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Natural frequencies of bonded and unbonded prestressed beams–prestress force effects

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Abstract

The paper rigorously presents the effect of the magnitude of the prestress force on the natural frequencies of prestressed beams with bonded and unbonded tendons. A nonlinear analytical model is formulated for the dynamic behavior of prestressed beams, in order to achieve this goal. The equations of motion for a prestressed beam and its associated boundary and continuity conditions are rigorously derived using the variational principle of virtual work following Hamilton's principle. The mathematical model is rigorous and general, and is valid for any kind of boundary and continuity conditions as well as any cable layout. The kinematic relations for the concrete are those of large displacements and moderate rotations, in order to take into account the compressive force effect caused by the prestress force. For the tendons, the same kinematic relation has been used including the cable profile as an imperfection. The effect of the magnitude of the prestress force on the natural frequencies is mathematically examined. Numerical examples are presented to illustrate the difference between natural frequencies determined by the proposed model and other models mentioned in the literature.

1. Introduction

In the case of bridge girders made of prestressed beams, it is necessary to predict the natural frequencies of such girders in order to determine the dynamic magnification factor of the vehicles' load and to follow some of the design constraints. Natural frequencies of non-prestressed beams are available in many textbooks; see for example Timoshenko et al. [1]. However, there are very few papers that deal with the natural frequencies of prestressed beams.

The behavior of a prestressed beam can be described as the combination of two substructures: a compressive concrete beam and a tensioned cable (see Fig. 1). The natural frequency of the compressed beam and the prestressed cable separately are available in the literature. The natural frequency of a simply supported axially compressed beam (see Ref. [1]) is

$$\omega_n = \frac{n\pi}{L_{\text{beam}}} \sqrt{\frac{1}{m_{\text{beam}}} \left[EI \left(\frac{n\pi}{L_{\text{beam}}} \right)^2 - N \right]} \tag{1}$$

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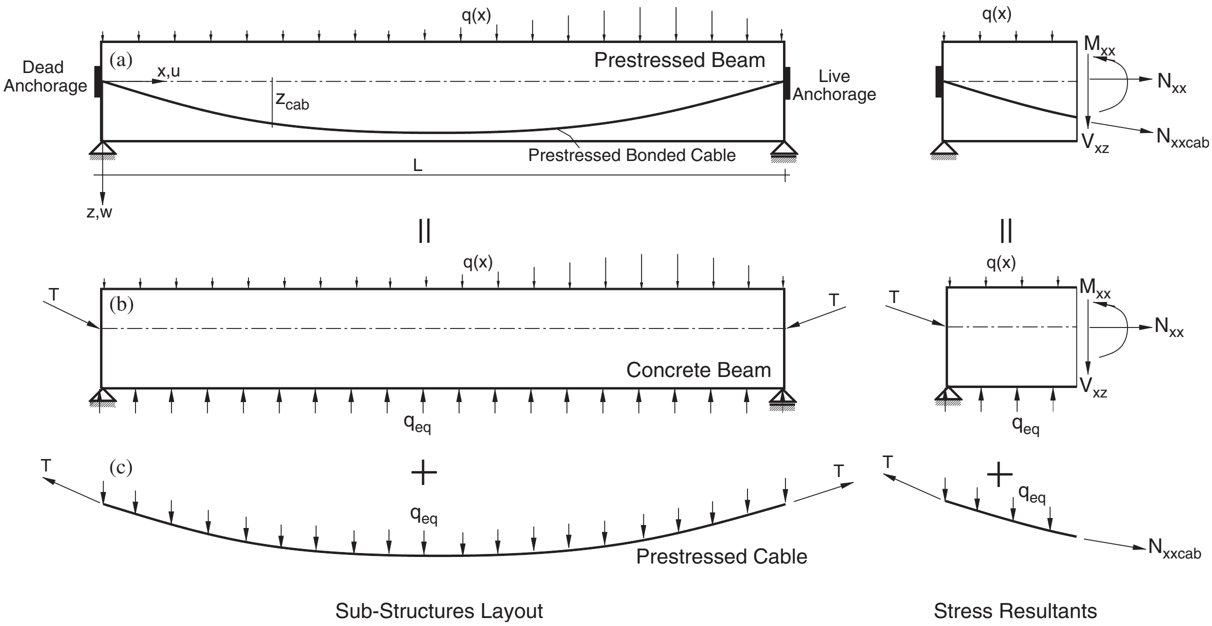

Fig. 1. Substructure layout and internal stress resultants of a typical prestressed beam: (a) prestressed beam; (b) concrete beam; and (c) tensioned cable.

where EI is the flexural rigidity of the beam, L_{beam} is the length of the beam, m_{beam} is the mass per unit length of the beam, n is the mode number and N is the axial compressive force. The natural frequency of a tensioned cable (string) (see Ref. [2]) is

$$\omega_n = \frac{n\pi}{L_{\text{cable}}}\sqrt{\frac{T}{m_{\text{cable}}}} \tag{2}$$

where L_{cable} is the length of the cable, m_{cable} is the mass per unit length of the cable and T is the magnitude of the tensile force. Eqs. (1) and (2) reveal that by increasing the magnitude of the axial force, the natural frequency decreases in the case of a compressed beam (‘‘compression softening’’), and increases in the case of a tensioned cable. Hence, the free vibration of a prestressed beam that combines the two substructures requires a full consideration of the free vibration behavior of the two substructures.

