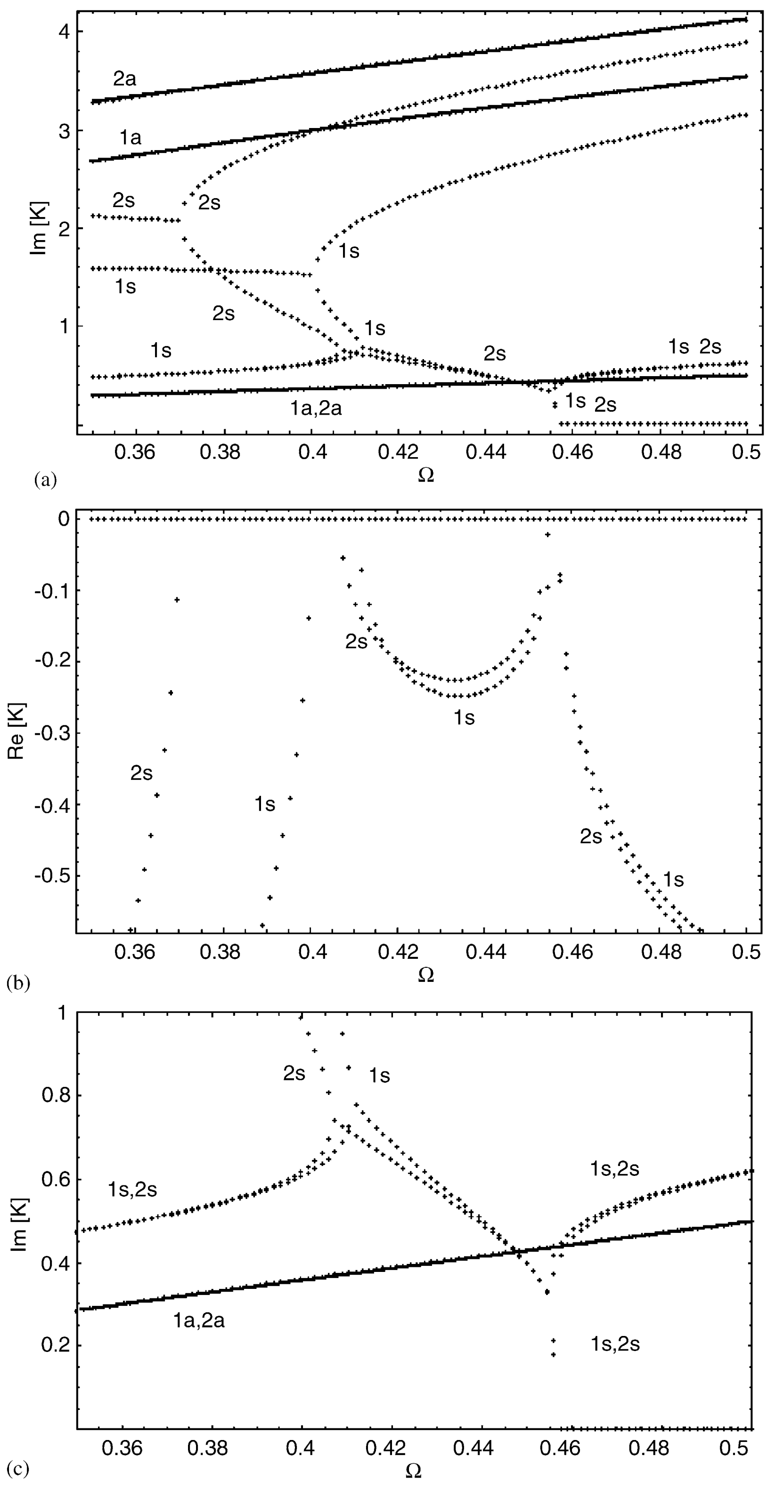

Fig. 2. Dispersion curves (a—imaginary parts; b—real parts; c—imaginary parts, zoomed) for a plate with (curves 1a—anti-phase modes, curves 1 s—in-phase modes) and without (curves 2a—anti-phase modes, curves 2 s—in-phase modes) pre-stress.

