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Influence of mass of cone spring on oscillatory period

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Abstract

In this paper, the wave equation of the cone spring with definite mass is presented. The effective elastic constant and the effective mass are given by means of the energy method, and the expression of the oscillatory period is obtained. Moreover, the analytical solutions of the wave equation are derived. It is proven that the cone spring exists an infinite number of circular frequencies, and this kind of spring moves quasi-periodically.

1. Introduction

The spring is widely applied in many domains, such as engineering, the manufacture of appliance, research and teaching, and so on. The spring mass is not negligible for enhancing data accuracy. The vibration problem of the spring of non-negligible mass is essentially the wave propagation problem in elastic medium. Therefore, this problem has attracted considerable attention in the scientific community, and many research efforts have been focused on this problem [1–7]. However, to the best of our knowledge, less work has been done to investigate the cone spring problem due to its complexity. The application of the cone spring is lagging behind owing to the absence of theoretical instruction. In this paper, the properties of the cone spring are analyzed theoretically, and a theoretical base for the practical application is provided. We can find that when the mass M of the suspended object is much larger than the mass m of the spring, the oscillatory period expression of the cone spring obtained by solving wave equation and the one by using energy method are almost identical. When $M<m$, the oscillatory period expression of the cone spring done is obtained by solving wave equation.

The paper is organized as follows. The wave equation of the cone spring with mass is presented in Section 2. The effective elastic constant and the effective mass of the vibrating spring are obtained by means of the energy method in Section 3. Moreover, the expression of the oscillation period is given in this section. In Section 4, we solve the wave equation analytically and find infinite number of circular frequencies in the cone spring. The circular frequency and the amplification A_n are given in Sections 5 and 6, respectively. The last section is a discussion about the interesting results. At the end, a brief summary is given.

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2. Wave equation of the spring

In Fig. 1, the spring of length L has the mass m, the total turn number is N, and the radii of small and large ends are R_0 and R_N, respectively.

On assuming that the radius of spring is R in the location x, then

$$R = R_0 + kx, \tag{1}$$

where

$$k = (R_N - R_0)/L. \tag{2}$$

Let the angle of rotation from the small end to x be θ, then $\theta/2N\pi = x/L$. Thus the portion of an x length suffers an elongation:

$$\mathrm{d}\theta = \frac{2N\pi}{L}\,\mathrm{d}x \tag{3}$$

and the elongation of the unit of length is

$$\mathrm{d}l = R\,\mathrm{d}\theta. \tag{4}$$

Substituting Eqs. (1)–(3) into Eq. (4), one can obtain

$$\mathrm{d}l = (R_0 + kx)\frac{2N\pi}{L}\,\mathrm{d}x. \tag{5}$$

Since the stiffness factor of the cylindrical spring [7] $K = G\pi r^4/2lR^2$, the stiffness factor corresponding to a dx length in location x is $K_{\mathrm{d}x}$

$$K_{\mathrm{d}x} = \frac{G\pi r^4}{2R^2\mathrm{d}l}, \tag{6}$$

where G is shear modulus of the spring, r is the radius of the wire, R is the radius of the spring. Inserting Eqs. (1)–(5) into Eq. (6), one yields

$$K_{\mathrm{d}x} = \frac{Gr^4L}{4(R_0 + kx)^3N\,\mathrm{d}x}, \tag{7}$$

which is the series connection of the spring with its length dx and stiffness factor K_x. From the stiffness factor relation of series connection and the integral mean value theorem, one gets

$$K_x = \frac{Gr^4L}{4(R_0 + kx)^3N}. \tag{8}$$

Since the total spring L is regarded as the series connection of the spring with its stiffness factor K_x then

$$\frac{1}{K} = \int_0^L \frac{\mathrm{d}x}{K_x} = \frac{N(R_N^2 + R_0^2)(R_N + R_0)}{Gr^4}, \tag{9}$$

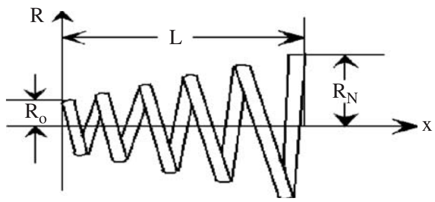

Fig. 1. Cross-section graphics of the spring.