Saiidi et al. [3] determined the natural frequencies of a prestressed concrete bridge using Eq. (1). The results indicate that an increase in the magnitude of the prestressed force reduces the natural frequencies of prestressed beams. The authors in [3] have not considered the perturbed cable tension and eccentricity due to the beam vibrations. Hence, their use of Eq. (1) to calculate the natural frequencies of the prestressed beam is erroneous. The paper by Saiidi et al. [3] has been followed by three discussions, as follows: Dallasta and Dezi [4] indicate that the effect of the prestress force on the beam bending vibration frequencies is negligible based on a linear model. Thus the nonlinear effects associated with the nonlinear change in the eccentricity of the compressive force in the concrete beam and the tension force in the cable are beyond the capabilities of the proposed model. Deak [5] claims that prestress force does not reduce the natural frequencies in either case. Nevertheless, this claim is not supported by any analytical nor mathematical proof. Jain and Goel [6] indicate that the prestress force is an internal force which does not cause ‘‘compression softening’’ effect. Again, this discussion lacks an appropriate mathematical model to verify their argument.

Also Kanaka and Venkateswara [7] have showed that the prestress force reduces the natural frequency of the lower modes, based on a Rayleigh–Ritz formulation that describes prestressing as an external axial compressive force only. Miyamoto et al. [8] dealt with the dynamic behavior of a prestressed beam strengthened with external tendons. They have considered the change in the tendon force along with the compressive force effect, by an incremental formulation of the equations of motion of the beam. All the calculated natural frequencies tend to decrease as the amount of the prestressing force increases. These results are due to the fact that the change in the tendon eccentricity has been ignored. Chan and Yung [9] indicate that the natural frequencies of a prestressed bridge decrease with the increase of the prestressing forces due to the "compression softening". It is seen that many studies agree that prestressing tends to decrease the natural frequencies, whereas some discussions [4–6] report that prestressing has no or negligible effect on the natural frequencies of prestressed beams. Hence, these opposite conclusions require further investigation.

Dallasta and Leoni [10] have presented a general formulation for the vibration of beams, prestressed by internal frictionless cables using kinematic relations of small displacements for the concrete beam. The formulation for the beam does not include the effect of the compressive force, and yields erroneous results. However, although Dallasta and Leoni [10] have not considered the compressive force effect, they indicated that the natural frequencies decrease as the prestress force increases. Kerr [11] studied experimentally and analytically the dynamic response of a prestressed beam. It was found that the magnitude of the prestress force for a cable that passes through the centroid of the beam cross-section has no effect upon the dynamic response of the beam. The analytical model uses only a linear formulation for the study of the dynamic response of the prestressed beam. This approach has been adopted for simplicity and has not been proven. Furthermore, the simple linear analysis that has been presented by Kerr [11] is not able to determine the change in the cable force and the cable eccentricity during the vibration of the beam, and it is limited to straight cables that pass through the centroid of the beam. Therefore, to the best of the authors' knowledge, there is no rigorous mathematical model that is able to properly define the effect of the prestress force on the natural frequencies of prestressed beams.

The present dynamic analysis of vertical free vibrations takes into account the compressive force effect, the changes in the prestressing force and cable eccentricity through a nonlinear model for bonded and unbonded prestressed beams. The analysis uses variational principles following Hamilton's principle, to formulate the nonlinear equations of motion of the prestressed beam along with the appropriate boundary and continuity conditions. The dynamic behavior of the prestressed beam is described by two substructures, which are interconnected through equilibrium and compatibility requirements (see Fig. 1). The two substructures consist of a concrete beam and a pretensioned cable. It is assumed that the behavior of the concrete beam follows Bernoulli–Euler's assumptions, and the substructures undergo moderate deformations, i.e., large displacements and moderate rotations in order to address the effect of the prestressed force on the vibration behavior of the beam. The constitutive relations of the concrete and the prestressed reinforcement follow Hooke's law. The effect of the longitudinal vibrations and the rotary inertia are negligible.

The mathematical formulation of the problem is presented for the case of bonded and unbonded tendons. Two numerical examples are subsequently presented, which demonstrate the difference between the natural frequencies determined by the proposed model and the other models that appear in the literature. At the end of the paper, a summary and conclusions are presented.

2. Mathematical formulation

2.1. Bonded tendons

The governing equations of motion of a typical prestressed beam with bonded tendons, after the prestress stage, and the appropriate boundary and continuity conditions are derived via Hamilton's principle, which requires that

$$\delta L = \int_{t_1}^{t_2} \delta(T - (U + V))\,\mathrm{d}t = 0 \tag{3}$$

where T is the kinetic energy, U is the internal potential energy, V is the external potential energy and equal zero in the case of free vibration, δ is the variational operator and t is the time coordinate.

The first variation of the kinetic energy is

$$\delta T = \int_{V_{\mathrm{beam}}} \rho \dot{w} \delta \dot{w}\,\mathrm{d}v \tag{4}$$

where ρ is the mass density of the concrete (the mass of the cable is neglected), $\dot{w}$ is the velocity of the concrete beam in the vertical direction, $(\dot{})$ denotes a derivative with respect to time, V_{beam} is the volume of the concrete beam. The sign convention and the stress resultants appear in Fig. 1.