very large identical pre-stress of both skins, $\sigma_1/E_1 = \sigma_3/E_1 = 0.001$. For convenience, a-curves are plotted in bold. As is seen, the location of the lowest branches of a-and s-dispersion curves is influenced very weakly by the pre-stress. Two propagating anti-phase waves exist in the whole frequency range considered, $0.35 \leqslant \Omega_{\text{cut-on}} \leqslant 0.5$. The location of the first one is not influenced by pre-stress, whereas the second one is shifted towards larger wave numbers, see Fig. 2a. The transformation of in-phase waves is the same in the cases of a pre-stressed plate and a plate with no pre-stress. However, the magnitude of a cut-on frequency of a pair of propagating in-phase waves in a pre-stressed plate, $\Omega_{\text{cut-on}} \approx 0.369$ is markedly lower, than in a plate with no pre-stress, $\Omega_{\text{cut-on}} \approx 0.403$ (these propagating waves are generated due to transformation of two complex conjugate wave numbers to a pair of purely imaginary ones). Then three propagating waves exist in the frequency range $0.403 \leqslant \Omega \leqslant 0.412$ for a plate with no pre-stress and in the range $0.369 \leqslant \Omega \leqslant 0.407$ for a pre-stressed plate. At the cut-off frequency (which is $\Omega_{\text{cut-off}} \approx 0.407$ for a pre-stressed plate and $\Omega_{\text{cut-off}} \approx 0.412$ for a plate with no pre-stress, see Fig. 2b), a pair of propagating waves transforms into a pair of evanescent waves (the negative real parts are shown versus excitation frequency in Fig. 2b) and in the range $0.407 \leqslant \Omega \leqslant 0.455$ (pre-stressed plate) or $0.412 \leqslant \Omega \leqslant 0.455$ (plate with no pre-stress) only one propagating in-phase wave exists. However, at the frequency $\Omega_{\text{cut-on}} \approx 0.455$ (which is the same for the plate with and without pre-stress) the pair of attenuated waves transform to a pair of propagating ones again and one of these waves cuts off and transforms into an evanescent wave almost immediately, so in the frequency range $\Omega \geqslant \Omega_{\text{cut-off},2} \approx 0.457$ two propagating in-phase waves exist (as in the low-frequency range). It is remarkable that one of these branches crosses the branch of the propagating anti-phase wave. This part of Fig. 2a is zoomed in Fig. 2c. Of course, there is no interaction between these intersecting brunches as long as the symmetry of static pre-stress is preserved. However, as soon as pre-stress becomes uneven it is necessary to set up both a singular expansion on the pre-stress parameter to describe the interaction between these branches, whereas a regular expansion is sufficient in the rest of the considered frequency range.

The regular expansion of a wavenumber on parameter α_2 is formulated as

$$k(\alpha^2, \omega, \ldots) = k(0, \omega, \ldots) + \left.\frac{\partial k(\alpha^2, \omega, \ldots)}{\partial(\alpha^2)}\right|_{\alpha^2=0} \alpha^2 + \cdots \tag{28}$$

The first-order sensitivity $\partial k/\partial(\alpha^2)$ is readily available from the derivative of Eq. (27) as

$$\left(\frac{\partial F_1}{\partial k} F_2 + \frac{\partial F_2}{\partial k} F_1\right) \frac{\partial k}{\partial(\alpha^2)} - 4\alpha^2 z_{22} z_{44} k^3 \frac{\partial k}{\partial(\alpha^2)} - z_{22} z_{44} k^4 = 0, \tag{29}$$

and for $\alpha^2 = 0$ it gives

$$\frac{\partial k}{\partial(\alpha^2)} = z_{22} z_{44} k^4 \left(\frac{\partial F_1}{\partial k} F_2 + \frac{\partial F_2}{\partial k} F_1\right)^{-1}. \tag{30}$$

This formula is valid in all cases when a wave number is not the root of both Eqs. (25).

In the case of a multiple root, expansion (28) is modified as

$$k(\alpha^2, \omega, \ldots) = k(0, \omega, \ldots) + k_1(0, \omega, \ldots)|\alpha| + \cdots \tag{31}$$

In Fig. 3, the dispersion curves for a plate of $h_1/h = 0.1$, $E/E_1 = 0.01$, $\rho/\rho_1 = 0.1$ are shown in the case of no pre-stress (curves 1a, 1 s) and in the case of one-side pre-stress ($\sigma_1/E_1 = 0.001$, $\sigma_3 = 0$), (curves 2a, 2 s). They are obtained by direct numerical solution of the dispersion Eq. (27) and their pattern is similar to the case illustrated in Fig. 2. In Fig. 4a these curves are zoomed and, although they are almost identical, there is a qualitative difference between them in the vicinity of crossing point. Further zooming in Fig. 4b of the curves plotted for the case of no pre-stress shows that a-curve simply crosses s-curve and no interaction occurs due to the orthogonal properties of these modes. In contrast, Fig. 4c for an uneven pre-stress shows the typical (see Refs. [7,8]) picture, when the dispersion curves do not intersect, but the modes are 'swapped' due to their interaction.