i.e. the stiffness factor of the cone spring is

$$K = \frac{Gr^4}{N(R_N^2 + R_0^2)(R_N + R_0)}. \tag{10}$$

If $R_0 = R_N = R$, then the stiffness factor of the cylindrical spring can be recovered, namely

$$K = \frac{Gr^4}{4NR^3}. \tag{11}$$

Let the density of the wire be ρ, then the mass corresponding to $\mathrm{d}x$ is

$$\mathrm{d}m = \rho \mathrm{d}l = \rho \frac{2N\pi}{L}(R_0 + kx)\,\mathrm{d}x. \tag{12}$$

While considering the forces as F_x and $F_{x+\mathrm{d}x}$ in x and $x+\mathrm{d}x$, respectively, the total force

$$\mathrm{d}F = F_{x+\mathrm{d}x} - F_x = K_x \frac{\partial U}{\partial x}\bigg|_{x+\mathrm{d}x} - K_x \frac{\partial U}{\partial x}\bigg|_x = \frac{\partial}{\partial x}\left(K_x \frac{\partial U}{\partial x}\right)\mathrm{d}x, \tag{13}$$

where U is the displacement in x. From the Newton's second law, $(\mathrm{d}F/\mathrm{d}m) = (\partial^2 U/\partial t^2)$, and Eqs. (12) and (13), one has

$$\frac{\partial/\partial x\left(K_x \partial U/\partial x\right)\mathrm{d}x}{\mathrm{d}m} = \frac{\partial^2 U}{\partial t^2}. \tag{14}$$

From Eqs. (8), (12) and (14), the wave equation of the cone spring is

$$\frac{\partial^2 U}{\partial t^2} = \frac{Gr^4 L^2}{8\pi\rho N^2 (R_0 + kx)} \frac{\partial}{\partial x}\left[\frac{1}{(R_0 + kx)^3}\frac{\partial U}{\partial x}\right]. \tag{15}$$

If $k = 0$, then the wave equation of the cylindrical spring can be recovered, i.e.

$$\frac{\partial^2 U}{\partial t^2} = \frac{Gr^4 L^2}{8\pi\rho R_0^4 N^2}\frac{\partial^2 U}{\partial x^2}. \tag{16}$$

3. Effective elastic constant and effective mass

Now we consider a system consisting of a mass M suspended to the large end of a spring of non-negligible mass m. When $R_0 = 0$, the special example is the cone spring. From Eqs. (2), (8), (10), (12) and (15), one gets

$$R_N = kL, \tag{17}$$

$$K_x = \frac{Gr^4 L}{4Nk^3 x^3}, \tag{18}$$

$$K = \frac{Gr^4}{NR_N^3}, \tag{19}$$

$$\mathrm{d}m = \frac{2\rho N k\pi}{L} x\,\mathrm{d}x, \tag{20}$$

$$\frac{\partial^2 U}{\partial t^2} = \frac{Gr^4 L^2}{8\pi\rho N^2 kx}\frac{\partial}{\partial x}\left[\frac{1}{(kx)^3}\frac{\partial U}{\partial x}\right]. \tag{21}$$

The substitution of Eqs. (17) and (19) into Eq. (18) yields

$$K_x = \frac{L^4 K}{4x^3}. \tag{22}$$

From Eq. (20), the total mass of the cone spring is

$$m = \int_0^L \mathrm{d}m = \int_0^L \rho k \frac{2\pi N}{L} x \, \mathrm{d}x = \rho N k L \pi. \tag{23}$$

The substitution of Eq. (23) back into Eq. (20) yields

$$\mathrm{d}m == \frac{2m}{L^2} x \, \mathrm{d}x. \tag{24}$$

Inserting Eqs. (17), (19) and (23) into Eq. (21), the wave equation is of the form

$$\frac{\partial^2 U}{\partial t^2} = \frac{KL^6}{8mx} \frac{\partial}{\partial x} \left[\frac{1}{x^3} \frac{\partial U}{\partial x} \right]. \tag{25}$$

A valid solution of Eq. (25) is

$$U(x, t) = f(x) z(t), \tag{26}$$

where $f(x)$ is "shape function" [4], which gives relative elongation as a function of x, and $z(t)$ gives the motion of the end point $x = L$, where mass M is attached. In general, $f(x)$ is unknown explicitly, yet clearly $f(L) = 1$, so that $U(L, t) = z(t)$. The total energy of this system is

$$E = (V + T)_{\text{spring}} + T_{\text{mass} \;\; M}, \tag{27}$$

where $(V + T)_{\text{spring}}$ depends on the way the spring is stretched that is on the "shape function" $f(x)$.

