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Vibration confinement in a general beam structure during harmonic excitations

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Abstract

In the present paper the Green's function is utilized to provide a simple, exact and direct analytical method for analysis of a beam structure of generic boundary conditions with attached springs and/or finite masses. The aim is to confine the vibration in a certain part of the structure. The beam equation is based on the Timoshenko beam theory with corrections for shear deformation and rotary inertia effects. In the analysis the beam is driven by a harmonic external excitation. The attached springs are modeled as simple reactions that provide transverse forces to the beam, while each added mass provides a transverse force in addition to a moment at its location. These forces (moments) act as secondary forces (moments) that reduce the response caused by the external force. Numerical simulations are conducted to find the optimal masses and/or springs that confine the vibration in a certain chosen region. The results were compared to that obtained from Euler–Bernoulli beam theory. Beams that are excited by a bi-harmonic force as well as dual excitation forces are analyzed. In addition, the case when the beams are excited near resonances is discussed. Also, a method is proposed to impose a node at any desired location along the structure.

1. Introduction

The need for lightweight structures for many applications has spurred the formulation of vibration confinement techniques to alleviate disturbance-induced structural vibration. These structures have inherently low structural damping; therefore, a disturbance may cause many vibration-induced problems such as structural degradation and failure, performance deterioration, malfunction of components and processes, material fatigue, noise transmission, human discomfort, hazard and various other problems. In addition, the occurrence of vibration in some structures that are equipped with sensitive elements may weaken their performance. Therefore, it may be of interest to eliminate vibration from a certain part of the structure more than the other. The large flexible space structures, for example, are usually built from lightweight materials with low damping. An excitation source may cause vibrations that propagate throughout the structure. In such case it is desirable to confine vibration in some chosen insensitive part while keeping other parts, for

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Nomenclature

a_i	location of added ith spring	L	beam length
A	cross-sectional area of the beam	m_r	rth added mass
b_r	location of added rth mass	$p(x, t)$	applied force per unit length
E	modulus of elasticity	t	time variable
F_0	excitation force amplitude	$w(x, t)$	transverse deflection
G	shear modulus	$W(x)$	transverse displacement amplitude
$G(x,u)$	Green's function	x	axial coordinate along the beam
I	area moment of inertia of cross section	γ	shear angle
J	mass moment of inertia of added mass	λ	spatial wavelength
k_i	ith spring stiffness	μ	mass ratio
k'	shear factor	v	poisson's ratio
k_b	wavenumber	ρ	beam material density
		ω	excitation frequency

instance an extremely sensitive antenna, relatively undisturbed. Therefore the vibration suppression of a structure excited by external forces has been the subject of numerous investigations because of its relevance to aeronautical, mechanical and structural engineering. The ideas suggested range from the use of passive control (utilizing added masses, springs, dampers, etc.) to the use of active control with sophisticated control strategies (utilizing sensors, actuators, smart materials, feedback devices, etc.). In periodic structures, some random disorders (interruption of periodicity) may be introduced intentionally to suppress the vibration utilizing mode localization phenomenon in which a structure simultaneously exhibits localized areas of relatively large amplitudes, where vibrational energy is contained, and isolated areas of small amplitudes. Mode localization has been known in solid state physics for over 40 years, being discovered by Anderson in his famous theoretical paper [1]. In the early 1980s, localization has been first studied as means for achieving vibration isolation in elastic structures by Hodges [2] who evidenced this phenomenon in structural dynamics by both theoretical investigation and experimental demonstration. Since then, many studies have been devoted to investigate the occurrence and the effects of this phenomenon in the area of structural dynamics. Among these studies, for example are Hodges and Woodhouse [3], Pierre and Dowell [4], Luongo [5] Bendiksen [6], and many others, see special issue on "localization phenomenon in physical and engineering sciences" [7]. These studies showed that mode localization is associated with frequency coalescence and frequency veering phenomena and the system exhibits a damping-like effect that could be used as passive control of vibration transmission. Localization theory is interesting because its prediction may run counter to our intuition and contradicts common engineering established views. The practical applications of localization have not been fully explored and the potential benefits have not yet been realized.

The purpose of the present analysis is to investigate the effect of added masses and their rotary inertia and/or springs on beams vibration. The equations of motion of the beam are based on Timoshenko beam theory in which the effects of rotary inertia and shear deformation are included. A technique utilizing the dynamic Green's function is presented to find the best arrangement of masses and/or springs, for vibration confinement in a certain part of a sinusoidally driven beam. The optimum mass is obtained at each external exciting frequency. Some of the objectives of the present work are similar to that of Keltie and Cheng [8] who used modal analysis approach to investigate the effects of added masses on vibrational behavior of a simply supported Euler–Bernoulli beam and furthermore utilized the results to control or reduce the vibration response. In their analysis, they considered only the mass locations as design parameters. In the present analysis a single or multiple rigid masses with rotary inertia and/or springs are attached to the beam at prescribed locations to maintain the vibration level significantly small at certain locations along the beam.