Three sets of curves, which present purely imaginary wavenumbers versus excitation frequency, are plotted in Fig. 5. The dispersion curve obtained numerically is designated as curve 1. The dispersion curve obtained by use of regular expansion (20) is shown in Fig. 5 as curve 2. As is seen, this formula becomes invalid in the vicinity of the intersection point. The dispersion curve obtained by use of singular expansion (31) is presented

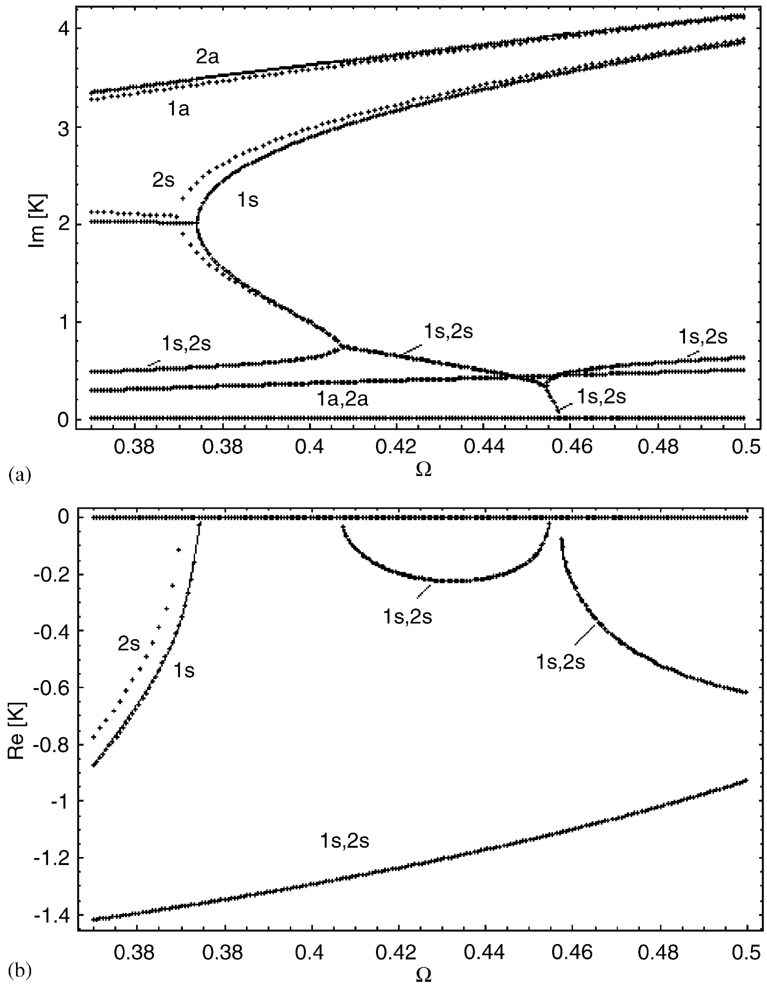

Fig. 3. Dispersion curves (a—imaginary parts; b—real parts) for a plate with an uneven pre-stress (curves 1) and without pre-stress (curves 2).

as curve 3. It is seen that this approximation totally recovers the correct behaviour at the intersection point, but it gives a systematic error outside this region. The asymptotic expansions (28) and (31) are truncated to the first-order corrections, because they yield perturbed wavenumbers with sufficient accuracy. The continuation of these expansions to higher-order corrections is standard, but it appears to be unnecessary in all considered cases.

5. The effect of one-side fluid loading

In naval engineering, it is typical to deal with plates and shells which are loaded on one side by water and on the other side by air, or a fluid with different properties (e.g., an oil). Then the symmetry may be broken both by an uneven static pre-stress and by an uneven fluid loading. As is well-known, the fluid loading parameter of density ratio is of order 10^{-1}, whereas the static pre-stress parameter is approximately two orders of magnitude weaker. Then the effect of fluid loading surely dominates the effect of static pre-stress so that the perturbation parameter may be reduced as $z_c = -\mathrm{i}\rho_+\omega^2h^2/E_1\gamma_+ + \mathrm{i}\rho_-\omega^2h^2/E_1\gamma_-$. In the case of loading by water at one side of a plate (say, underneath a plate, i.e., $\rho_- = 10^3\,kg/m^3$) and by an air at another side it is realistic to put $\rho_+ = 0$. Consider firstly the limit case of an incompressible fluid ($c_- \to \infty$). Then $\gamma_-|k|$ and $z_c^2 = -\rho_-^2\omega^4h^4/E_1^2k^2$.