V_{spring} can be obtained as follows. By using the relation $F(x) = K_x(\partial U/\partial x)$, the potential of the segment $\mathrm{d}U$ is

$$\mathrm{d}V = F(\text{average}) \, \mathrm{d}U = \frac{F(x)}{2} \, \mathrm{d}U = \frac{K_x}{2} \left(\frac{\partial U}{\partial x} \right)^2 \mathrm{d}x. \tag{28}$$

Substituting the differential of Eq. (26) to argument x into Eq. (28), and integrating once to argument x with Eq. (22), we have

$$V_{\text{spring}} = \frac{z^2(t) L^4 K}{8} \int \frac{[\mathrm{d}f/\mathrm{d}x]^2}{x^3} \, \mathrm{d}x = \frac{aKz^2(t)}{2}, \tag{29}$$

where

$$a = \frac{L^4}{4} \int_0^L \frac{[\mathrm{d}f/\mathrm{d}x]^2}{x^3} \, \mathrm{d}x. \tag{30}$$

Now we discuss the kinetic energy of the spring. As for kinetic energy, each element $\mathrm{d}x$ with mass $\mathrm{d}m$ and velocity $\mu(x, t)$ possesses an energy:

$$\mathrm{d}T_{\text{spring}} = \frac{1}{2} v^2(x, t) \, \mathrm{d}m = \frac{1}{2} v^2(x, t) \frac{2m}{L^2} x \, \mathrm{d}x. \tag{31}$$

By differentiating Eq. (26) to argument t, the velocity is

$$v(x, t) = z'(t) f(x). \tag{32}$$

where "′" denotes differential to argument t. At the location $x = L$, the velocity

$$v(L, t) = z'(t) f(L) = z'(t). \tag{33}$$

Substituting Eq. (33) back into Eq. (32), one gets

$$v(x, t) = v(L, t) f(x). \tag{34}$$

Inserting Eq. (34) in Eq. (31) and integrating once, the total kinetic energy of the spring reads

$$T_{\text{spring}} = \tfrac{1}{2} v^2(L, t) m b, \tag{35}$$

with

$$b = \frac{2}{L^2}\int_0^L x f^2(x)\,\mathrm{d}x. \tag{36}$$

Moreover, the kinetic energy of the mass M is

$$T_{\text{mass}\ \ M} = \tfrac{1}{2}v^2(L,t)M. \tag{37}$$

On substituting Eqs. (29), (35) and (37) into Eq. (27) obtains the total energy of the system

$$E = \frac{aKz^2(t)}{2} + (M+bm)v^2(L,t)/2, \tag{38}$$

where aK and bm are the effective elastic constant and the effective mass of the spring, respectively.

Since the energy $E(t) =$ constant due to the energy conservation law, its time derivative must be zero:

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \frac{aKz\mathrm{d}z}{\mathrm{d}t} + \frac{(M+bm)v(L,t)\mathrm{d}v(L,t)}{\mathrm{d}t} = 0. \tag{39}$$

The derivative of Eq. (33) to argument t substitutes into Eq. (39), to yield

$$\frac{\mathrm{d}^2 z}{\mathrm{d}t^2} = -\left[aK/(M+bm)\right]z = -\omega^2 z, \tag{40}$$

with

$$\omega = \left[aK/(M+bm)\right]^{1/2}. \tag{41}$$

Since a and b are constant for a given shape function $f(x)$, we see that $z(t)$ must be harmonic with period

$$T = \frac{2\pi}{\omega} = 2\pi\left[(M+bm)/aK\right]^{1/2}, \tag{42}$$

where a and b are defined by Eqs. (30) and (36). Let us evaluate a and b for the limit case $M \gg m$.

For this case, the elongation is quasi-static; that is, the spring stretches uniformly. The elongation in the location x is

$$y(x) = \int_0^x \mathrm{d}y = \int_0^x \frac{F}{K_x}\,\mathrm{d}x = \frac{Fx^4}{KL^4}$$

and the ‘‘shape function’’

$$f(x) = \frac{y(x)}{y(L)} = \frac{x^4}{L^4}. \tag{43}$$

In terms of Eqs. (30), (36) and (43), constants $a = 1, b = \frac{1}{5}$. The substitution in Eq. (42) gives

$$T = 2\pi\left[\left(M + \tfrac{m}{5}\right)\Big/K\right]^{1/2}, \tag{44}$$

which is different from the known spring period $T = 2\pi(M/K)^{1/2}$ with the spring mass being ignored.