The first variation of the internal potential energy is given as

$$\delta U = \int_{V_{\text{beam}}} \sigma_{xx}\delta\varepsilon_{xx}\,\mathrm{d}v + \int_{V_{\text{cab}}} \sigma_{xx\text{cab}}\delta\varepsilon_{xx\text{cab}}\,\mathrm{d}v + \int_0^{\mathrm{L}} \delta[\lambda_1(x)(w - w_{\text{cab}})]\,\mathrm{d}x + \int_0^{\mathrm{L}} \delta[\lambda_2(x)(u - u_{\text{cab}})]\,\mathrm{d}x \tag{5}$$

where σ_{xx} and ε_{xx} are the longitudinal normal stress and strain in the concrete beam respectively, σ_{xxcab} and $\varepsilon_{xx\text{cab}}$ are the stress and strain in the cable respectively, V_{cab} is the volume of the cable, $\lambda_1(x)$ and $\lambda_2(x)$ are Lagrange multipliers which are actually the vertical and longitudinal components of the force that the prestress cable exerts on the concrete beam, and impose identical deformations for the concrete beam and the cable along the beam, w_{cab} and u_{cab} are the vertical and longitudinal deformations of the cable respectively, w is the vertical deformation of the concrete beam and u is the longitudinal deformation of the concrete beam at the cable profile along the beam, i.e.,

$$u = u_0 - z_{\text{cab}} w_{,x} \tag{6}$$

where u_0 is the longitudinal deformation of the concrete beam at its centroid, z_{cab} is the cable eccentricity measured downwards from the center of gravity of the concrete beam, $(\,)_{,x}$ denotes differentiation with respect to x, see Fig. 1.

The kinematic relations for the concrete beam are based on large displacements with moderate rotations, thus

$$\varepsilon_{xx}(x, z) = u_{0,x} + \tfrac{1}{2}(w_{,x})^2 - z w_{,xx} \tag{7}$$

The kinematic relation for the cable includes also the cable profile as an imperfection, thus

$$\varepsilon_{xx\text{cab}}(x, z_{\text{cab}}) = u_{\text{cab},x} + \tfrac{1}{2}(w_{\text{cab},x})^2 + z_{\text{cab},x} w_{\text{cab},x} \tag{8}$$

The governing equations of motion and the associated boundary and continuity conditions are derived using Eqs. (3)–(6), along with the use of the kinematic relations (7)–(8). Hence, after some algebraic manipulations and integration by parts, the governing equations lead

$$-m\ddot{w} + (N_{xx} w_{,x})_{,x} + M_{xx,xx} - \lambda_1 - (\lambda_2 z_{\text{cab}})_{,x} = 0 \tag{9}$$

$$N_{xx,x} - \lambda_2 = 0 \tag{10}$$

$$N_{xx\text{cab},x} + \lambda_2 = 0 \tag{11}$$

$$\left(N_{xx\text{cab}} w_{\text{cab},x}\right)_{,x} + \left(N_{xx\text{cab}} z_{\text{cab},x}\right)_{,x} + \lambda_1 = 0 \tag{12}$$

$$-w + w_{\text{cab}} = 0 \tag{13}$$

$$-u_0 + z_{\text{cab}} w_{,x} + u_{\text{cab}} = 0 \tag{14}$$

where m is the mass per unit length of the concrete beam, N_{xx} and $N_{xx\text{cab}}$ are the stress resultants in the longitudinal direction in the concrete beam and the cable respectively, and M_{xx} is the bending moment stress resultant in the concrete beam, which take the following form:

$$N_{xx} = \int_{A_{\text{beam}}} \sigma_{xx}\,\mathrm{dA}, \quad N_{xx\text{cab}} = \int_{A_{\text{cab}}} \sigma_{xx\text{cab}}\,\mathrm{dA}, \quad M_{xx} = \int_{A_{\text{beam}}} \sigma_{xx} z\,\mathrm{dA} \tag{15}$$

Note that the global bending moment of the prestressed beam equals

$$M_{xx,\text{global}} = M_{xx} + N_{xx\text{cab}} z_{\text{cab}} \tag{16}$$

The boundary conditions lead

$$N_{xx}+N_{xx\text{cab}}=0 \quad \text{or} \quad u_0=0 \tag{17}$$

$$N_{xx}w_{,x}+M_{xx,x}+N_{xx\text{cab},x}z_{\text{cab}}+N_{xx\text{cab}}\left(w_{,x}+z_{\text{cab},x}\right)=0 \quad \text{or} \quad w=0 \tag{18}$$

$$M_{xx}+N_{xx\text{cab}},z_{\text{cab}}=0 \quad \text{or} \quad w_{,x}=0 \tag{19}$$

The unknown deformations may take the following form in the case of free vibrations:

$$w(x,t)=w(x)\sin(\omega t+\phi), \quad u(x,t)=u(x)\sin(\omega t+\phi) \tag{20}$$

$$w_{\text{cab}}(x,t)=w_{\text{cab}}(x)\sin(\omega t+\phi), \quad u_{\text{cab}}(x,t)=u_{\text{cab}}(x)\sin(\omega t+\phi) \tag{21}$$

where ω is the natural frequency of the prestressed beam and ϕ is a phase angle.

The equations of motion are algebraic and nonlinear differential equations. Hence, after some algebraic manipulations and substitution of Eqs. (20)–(21) in Eqs. (9)–(14), the equations of motion are reduced to two nonlinear governing equations along with the appropriate boundary and continuity conditions (17)–(19), as follows

$$m\omega^2 w+M_{xx,xx}+((N_{xx\text{cab}}+N_{xx})w_{,x})_{,x}+\left(N_{xx\text{cab}}z_{\text{cab},x}\right)_{,x}-\left(N_{xx,x}z_{\text{cab}}\right)_{,x}=0 \tag{22}$$

$$N_{xx,x}+N_{xx\text{cab},x}=0 \tag{23}$$

where

$$N_{xx\text{cab}}=EA_{\text{cab}}\left[\frac{N_{xx}}{EA}-z_{\text{cab}}w_{,xx}\right] \tag{24}$$

where EA and EA_{cab} are the inplane rigidities of the concrete beam and the cable, respectively.