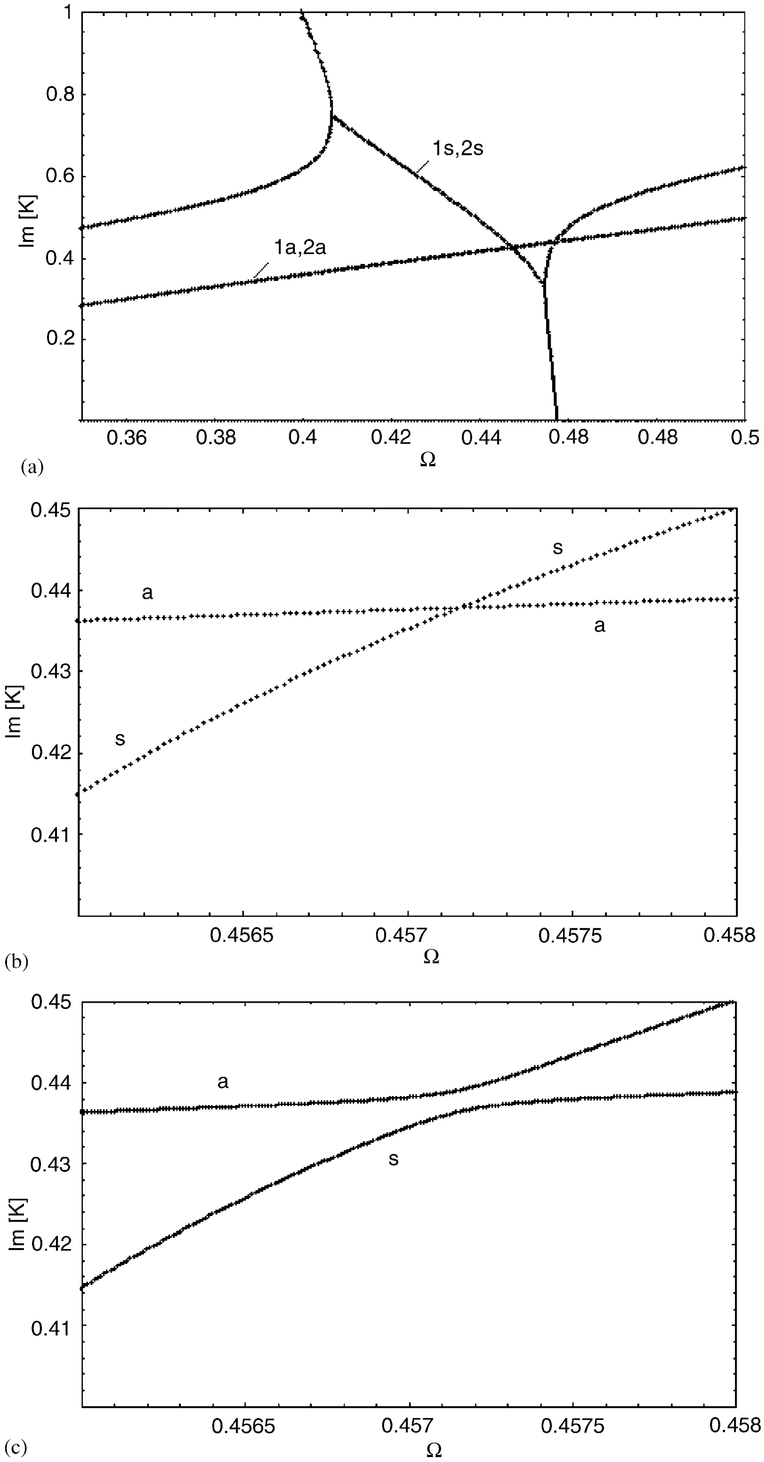

Fig. 4. Zoomed dispersion curves (a—with an even pre-stress and with an uneven pre-stress; b—with an even pre-stress; c—with an uneven pre-stress).