Compared with the oscillation period of the cylinder spring [2,4,5]

$$T = 2\pi\left[\left(M + \tfrac{m}{3}\right)\Big/K\right]^{1/2}, \tag{45}$$

it is clearly seen that the influence of the mass of the cone spring on vibration is smaller than on the cylinder spring. Moreover, it is easily proven that effective elastic constant of the circular truncated cone spring is between the cylindrical spring and the cone spring, i.e. $\frac{1}{5} \leqslant b \leqslant \frac{1}{3}$. However, for other cases, except for $M \gg m$, the shape function $f(x)$ is difficult to describe. In these cases, we should strictly solve the wave equation to discuss the properties of the cone spring.

4. Finite solution problem to wave equation

While considering the initial condition and boundary condition, if the spring is fixed at the location $x = 0$, then $U(0, t) = 0$. And if it is connected to an object of mass M in the location $x = L$, then $MU_{tt}(L, t) = -(LK/4)U_x(L, t)$ due to Newton's second law and Hooke's law. When $t = 0$, the static spring has tensile displacement U_0, i.e. $U(x, t) = U_0\left(x^4/L^4\right)$, and $U_t(L, t) = 0$, thus the statement of the problem is

$$\begin{aligned}
&U_{tt} = \frac{c^2}{x}\left(\frac{1}{x^3}U_x\right)_x,\\
&U(0, t) = 0,\\
&MU_{tt}(L, t) = -\frac{LK}{4}U_x(L, t),\\
&U(x, 0) = U_0\frac{x^4}{L^4},\\
&U_t(x, 0) = 0,
\end{aligned} \tag{46}$$

with

$$c^2 = \frac{L^6K}{8m}. \tag{47}$$

On inserting Eq. (26) into the first equation in Eq. (46), the wave equation has the form

$$z_{tt}(t)f(x) = \frac{c^2}{x}\left[\frac{1}{x^3}f_x(x)\right]_x z(t).$$

By variable separation, one can get

$$\begin{aligned}
&z_{tt}(t) + \omega^2 z(t) = 0,\\
&f_{xx}(x) - \frac{3}{x}f(x) + \frac{\omega^2}{c^2}x^4 f(x) = 0.
\end{aligned} \tag{48}$$

From the initial condition in Eq. (46) and the first expression in Eq. (48), it is easily proven that

$$z(t) = D_n\cos(\omega t). \tag{49}$$

And the second expression in Eq. (48) is a Bessel equation [8]

$$f_{xx}(x) + \frac{1-2\alpha}{x}f_x(x) + \left[\left(\beta\gamma x^{\gamma-1}\right)^2 + \frac{\alpha^2 - m^2\gamma^2}{x^2}\right]f(x) = 0,$$

with its solution $f(x) = Cx^\alpha J_m(\beta x^2)$ because $f(0)$ is zero, where $\alpha = 2, \gamma = 3, \beta = \omega/3c, m = 2/3$. Thus the solution of the second expression in Eq. (48) reads

$$f(x) = C_\omega x^2 J_{2/3}\left(\frac{\omega}{3c}x^3\right). \tag{50}$$

Clearly, the coefficient C_ω has the dimension of m^{-2}. If $B_\omega = C_\omega L^2$, the dimensionless form is

$$f(x) = \frac{B_\omega x^2}{L^2}J_{2/3}\left(\frac{\omega}{3c}x^3\right). \tag{51}$$

Furthermore, let $A_\omega = B_\omega D_\omega$, from Eqs. (49) and (51), then the wave equation has the solution as

$$U = A_\omega\frac{x^2}{L^2}J_{2/3}\left(\frac{\omega}{3c}x^3\right)\cos(\omega t). \tag{52}$$

5. Circular frequency ω

Substituting the boundary condition $MU_{tt}(L,t) = -(LK/4)U_x(L,t)$ in Eq. (46) into Eq. (52) yields

$$-\omega^2 Mx^2 J_{2/3}\left(\frac{\omega}{3c}x^3\right)\Big|_{x=L} = -\frac{LK}{4}\frac{\mathrm{d}}{\mathrm{d}x}\left[x^2 J_{2/3}\left(\frac{\omega}{3c}x^3\right)\right]\Big|_{x=L}. \tag{53}$$