The problem being studied falls in the class of the dynamics of combined systems, which consist of linear elastic structures carrying lumped attachments. The free vibrations of such systems are studied extensively in the literature [9–27]. Literature review indicates that the authors of these papers have generally directed their investigations into finding the natural frequencies and the corresponding mode shapes. The most common analytical approach used is the assumed mode methods. Other approach such as finite element, Laplace transform, transfer matrix, sub-structuring method, Lagrange's multipliers and analytical and numerical combined method are also used. In addition, a pure analytical (closed form) solution for a few simple special cases was reported.

For any type of boundary conditions, the exact natural frequencies and mode shapes for a Timoshenko beam with attached springs and/or rigid masses, each associated with mass moment of inertia can be obtained as a result of the present analysis. However, in order not to disturb the focus of the present paper we will neither discuss the natural frequencies nor the mode shapes of the system being studied.

A key feature of the present work is that the problem is formulated in terms of the Green's functions. This method is exact and straightforward. It was chosen for its freedom from numerical accuracy when compared to the standard application of modal superposition technique. The boundary conditions are embedded in the Green's function of the corresponding beam and it is not necessary to solve the free vibration problem in order to obtain the eigenvalues and the corresponding eigenfunctions, which are required for modal superposition solution. Equally important, this procedure exhibits appreciably greater computational efficiency when compared with other methods. Therefore any engineer without any difficulty of pragmatic nature can use it. The present paper is concerned essentially with a system similar to that investigated in Ref. [28] but in this analysis the Timoshenko beam model is used and the rotary inertia of the added rigid masses is included as a counter part of that investigation.

2. Dynamic modeling

The Euler–Bernoulli beam theory, in which the effects of rotary inertia and shear deformation are neglected, represents an engineering approximation that substantially simplifies any further analysis. But neglecting the rotational inertia is not always justified because while the magnitude of the mass moment of inertia per unit length may be small, at very high frequencies the contribution of the angular acceleration could be significant. On the other hand, neglecting the shear deformation even though the cross sections of the beam carry a resultant shear force represents an anomaly. In actuality, the shear stress and shear strain vary over the cross section, because the shear stress must be zero at the upper and lower surfaces of the beam. As a consequence of these simplifying assumptions the obtained equation of the beam is only adequate for slender beams of lower vibrational modes. In other words, the Euler–Bernoulli beam model is inadequate for short and thin-webbed beams and beams where higher modes are excited, as well as for beams that are made of material sensitive to shear stress. From wave point of view, the Euler–Bernoulli beam theory predicts unrealistic wave speed because it approaches infinity for very high frequencies. This unrealistic limit is corrected by subsequent beam theories.

Lord Rayleigh [29] incorporated the effect of rotational movements of beam element as a first correction to the simple theory. The partially refined beam equation is known as Rayleigh beam. Later on, another engineering refinement was made by Timoshenko [30,31] who proposed a beam theory, which adds the effect of shear deformation as well as the effect of rotational inertia to the Euler–Bernoulli beam model. As it is easy to see that during vibration, a typical element of beam performs not only transverse motion but also rotates. Timoshenko approximated the effect of shear as an average over the cross section. This entails allowing each cross section to rotate independently of the slope of the centroidal axis in the deformed state. After the two classical papers on beam vibrations by Timoshenko, hundreds of papers have been published considering Timoshenko beam model. These studies (e.g. [32]) show that the effects of rotary inertia and shear deformation (Timoshenko effects) on the first mode of vibration are small, but these effects increase rapidly for second and higher modes.

The system to be considered consists of a uniform elastic beam of finite length L originally at rest with different classical or unconventional boundary conditions at $x = 0$ and L (not shown in Fig. 1). There is R finite rigid masses each having mass m_r with associated mass moment of inertia J_r. In addition there are N translational springs each has a stiffness k_i attached to the beam at different locations. The beam is driven by a sinusoidal external force. Upon neglecting beam axial inertia, the expression of the total kinetic energy of the