In general, the governing equations are nonlinear with variable coefficients, and are solved using the Multiple Shooting method (see Ref. [12]). The natural frequency ω is determined using Yuan [13] approach, which adds an additional unknown Φ that equals:

$$\Phi=\int_x mw^2\,\mathrm{d}x \tag{25}$$

The function Φ and the natural frequency ω are determined through the use of additional two first order differential equations and their associated two boundary conditions, thus

$$\Phi_{,x}=mw^2, \quad \omega_{,x}=0 \tag{26}$$

$$\Phi(0)=0, \quad \Phi(L)=C \tag{27}$$

where C is the value of the normalized eigenmodes, and L is the length of the prestressed beam. Here, the governing equations of motion are replaced by a set of seven first-order differential equations along with their associated seven boundary conditions.

Note that Eq. (23) along with the boundary condition (17) for the movable edge of the beam yields

$$N_{xx}(x)+N_{xx\text{cab}}(x)=0 \tag{28}$$

Hence, the global axial force in the prestressed beam is null through the length of the beam, which changes Eq. (22) into a linear differential equation. Although the last two terms of Eq. (22) still survive, those are linear terms with variable coefficients, since the eccentricity of the cable (z_{cab}) is a defined function. Hence, the superposition principle may be used for all solution steps, and the solution of the free vibration problem of a prestressed beam does not require considering the stresses that exist in the beam prior to the vibration process. It means that the magnitude of the prestress forces does not affect the natural frequencies of prestressed beams.

The above result can also be proved by an incremental formulation that considers the prestress forces and stresses that exist in the beam prior to the vibration stage. The incremental formulation is also achieved through Hamilton's principle, where it is assumed that all stresses and internal forces prior to the vibration stage are known functions. The incremental variational formulation yields the following incremental equations of motion:

$$
\begin{aligned}
&-m\Delta\ddot{w}+((N_{xx}+\Delta N_{xx})(\Delta w_{,x}+w_{,x}))_{,x}+M_{xx,xx}+\Delta M_{xx,xx}-(\lambda_1+\Delta\lambda_1)\\
&-((\lambda_2+\Delta\lambda_2)z_{\mathrm{cab}})_{,x}+q=0
\end{aligned} \tag{29}
$$

$$\left(N_{xx,x}+\Delta N_{xx,x}\right)-(\lambda_2+\Delta\lambda_2)=0 \tag{30}$$

$$\left(N_{xx\mathrm{cab},x}+\Delta N_{xx\mathrm{cab},x}\right)+(\lambda_2+\Delta\lambda_2)=0 \tag{31}$$

$$\left((N_{xx\mathrm{cab}}+\Delta N_{xx\mathrm{cab}})\left(\Delta w_{\mathrm{cab},x}+z_{\mathrm{cab},x}+w_{\mathrm{cab},x}\right)\right)_{,x}+(\lambda_1+\Delta\lambda_1)=0 \tag{32}$$

$$-\Delta w+\Delta w_{\mathrm{cab}}=0 \tag{33}$$

$$-\Delta u_0+z_{\mathrm{cab}}\Delta w_{,x}+\Delta u_{\mathrm{cab}}=0 \tag{34}$$

where the Δ operator refers to the forces and the deformations at the present increment, while all other functions are known functions, q is the static load prior to the vibration stage. These equations may be simplified through substitution of the equations of static equilibrium, and neglecting the second order terms (Δ^2). Hence, the simplified incremental equations of motion read:

$$-m\Delta\ddot{w}+(N_{xx}\Delta w_{,x})_{,x}+(\Delta N_{xx}w_{,x})_{,x}+\Delta M_{xx,xx}-\Delta\lambda_1-(\Delta\lambda_2 z_{\mathrm{cab}})_{,x}=0 \tag{35}$$

$$\Delta N_{xx,x}-\Delta\lambda_2=0 \tag{36}$$

$$\Delta N_{xx\mathrm{cab},x}+\Delta\lambda_2=0 \tag{37}$$

$$\left(N_{xx\mathrm{cab}}\Delta w_{\mathrm{cab},x}\right)_{,x}+\left\lfloor\Delta N_{xx\mathrm{cab}}\left(z_{\mathrm{cab},x}+w_{\mathrm{cab},x}\right)\right\rfloor_{,x}+\Delta\lambda_1=0 \tag{38}$$

$$-\Delta w+\Delta w_{\mathrm{cab}}=0 \tag{39}$$

$$-\Delta u_0+z_{\mathrm{cab}}\Delta w_{,x}+\Delta u_{\mathrm{cab}}=0 \tag{40}$$

These incremental equations reveal that the axial forces N_{xx} and $N_{xx\mathrm{cab}}$ may affect the dynamic response of the beam. However, after some algebraic manipulations, while keeping in mind that the axial forces prior to that increment are $N_{xx\mathrm{cab}}=-N_{xx}=T_\infty$ (T_∞ is the prestress force after losses), the equations of motion change into the following two governing differential equations:

$$m\omega^2\Delta w+\Delta M_{xx,xx}+\left(\Delta N_{xx\mathrm{cab}}z_{\mathrm{cab},x}\right)_{,x}-\left(\Delta N_{xx,x}z_{\mathrm{cab}}\right)_{,x}=0 \tag{41}$$

$$\Delta N_{xx,x}+\Delta N_{xx\mathrm{cab},x}=0 \tag{42}$$

where

$$\Delta N_{xx\mathrm{cab}}=EA_{cab}\left[\frac{\Delta N_{xx}}{EA}-z_{\mathrm{cab}}\Delta w_{,xx}\right] \tag{43}$$

These equations reveal that the effect of the static axial forces disappears. Hence, the magnitude of the prestress force does not affect the natural frequencies of prestressed beam, as opposed to some of the papers mentioned in the literature survey.