That is

$$L^3 K J_{-1/3}\left(\frac{\omega}{3c}L^3\right) = 4c\omega M J_{2/3}\left(\frac{\omega}{3c}L^3\right).$$

On substituting c in Eq. (47) into above equation, ω satisfies

$$\frac{\sqrt{mK}}{\sqrt{2}\omega M} = \frac{J_{2/3}\left(\frac{2\sqrt{2}}{3}\sqrt{(m/K)}\omega\right)}{J_{-1/3}\left(\frac{2\sqrt{2}}{3}\sqrt{(m/K)}\omega\right)}, \tag{54}$$

which is similar to the expression of the cylindrical spring $\left(\sqrt{Km}/\omega M\right) = \tan\left(\omega\sqrt{m/K}\right)$ in Refs. [1–3,5,6]. They both admit an infinity of circular frequencies, yet Eq. (54) is more complex than that of cylindrical spring. The wave equation has the solution

$$U = \sum_n A_n \frac{x^2}{L^2} J_{2/3}\left(\frac{\omega_n}{3c}x^3\right)\cos(\omega_n t), \tag{55}$$

where ω_n is the frequency of the nth solution, A_n is the corresponding amplification.

6. Amplification A_n

The substitution of the initial condition $U(x,0) = U_0(x^4/L^4)$ in Eq. (46) into Eq. (55) yields

$$U_0\frac{x^2}{L^2} = \sum_{n=1}^{\infty} A_n J_{2/3}\left(\frac{2\sqrt{2}}{3}\sqrt{\frac{m}{K}}\frac{\omega_n}{L^3}x^3\right). \tag{56}$$

Through the following variable substitution,

$$y = x^3, \tag{57}$$

$$\alpha_n = \frac{2\sqrt{2}}{3}\sqrt{\frac{m}{K}}\omega_n, \tag{58}$$

$$\tau = L^3, \tag{59}$$

Eq. (56) reads

$$U_0\frac{y^{2/3}}{\tau^{2/3}} = \sum_{n=1}^{\infty} A_n J_{2/3}\left(\frac{\alpha_n}{\tau}y\right). \tag{60}$$

In light of the orthogonal and unitary properties and integral formula of Bessel function, it is easily proven that

$$A_n = \frac{2U_0 J_{5/3}(\alpha_n)}{\alpha_n\left[J_{2/3}(\alpha_n)^2 - J_{-1/3}(\alpha_n)J_{5/3}(\alpha_n)\right]}. \tag{61}$$

Upon inserting Eqs. (57)–(59) back in Eq. (61), the amplification A_n has the form

$$A_n = \frac{2U_0 J_{5/3}\left(\frac{2\sqrt{2}}{3}\sqrt{\frac{m}{K}}\omega_n\right)}{\frac{2\sqrt{2}}{3}\sqrt{(m/K)}\omega_n\left[J_{2/3}\left(\frac{2\sqrt{2}}{3}\sqrt{\frac{m}{K}}\omega_n\right)^2 - J_{-1/3}\left(\frac{2\sqrt{2}}{3}\sqrt{(m/K)}\omega_n\right)J_{5/3}\left(\frac{2\sqrt{2}}{3}\sqrt{\frac{m}{K}}\omega_n\right)\right]}. \tag{62}$$

Substituting Eq. (62) into Eq. (55), the solution of wave equation reads

$$U=\sum_n \frac{2U_0\left(\frac{x^2}{L^2}\right)J_{5/3}\left(\frac{2\sqrt{2}}{3}\sqrt{(m/K)}\omega_n\right)J_{2/3}\left(\frac{2\sqrt{2}}{3}\sqrt{(m/K)}\omega_n\left(\frac{x^3}{L^3}\right)\right)}{\frac{2\sqrt{2}}{3}\sqrt{(m/K)}\omega_n\left[J_{2/3}\left(\frac{2\sqrt{2}}{3}\sqrt{(m/K)}\omega_n\right)^2-J_{-1/3}\left(\frac{2\sqrt{2}}{3}\sqrt{(m/K)}\omega_n\right)J_{5/3}\left(\frac{2\sqrt{2}}{3}\sqrt{(m/K)}\omega_n\right)\right]}\cos(\omega t). \tag{63}$$

7. Discussion and summary

Now we discuss our interesting results in three different aspects (i.e. m, M and M/m). For simplicity, we let

$$\eta_1(\omega)=\frac{J_{2/3}\left(\frac{2\sqrt{2}}{3}\sqrt{(m/K)}\omega\right)}{J_{-1/3}\left(\frac{2\sqrt{2}}{3}\sqrt{(m/K)}\omega\right)}, \tag{64}$$

$$\eta_2(\omega)=\frac{\sqrt{mK}}{\sqrt{2}\omega M}. \tag{65}$$

(1) Discussion about m:

When $m\to 0$, the expression of simple harmonic vibration without considering mass of the spring can be recovered.