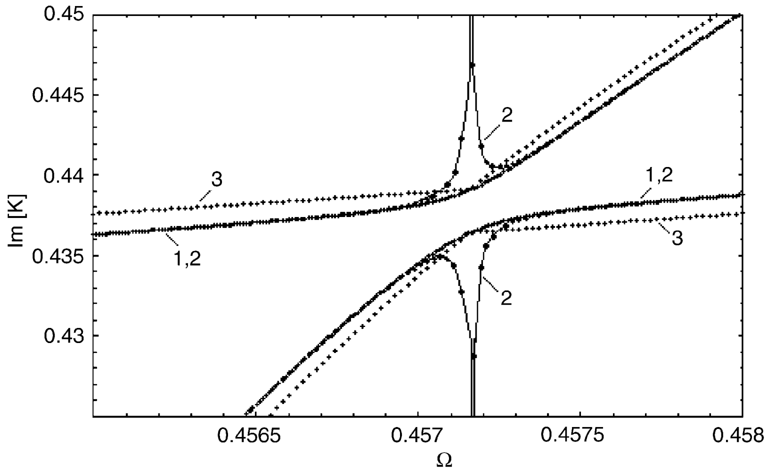

Fig. 5. Zoomed dispersion curves for a plate with an uneven pre-stress. Curves 1—numerical solution; curves 2—regular asymptotic expansion; curves 3—singular asymptotic expansion.

The small parameter is introduced as $\beta^2 = \rho_-^2\omega^4h^4/E_1^2$ and Eq. (24) becomes

$$F_1F_2 + \beta^2 z_{22}z_{44}k^{-2} = 0. \tag{32}$$

This is a case of singular perturbation, but in the low-frequency range the propagating modes are described by perturbed roots of the factorised dispersion Eqs. (25). Then they are conveniently presented in a regular expansion on small parameter β.

In Fig. 6, the dispersion curves are plotted for the same parameters of sandwich plate composition as in Section 4, $h_1/h = 0.1$, $E/E_1 = 0.01$, $\rho/\rho_1 = 0.1$ in a low-frequency range. The density ratio is taken as $\rho_-/\rho_1 = 0.128$, which corresponds to steel skins loaded by water. Both the direct solution and the perturbation theory suggest that two propagating waves (dominantly in-phase one, curve 1 and dominantly anti-phase one, curve 2 in Fig. 6a) exist at any frequency and they do not interact with each other. The second anti-phase wave and the second in-phase wave are generated at $\Omega_{\text{cut-on}} \approx 0.193$ and $\Omega_{\text{cut-on}} \approx 0.299$, see curves 3, 4 in Fig. 6a. Their interaction results in the transformation of this pair of propagating waves at $\Omega_{\text{cut-on}} \approx 0.458$ into a pair of attenuated waves, see also Fig. 6b, where a real part of the complex-valued wavenumber is presented.

Finally, the case of heavy fluid loading on one side of a plate is considered with compressibility taken into account. The dispersion equation is

$$F_1F_2 + \beta^2 z_{22}z_{44}\left|k^2 - \left(\frac{\omega h}{c_-}\right)\right|^{-1} = 0. \tag{33}$$

It is convenient to transform this equation as

$$\left[F_1F_2\left(k^2 - \left(\frac{\omega h}{c_-}\right)\right)\right]^2 - \left(\beta^2 z_{22}z_{44}\right)^2 = 0. \tag{34}$$

Then the perturbation technique is applied. This equation has twice as many roots as the original one (33), so that each root obtained either directly or via perturbation method is checked whether it satisfies Eq. (33) and whether it satisfies the Sommerfeld principle. Then it appears that in the case of loading of a plate with $h_1/h = 0.1$, $E/E_1 = 0.01$, $\rho/\rho_1 = 0.1$ by water $\rho_-/\rho_1 = 0.128$, $c_-/c_1 = 0.307$, only two propagating waves exists—in contrast with the predictions obtained when a model of an incompressible fluid is used. The dispersion curves are shown in Fig. 7, curve 1 presents the propagating in-phase wave and curve 2 presents the propagating anti-phase wave. Their location is not markedly altered by the effect of compressibility. This