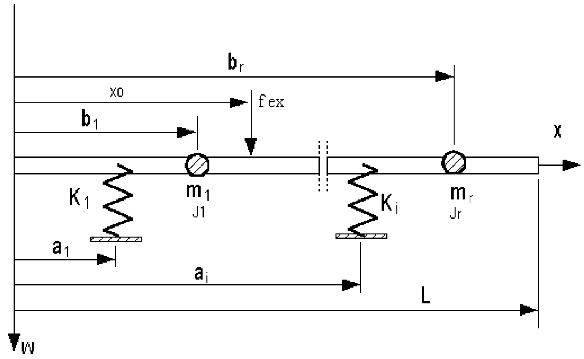

Fig. 1. A uniform beam with attached finite masses and springs.

system can be written as

$$T = \frac{1}{2}\int_0^L \rho A\left(\frac{\partial w(x,t)}{\partial t}\right)^2 \,\mathrm{d}x + \frac{1}{2}\int_0^L \rho I\left(\frac{\partial \psi(x,t)}{\partial t}\right)^2 \,\mathrm{d}x + \frac{1}{2}\sum_{r=1}^{r=R}\left[m_r\left(\frac{\partial w(b_r,t)}{\partial t}\right)^2 + J_r\left(\frac{\partial \psi(b_r,t)}{\partial t}\right)^2\right]\delta(x-b_r]\,\mathrm{d}x, \tag{1}$$

where ρ is the mass per unit volume of beam material, A is the cross-sectional area, I is area moment of inertia of cross section, $w(x,t)$ and $\psi(x,t)$ are the transverse deflection and bending slope at spatial point x along the beam and at time t, respectively. b_r is the location of concentrated mass m_r and $\delta(.)$ is the Dirac delta singularity function.

The potential energy can be written as

$$U = \frac{1}{2}\int_0^L \left[EI\left(\frac{\partial \psi}{\partial x}\right)^2 + k' AG\gamma(x,t)^2\right]\,\mathrm{d}x + \frac{1}{2}\sum_{i=1}^{i=N} k_i w(a_i,t)^2\delta(x-a_i), \tag{2}$$

where E is modulus of elasticity, G is the shear modulus of rigidity which equals to $E/(2+2v)$ where v is Poisson's ratio, γ is the shear strain and a_i is the location of the ith spring. k' is shear correction factor, being defined as the ratio of the averaged shear strain within the cross section to the shear strain at the section centroid [33]. Some references take the numerical values of k' as $\frac{5}{6}$ for a rectangular cross section and $\frac{9}{11}$ for a circular one.

The equations of motion can now be derived by applying Hamilton's principle

$$\int_{t_1}^{t_2} \delta(T-U)\,\mathrm{d}x + \int_{t_1}^{t_2} \delta W_{\mathrm{nc}}\,\mathrm{d}t = 0, \tag{3}$$

where δW_{nc} denotes the virtual work done by the non-conservative forces and can be expressed as

$$W_{\mathrm{nc}} = \int_0^L f_{\mathrm{ex}}(x,t)w(x,t)\,\mathrm{d}x, \tag{4}$$

where $f_{\mathrm{ex}}(x,t)$ represents the external applied force per unit length of the beam.

Substituting the expressions for T, U and W_{nc} in the standard form of Hamilton's principle, Eq. (3), and then performing the usual variation, the following field displacements equations are derived

$$EI\frac{\partial^2 \psi}{\partial x^2} + k' AG\left(\frac{\partial w}{\partial x} - \psi\right) - \rho I\frac{\partial^2 \psi}{\partial t^2} - \sum_{r=1}^{r=R} J_r\frac{\partial^2 \psi}{\partial t^2}\delta(x-b_r) = 0, \tag{5}$$

$$\rho A\frac{\partial^2 w}{\partial t^2} - k' AG\left(\frac{\partial^2 w}{\partial x^2} - \frac{\partial \psi}{\partial x}\right) + \sum_{r=1}^{r=R} m_r\frac{\partial^2 w}{\partial t^2}\delta(x-b_r) + \sum_{i=1}^{i=N} k_i w\delta(x-a_i) - f_{\mathrm{ex}}(x,t) = 0. \tag{6}$$

Table 1
Boundary conditions

End support	Euler–Bernoulli theory	Timoshenko theory
Simple	$w(x,t)=0$	$w(x,t)=0$
	$w''(x,t)=0$	$w''(x,t)=0$
Clamped	$w(x,t)=0$	$w(x,t)=0$
	$w'(x,t)=0$	$w'''(x,t)-\frac{\rho}{k'G}\ddot{w}'(x,t)+\frac{k'AG}{EI}w'(x,t)=0$
Free	$w''(x,t)=0$	$w''(x,t)-\frac{\rho}{k'G}\ddot{w}(x,t)=0$
	$w'''(x,t)=0$	$w'''(x,t)-\left(\frac{\rho}{k'G}+\frac{\rho}{E}\right)\ddot{w}'=0$