From the asymptotic formula of Bessel function [8]

$$J_\gamma(x)\overset{x\to 0}{\sim}\frac{1}{\Gamma(\gamma+1)}\left(\frac{x}{2}\right)^\gamma, \tag{66}$$

we have $\eta_1(\omega)=\frac{1}{\sqrt{2}}\sqrt{(m/K)}\omega$. Substituting this expression into Eq. (65), one gets

$$\omega=\sqrt{\frac{K}{M}}. \tag{67}$$

In light of the asymptotic formula of Bessel function , Eq. (63) reads

$$U(L,t)=\frac{U_0\frac{1}{\Gamma\left(\frac{5}{3}+1\right)\Gamma\left(\frac{2}{3}+1\right)}}{\frac{1}{\Gamma\left(\frac{2}{3}+1\right)^2}-\frac{1}{\Gamma\left(-\frac{1}{3}+1\right)\Gamma\left(\frac{5}{3}+1\right)}}\cos(\omega t). \tag{68}$$

Substituting Eq. (67) into Eq. (68) with the relation $\Gamma(\alpha+1)=\alpha\Gamma(\alpha)$, the spring of negligible mass satisfies the simple harmonic vibration:

$$U(L,t)=U_0\cos\left(\sqrt{\frac{K}{M}}t\right), \tag{69}$$

which indicates the correctness of our results, and it is a special example of the general results (63) in this paper.

(2) Discussion about M:

The circular frequency ω satisfies

$$\frac{\sqrt{mK}}{\sqrt{2}\omega M}=\frac{J_{2/3}\left(\frac{2\sqrt{2}}{3}\sqrt{(m/K)}\omega\right)}{J_{-1/3}\left(\frac{2\sqrt{2}}{3}\sqrt{(m/K)}\omega\right)},$$

which is a transcendental equation. For avoiding the complicated operation, we employ the graphic analysis method. In Fig. 2, the abscissa is ω with unit $\frac{3}{2\sqrt{2}}\sqrt{(K/m)}$. We take $(3K/4M)=20$ for example, F_2 and the graphics of η_1 and η_2 are shown in Fig. 2.

From Fig. 2, we infer that η_2 and η_1 have infinity crossing points, and the frequency related to the first one is basic frequency, others are related to corresponding order harmonic frequencies, respectively. Strictly

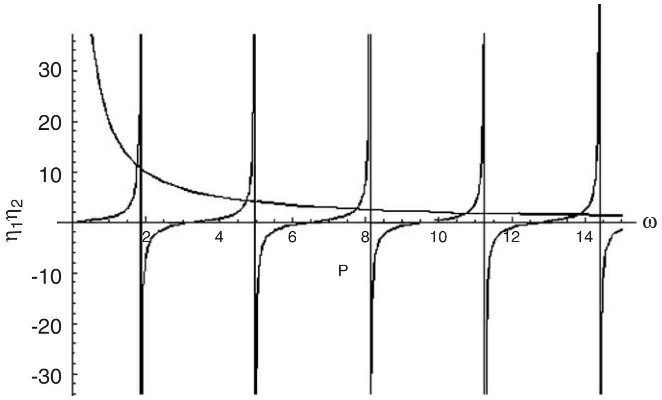

Fig. 2. Graphical solution of Eqs. (64) and (65).

speaking, the motion of the spring is not simple harmonic vibrating, but quasi-period movement based mainly on the basic frequency.

When M increases, the rectangular hyperbola moves in the paraxial direction, thus basic frequency and all order frequencies decrease. For the limited case $M = \infty$, basic frequency should be the frequency corresponding to the point P in Fig. 2 due to the request that basic frequency is not zero. When $M = 0$, the basic frequency is the frequency related to the first asymptotic line.