The proposed model offers an enhancement and generalization of the models that appear in the literature. The compressed beam model [3,7–9] does not consider the prestressing cable at all. Hence, the equations of this model can be generated from the proposed model by omitting the equations of motion of the cable (Eqs. (37,38)) and the compatibility equations (Eqs. (39,40)), and by neglecting the Lagrange multipliers from the other equations. Thus, the equations of motion of this model are as follows:

$$-m\Delta\ddot{w}+(N_{xx}\Delta w_{,x})_{,x}+(\Delta N_{xx}w_{,x})_{,x}+\Delta M_{xx,xx}=0 \tag{44}$$

$$\Delta N_{xx,x} = 0 \tag{45}$$

The second model that appears in literature [4], uses kinematic relations of small displacements for the concrete beam as follows:

$$\varepsilon_{xx}(x,z) = u_{0,x} - zw_{,xx} \tag{46}$$

The use of this kinematic relation cancels the nonlinear terms in the governing equation of the concrete beam, and leads to the following equations of motion:

$$-m\Delta\ddot{w} + \Delta M_{xx,xx} - \Delta\lambda_1 - (\Delta\lambda_2 z_{\text{cab}})_{,x} = 0 \tag{47}$$

$$\Delta N_{xx,x} - \Delta\lambda_2 = 0 \tag{48}$$

$$\Delta N_{xx\text{cab},x} + \Delta\lambda_2 = 0 \tag{49}$$

$$\left(N_{xx\text{cab}}\Delta w_{\text{cab},x}\right)_{,x} + \left(\Delta N_{xx\text{cab}}(z_{\text{cab},x} + w_{\text{cab},x})\right)_{,x} + \Delta\lambda_1 = 0 \tag{50}$$

$$-\Delta w + \Delta w_{\text{cab}} = 0 \tag{51}$$

$$-\Delta u_0 + z_{\text{cab}}\Delta w_{,x} + \Delta u_{\text{cab}} = 0 \tag{52}$$

These equations can be solved using the Multiple Shooting method. It should be noted that the magnitude of the static axial force in the cable $N_{xx\text{cab}}$ prior to the vibration stage, affects the dynamic behavior of the beam in this model. Hence, although the prestressed beam vibrates in small displacements at service condition, a nonlinear approach that accounts for large displacements and moderate rotations should be conducted, in order to describe the effect of the axial forces in the concrete beam and the tendons on the overall vibration behavior of the prestressed beam. Such a model has been developed earlier in this paper.

The difference between the natural frequencies determined by the proposed model and the other models that appear in the literature is presented in Section 3 through some numerical examples.

2.2. Unbonded tendons

Following the same procedure, the governing equations of motion and the appropriate boundary conditions of a typical prestressed beam with unbonded tendons after the prestress stage are derived. The first variation of the kinetic energy is

$$\delta T = \int_{V_{\text{beam}}} \rho\dot{w}\delta\dot{w}\,\mathrm{d}v \tag{53}$$

The first variation of the internal potential energy takes into account the change in the prestress force of the unbonded tendon, through a constraint that states that the change in the length of the concrete fiber that is adjacent to the cable equals the change in the length of the unbonded tendon, see fourth term in the equation that follows:

$$\begin{aligned}\delta U = &\int_{V_{\text{beam}}} \sigma_{xx}\delta\varepsilon_{xx}\,\mathrm{d}v + \int_{V_{\text{cab}}} \sigma_{xx\text{cab}}\delta\varepsilon_{xx\text{cab}}\,\mathrm{d}v + \int_0^L \delta[\lambda_1(x)(w - w_{\text{cab}})]\,\mathrm{d}x \\ &+ \delta\left[\lambda\left(\int_0^L \varepsilon_{xx}(z = z_{\text{cab}})\,\mathrm{d}x - \int_0^L \varepsilon_{xx\text{cab}}\,\mathrm{d}x\right)\right]\end{aligned} \tag{54}$$

where λ is a Lagrange multiplier that represents the increase in the prestressing force, which is assumed to be uniform along the beam since there is no friction between the tendon and the concrete beam in the case of unbonded tendon, $\varepsilon_{xx}(z = z_{\text{cab}})$ is the longitudinal strain of the concrete beam at the cable profile along the beam (see Eq. (7)). Following Hamilton's principle (see Eq. (3)), the equations of motion are derived by integration by parts of Eqs. (53)–(54) along with the use of the kinematic relations (see Eqs. (7–8)), and they read

$$-m\ddot{w} + (N_{xx}w_{,x})_{,x} + M_{xx,xx} - \lambda_1 = 0 \tag{55}$$

$$N_{xx,x}=0 \tag{56}$$

$$N_{xx\text{cab},x}=0 \tag{57}$$

$$\big(N_{xx\text{cab}}(z_{\text{cab},x}+w_{\text{cab},x})\big)_{,x}+\lambda_1=0 \tag{58}$$

$$-w+w_{\text{cab}}=0 \tag{59}$$

Integration by parts of the fourth term of Eq. (54) yields the following expression:

$$\begin{aligned}&\lambda\left(\delta u_{\text{cab}}\big|_0^L+\int_0^L w_{\text{cab},x}\delta w_{\text{cab},x}+z_{\text{cab}}\delta w_{\text{cab},x}\big|_0^L-z_{\text{cab}}\delta w_{\text{cab},xx}\big|_0^L-\delta u_0\big|_0^L-\int_0^L w_{,x}\delta w_{,x}+z_{\text{cab}}\delta w_{,xx}\big|_0^L\right)\\&+\delta\lambda\left(u_{\text{cab}}\big|_0^L+\int_0^L \tfrac{1}{2}w_{\text{cab},x}^2+z_{\text{cab}}w_{\text{cab},x}\big|_0^L-z_{\text{cab}}w_{\text{cab},xx}\big|_0^L-u_0\big|_0^L-\int_0^L \tfrac{1}{2}w_{,x}^2+z_{\text{cab}}w_{,xx}\big|_0^L\right)=0\end{aligned} \tag{60}$$