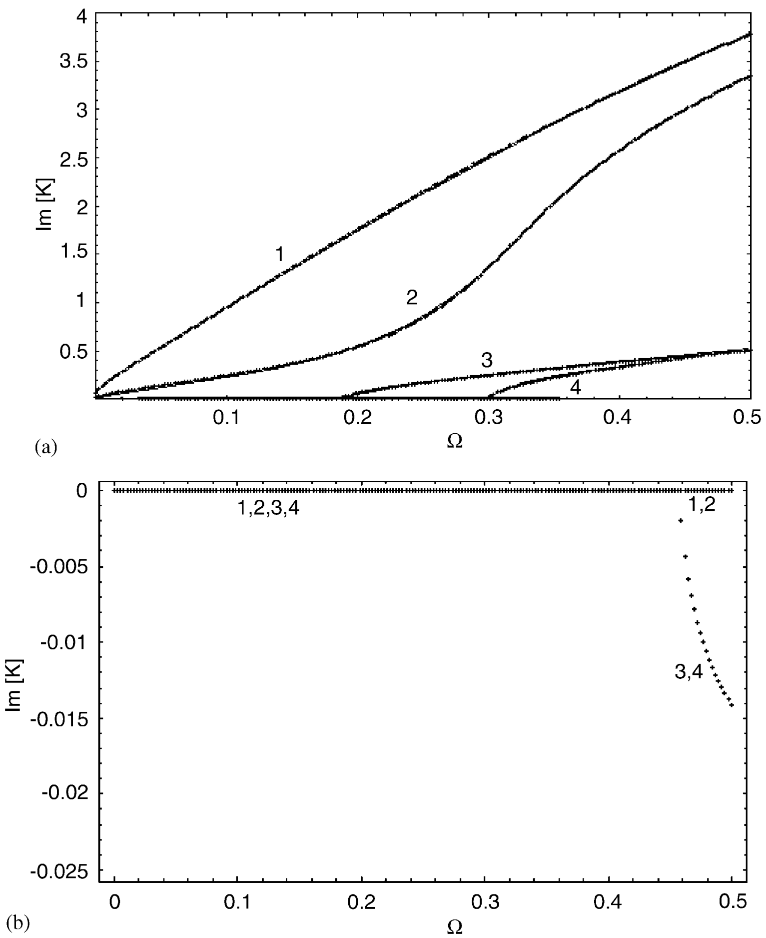

Fig. 6. Dispersion curves (a—imaginary parts; b—real parts) for a sandwich plate with one side loading by an incompressible fluid. Curves 1, 3—'in-phase' waves; curves 2, 4—'anti-phase' waves.

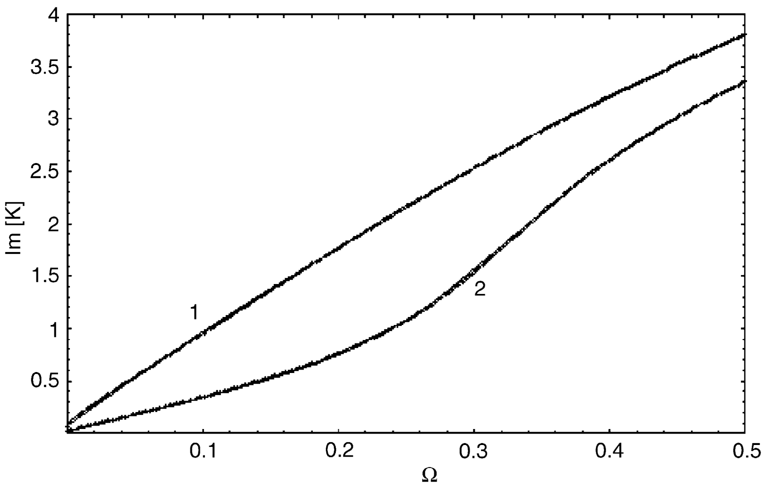

Fig. 7. Dispersion curves for a sandwich plate with one side loading by a compressible fluid. Curve 1—'in-phase' wave; curve 2—'anti-phase' wave.

result has been already observed in Refs. [1,11] and the location of a dispersion curve for the in-phase mode agrees very well with the predictions for the wave of the same type reported in Ref. [11].

6. Conclusions

The system of equations which describes propagation of acoustic and elastic harmonic waves in a sandwich plate of arbitrary composition with heavy fluid loading on both sides is derived. This system is specialised for the practically meaningful case of symmetric composition of such a plate and a classic perturbation theory is applied to study the symmetry-breaking coupling effects in wave propagation introduced by uneven pre-stress or by uneven fluid loading. A very good agreement between the results of asymptotic analysis in the cases of regular and singular perturbations and the results of direct solution is observed.

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