(3) Discussion about m and M:

For simplification, we assume that m and K are equal. Eq. (54) reads

$$\frac{m}{\sqrt{2}\omega M} = \frac{J_{2/3}\left(\frac{2\sqrt{2}}{3}\omega\right)}{J_{-1/3}\left(\frac{2\sqrt{2}}{3}\omega\right)}. \tag{70}$$

Further considering the vibration of the top end $U(L,t)$, Eq. (63) turns into

$$U(L,t) = \sum_n A_n(L)\cos(\omega t)$$

and Eq. (62) reads

$$A_n(L) = \frac{2U_0 J_{5/3}\left(\frac{2\sqrt{2}}{3}\omega_n\right) J_{2/3}\left(\frac{2\sqrt{2}}{3}\omega_n\right)}{\frac{2\sqrt{2}}{3}\omega_n\left[J_{2/3}\left(\frac{2\sqrt{2}}{3}\omega_n\right)^2 - J_{-1/3}\left(\frac{2\sqrt{2}}{3}\omega_n\right) J_{5/3}\left(\frac{2\sqrt{2}}{3}\omega_n\right)\right]}.$$

Table 1 lists the corresponding results of circular frequency of the cone spring for Eq. (70) obtained by solving wave equation, Eq. (44) obtained by energy method and the approximate formula $\omega = \sqrt{K/M}$ with ignoring inherent mass of the cone spring via different values M/m respectively. In this table, $\sigma_n = \left((A_n(L))/U_0\right)^2$ denotes the ratio of all kinds of frequency to total energy.

From Table 1, we can find some novel results:

(1) The energy corresponding to the basic frequency accounts for very high percents in the total energy. When M/m increases, this ratio increases in the total energy. When $M/m \geqslant 2$, this ratio surpasses 95%, which indicates basic frequency, namely the circular frequency observed in the experiment, is principal macroscopic representation during the vibration. And in this case, the first harmonic frequency is less than 0.7%. Moreover, when M/m decreases, this ratio does not always increase, and may also decrease. For example, the ratio of $M/m = 0.01$ is smaller than that of $M/m = 0.1$, whose reason is that the decreasing energy may transfer to other harmonic frequencies.

(2) When $M > m$, the basic frequency ω obtained from energy method and by solving wave equation is almost identical. The formula of circular frequency by energy method possesses high precision.

Table 1

$n = \frac{M}{m}$	$n = \infty$, i.e. $m = 0$	100	50	30	10	2	1	0.1	0.01
Eq. (70) obtained by solving wave equation									
Basic frequency ω_1	$\omega = \sqrt{\frac{K}{M}}$ no basic frequency	0.09990	0.14113	0.18197	0.31310	0.67348	0.90954	1.72043	1.95024
Energy percent of basic frequency σ_1	100	99.9002	99.8007	99.6687	99.0177	95.4231	91.5935	68.9712	59.8520
Harmonic frequency ω_2	Does not exist	3.5825	3.5846	3.5874	3.6012	3.6824	3.7783	4.6604	5.2122
Energy percent of the first harmonic frequency σ_2	Does not exist	0.0001	0.0005	0.0015	0.1337	0.3002	1.0253	1.7197	0.6661
Eq.(44) obtained by energy method									
$M > m$ $\omega = \left[1 \Big/ (n+0.2)\right]^{1/2}$	$\omega = \sqrt{\frac{K}{M}}$	0.09990	0.14113	0.18197	0.31159	0.67420			
General approximation with ignoring mass									
$M > m$ $\omega = \left(1 \Big/ n\right)^{1/2}$	$\omega = \sqrt{\frac{K}{M}}$	0.10000	0.14142	0.18257	0.31623	0.70711			

(3) ω , which is given by general approximate formula with ignoring mass in its applicability with negligible mass, is larger than that by the energy method. This conveys us that the energy method is superior to general approximate formula in the applicability and precision.

In summary, we discuss the oscillatory properties of the cone spring by energy method and directly solving the wave equation. These results from different methods are identical. For another opposite case, i.e. the large is fixed and the mass M is suspended to the small end, we can deal with it similarly to this paper except for the difference of the influence of the mass of the spring on oscillatory period. Moreover, for a circular truncated cone spring with given R_0, R_N, and L, one can deal with it closely to this paper. We think that our worthy work will provide a theoretical base for the cone spring in the practical application. Further work will be discussed.

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