Substitution of Eq. (59) into Eq. (60) yields the following expression:

$$\lambda\left(\delta u_{\text{cab}}\big|_0^L+z_{\text{cab}}\delta w_{,x}\big|_0^L-\delta u_0\big|_0^L\right)+\delta\lambda\left(u_{\text{cab}}\big|_0^L+z_{\text{cab}}w_{,x}\big|_0^L-u_0\big|_0^L\right)=0 \tag{61}$$

Eq. (61), along with integration by parts of Eq. (54) yield the boundary conditions of the problem, as follows:

$$N_{xx\text{cab}}-\lambda=0 \quad \text{or} \quad u_{\text{cab}}=u_0-z_{\text{cab}}w_{,x} \tag{62}$$

$$N_{xx}+\lambda=0 \quad \text{or} \quad u_0=0 \tag{63}$$

$$N_{xx}w_{,x}+M_{xx,x}+N_{xx\text{cab},x}z_{\text{cab}}+N_{xx\text{cab}}\left(w_{,x}+z_{\text{cab},x}\right)=0 \quad \text{or} \quad w=0 \tag{64}$$

$$M_{xx}+\lambda\, z_{\text{cab}}=0 \quad \text{or} \quad w_{,x}=0 \tag{65}$$

Notice that λ is actually the change in the magnitude of the force in the tendon at its edges only. However, since the change in the force of the tendon is independent of x, see Eq. (57), any change in the force at the edges affects the prestressing force throughout the length of the beam.

The equations of motion are algebraic and nonlinear differential equations. After some algebraic manipulations, the equations of motion are reduced to three nonlinear governing equations as follows:

$$m\omega^2 w+M_{xx,xx}+((N_{xx\text{cab}}+N_{xx})w_{,x})_{,x}+N\left(N_{xx\text{cab}}z_{\text{cab},x}\right)_{,x}=0 \tag{66}$$

$$N_{xx,x}=0 \tag{67}$$

$$N_{xx\text{cab},x}=0 \tag{68}$$

Please notice that Eqs. (67)–(68) along with the boundary conditions (62)–(63) for the movable edge of the beam yields

$$N_{xx}(x)+N_{xx\text{cab}}(x)=0 \tag{69}$$

Hence, the global axial force in the prestressed beam is null through the length of the beam, which changes Eq. (66) into a linear differential equation. Again the solution of the free vibration problem of a prestressed beam with unbonded tendons does not require considering the stresses that exist in the beam prior to the vibration process, since the equations of motion are linear ones. It leads to the conclusion that the magnitude of the prestress forces *does not* affect the natural frequencies of prestressed beams with unbonded tendons also.

For comparison of the proposed model with other models that appear in the literature, the equations of motion that are based on the small deformations theory for the concrete beam, and those based on the compressed beam model are derived using the proposed general model. The governing equation of motion in the case of small deformations are nonlinear ones, hence an incremental formulation is used. These equations are

$$m\omega^2\Delta w+\Delta M_{xx,xx}+N_{xx\text{cab}}\Delta w_{,x}+\Delta N_{xx\text{cab}}w_{,x}+\left((N_{xx\text{cab}}+\Delta N_{xx\text{cab}})z_{\text{cab},x}\right)_{,x}=0 \tag{70}$$

$$\Delta N_{xx,x} = 0 \tag{71}$$

$$\Delta N_{xx\text{cab},x} = 0 \tag{72}$$

where the Δ operator refers to the forces and the deformations at the present increment, while all other functions are known functions. It is seen that the magnitude of the prestress force prior to the vibration of the beam, affects the bending vibration behavior when the small deformations theory is conducted. However, this approach does not account for the nonlinear effects of the compressed force on the behavior of the concrete beam.

The equations of motion of the compressed beam model are similar to those presented in the bonded case (see Eqs. (44,45)). Numerical examples are presented ahead, that illustrate the difference between the natural frequencies determined by the proposed model and the other models that appear in the literature.

3. Numerical examples

Two numerical examples are presented. The first deals with free vibration of prestressed bridge beam with straight bonded tendons and the second one considers a prestressed beam with a parabolic bonded and unbonded cable. A quantitative comparison between the natural frequencies calculated by the proposed model, which are independent of the prestressing force, and the ones calculated by other models that appear in the literature is presented for the two numerical examples. Such models, see mathematical formulation, include the compressed beam model and the one that is based on the small displacements theory. Note that the value of the natural frequencies that have been determined based on these models depends on the magnitude of the prestressing force. Hence, the numerical study quantitatively describes the effect of the prestressing force on the natural frequencies, as determined by the other models, and compares them with those determined by the proposed model.

3.1. Example 1

A simply supported prestressed bridge beam with straight tendons is considered, see Fig. 2 (see Ref. [14]). The natural frequencies determined by the proposed model are

$$\omega_1 = 35.3225\ \text{rad/s},$$
$$\omega_2 = 141.29\ \text{rad/s},$$
$$\omega_3 = 317.9025\ \text{rad/s}.$$

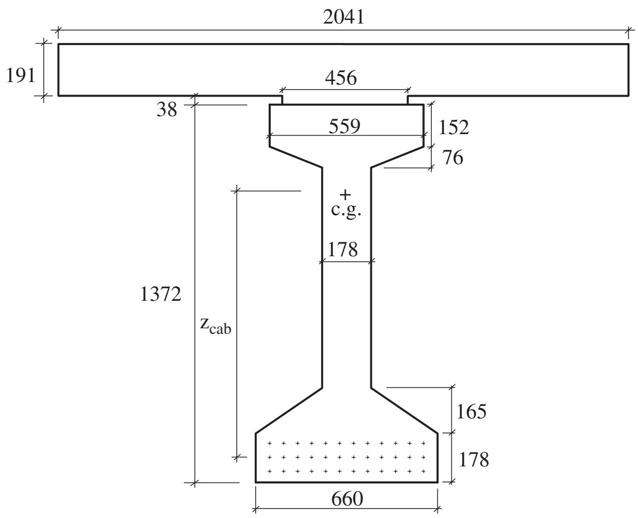

Fig. 2. Cross-section of a prestressed beam with straight tendons (dimensions in mm, $L = 25\,\text{m}$, $A_c = 848121\,\text{mm}^2$, $I_c = 2646\text{e}11\,\text{mm}^4$, $E_c = 35\,\text{GPa}$, $z_{\text{cab}} = 940\,\text{mm}$, $E_{\text{cab}} = 195\,\text{GPa}$, $T_\infty = 2000\,\text{kN}$).

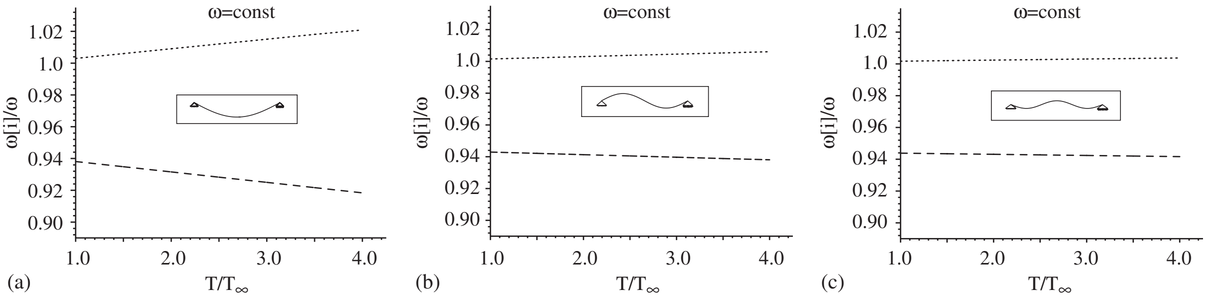

Fig. 3. First three natural frequencies of a prestressed beam with straight tendons versus prestress force, calculated by the proposed model and the other models, see Refs. [3–6]: (a) first mode; (b) second mode; (c) third mode (legend: ω—proposed model, ω[1]—compressed beam model, ω[2]—small displacement model, - - - - ω[1]/ω, …… ω[1]/ω).

Comparison of the natural frequencies calculated by the proposed model and the aforementioned models appear in Fig. 3. The comparison indicates that as the magnitude of the prestress force increases, the natural frequencies based on the compressed beam model tend to decrease and diminish to approximately 92% of the first natural frequency determined by the proposed model. This is a result of the improper modeling of the effect of the prestress force conducted by the compressed beam model, which describes the prestress force as an external axial compressed load that tends to decrease the natural frequencies of the prestressed beam. On the other hand, the natural frequencies that are determined based on the small displacements theory tend to increase, and are about 103% of the first natural frequency of the proposed model. This model does not consider the nonlinear effects of the compressed prestress force on the behavior of the concrete beam, and hence leads to magnified natural frequencies as a result of the stiffening effect caused by the tensioned cable. Notice that the variation of the prestress force predominantly affects the first natural frequency. In the higher modes, the compressed beam model predicts natural frequencies that are less than 95% of those evaluated by the proposed model, already for small magnitudes of prestress force. This indicates a fundamental error in this model.

However, it is possible to determine the natural frequencies of prestressed beams by the linear elastic beam theory [1]. Hence, the natural frequencies that are determined through the following equation are

$$\omega_n = n^2\pi^2\sqrt{\frac{EI}{mL^4}} \tag{73}$$

$$\omega_1 = 33.3642\ \mathrm{rad/s},$$
$$\omega_2 = 133.4568\ \mathrm{rad/s},$$
$$\omega_3 = 300.2778\ \mathrm{rad/s}.$$

The difference between the proposed model and the linear beam model is that the linear model does not consider the change in the cable force during vibration. Hence, it should be noted that identical results with those achieved by the proposed model may be determined by using EI_{eq} instead of EI in Eq. (73), where EI_{eq} is the equivalent flexural rigidity of the composite section, concrete and cable.

3.2. Example 2

The second example deals with a simply supported prestressed roof beam, with parabolic cable, see Fig. 4 [14]. Two cases are studied: bonded cable and unbonded one. The natural frequencies of the prestressed beam with bonded cable that are determined by the proposed model are

$$\omega_1 = 16.6378\ \mathrm{rad/s},$$
$$\omega_2 = 65.3338\ \mathrm{rad/s},$$
$$\omega_3 = 146.8435\ \mathrm{rad/s}.$$

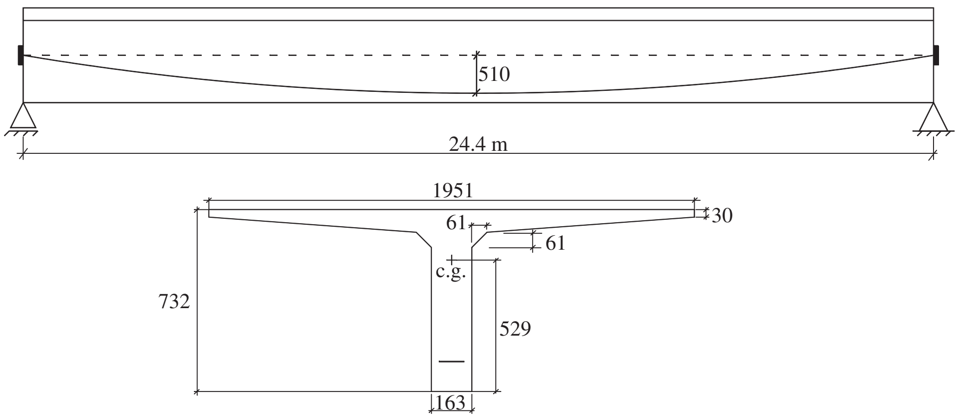

Fig. 4. Geometry, cross-section and mechanical properties of a prestressed beam with a parabolic tendon (dimensions in mm, $A_c = 367441\,\text{mm}^2$, $I_c = 2.868\text{e}10\,\text{mm}^4$, $E_c = 27.8\,\text{GPa}$, $E_{\text{cab}} = 195\,\text{GPa}$, $T_\infty = 1000$ kN).

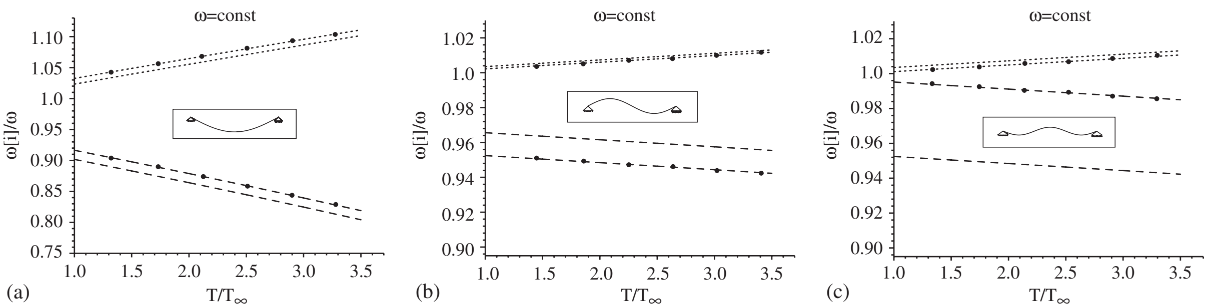

Fig. 5. First three natural frequencies of a prestressed beam with a parabolic cable versus prestress force, calculated by the proposed model and compared to the other models see Refs. [3–6]: (a) first mode; (b) second mode; (c) third mode (legend: ω—proposed model, $\omega[1]$—compressed beam model, $\omega[2]$—small displacement model, - - - - $\omega[1]/\omega$ (bonded case), …… $\omega[2]/\omega$ (bonded case), –●– $\omega[1]/\omega$ (unbonded case), · · ● · · $\omega[2]/\omega$ (unbonded case)).

Comparison of the results with other models appears in Fig. 5, and it reveals that the other models, which appear in the literature, predict natural frequencies that deviate by about $\pm(10\text{–}20)\%$ from the natural frequencies that are determined through the proposed model. This deviation is notable, and points out the deficiency of the models appear in the literature for the determination of the natural frequencies of prestressed beams. The natural frequencies in the case of unbonded tendon are

$$\omega_1 = 16.3256\,\text{rad/s},$$
$$\omega_2 = 64.3404\,\text{rad/s},$$
$$\omega_3 = 140.4626\,\text{rad/s}.$$

It is clear that the natural frequencies of the unbonded prestressed beam are smaller than those of the bonded case. Note that these frequencies cannot be determined by the use of Eq. (73) with an equivalent flexural rigidity. It is also seen in Fig. 5 that the other models that appear in the literature predict a change of $\pm(10\text{–}20)\%$ in the natural frequencies.

4. Summary and conclusions

A nonlinear analytical model has been developed for the dynamic behavior of prestressed beams with bonded and unbonded tendons. The governing equations of motion of the analytical model and their associated boundary and continuity conditions are rigorously derived using the variational principle of virtual work following Hamilton's principle. The proposed model is general, and is valid for any kind of boundary and continuity conditions as well as any cable profile. The model includes the change in the cable eccentricity and the cable force during the vibration of the beam, as well as the effect of the compressive axial load caused by the prestressed tendons on the vibrations of the beam. Based on the derived governing equations it has been mathematically rigorously proven that the magnitude of prestressed force does not affect the natural frequencies of bonded or unbonded prestressed beams as opposed to some research works. Two numerical examples are presented with a straight and a parabolic cable layout. The natural frequencies based on the models that appear in the literature lead to erroneous values that deviate by $\pm(5–20)\%$ from the correct natural frequencies of the prestressed beam.

The proposed model has mathematically proved that the prestress force does not affect the dynamic behavior of general prestressed beams, and reveals that the natural frequencies of bonded prestressed beams can be determined through a linear elastic beam theory with an equivalent moment of inertia of the composite section, while those of the unbonded prestressed beams can be determined by the proposed model.

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