Eqs. (5) and (6) can be decoupled and one can obtain an equation which governs the beam transverse deflection $w(x,t)$ and another equation that governs the rotation of the cross section $\psi(x,t)$. The equation governing $w(x,t)$ can be obtained by solving Eq. (6) for $\partial\psi/\partial x$ and substituting the result in Eq. (5). This procedure leads to

$$EI\frac{\partial^4 w}{\partial x^4}+\rho A\frac{\partial^2 w}{\partial t^2}-\rho I\left(1+\frac{EA}{K_s}\right)\frac{\partial^4 w}{\partial x^2\partial t^2}+\frac{\rho^2 AI}{K_s}\frac{\partial^4 w}{\partial t^4}$$
$$=f+\frac{\rho I}{K_s}\frac{\partial^2 f}{\partial t^2}-\frac{EI}{K_s}\frac{\partial^2 f}{\partial x^2}+\sum_{r=1}^{r=R}J_r\frac{\partial^3 w}{\partial x\partial t^2}\delta(x-b_r), \tag{7}$$

where $K_s=k'AG$ is the shear rigidity and f is given by

$$f=f_{\rm ex}(x,t)-\sum_{r=1}^{r=R}m_r\frac{\partial^2 w}{\partial t^2}\delta(x-b_r)-\sum_{i=1}^{i=N}K_i w\delta(x-a_i). \tag{8}$$

It is to be noted that the coupling between the transverse deflection $w(x,t)$ and the bending slope $\psi(x,t)$ is retained via the boundary conditions at the two ends of the beam as shown in Table 1 for conventional end supports. In this table the primes denote differentiation with respect to x, while the dots represent differentiation with respect to time.

In essence, the Timoshenko beam model results in two fourth-order partial differential equations in time and space for the transverse deflection and the bending slope. Consequently, solving the boundary value problem yields two independent sequences of natural frequencies and two corresponding sequences of mode shapes. A particular natural frequency and its corresponding mode shape describe one particular solution to the boundary value problem of the beam. From the eigenfunction expansion sense, all these possible solutions have to be considered in the complete series expansion of the beam field displacements. However, the question of whether the two independent sequences of natural frequencies imply the existence of two distinct spectra of frequencies has been a long standing topic of debate, and hitherto has not been solved completely [34–38]. Fortunately, the use of the Green's functions method to solve the cited problem annihilates sharing the ongoing debate.

We proceed by assuming that the external force $f_{\rm ex}(x,t)$ to be concentrated at location $x=x_0$. Hence it may be expressed formally using the Dirac delta singularity function

$$f_{\rm ex}(x,t)=F_0{\rm e}^{{\rm i}\omega t}\delta(x-x_0), \tag{9}$$

where ω is the excitation frequency.

As may be confirmed by a direct substitution, Eqs. (7) and (8) have a particular harmonic solution of the following form:

$$w(x,t)=W(x){\rm e}^{{\rm i}\omega t}. \tag{10}$$

This allows Eqs. (7) and (8) to be reduced to

$$\frac{d^4W}{dx^4}+\frac{\rho\omega^2}{E}\left(1+\frac{EA}{K_s}\right)\frac{d^2W}{dx^2}-\frac{\rho A\omega^2}{EI}\left(1-\frac{\rho I\omega^2}{K_s}\right)W=\frac{1}{EI}\left(1-\frac{\rho I\omega^2}{K_s}\right)F-\frac{1}{K_s}\frac{d^2F}{dx^2}-\frac{\omega^2}{EI}\sum_{r=1}^{r=R}J_r\frac{dW}{dx}\delta'(x-b_r), \tag{11}$$

where

$$F=F_0\delta(x-x_0)+\omega^2\sum_{r=1}^{r=R}m_rW(b_r)\delta(x-b_r)-\sum_{i=1}^{i=N}k_iW(a_i)\delta(x-a_i). \tag{12}$$

It will be convenient for further progress if, at this stage, we collect together some terms, simplify Eq. (11) and put it in the following form

$$W''''-\beta W''-\alpha^4W=K_1F+K_2F''+K_3\sum_{r=1}^{r=R}J_rW'(b_r)\delta'(x-b_r), \tag{13}$$

where

$$\beta=-\frac{\rho\omega^2L^2}{E}\left(1+\frac{EA}{k_s}\right),\quad \alpha^4=\frac{\rho A\omega^2L^4}{EI}\left(1-\frac{\rho I\omega^2}{k_s}\right),$$
$$K_1=\frac{L^3}{EI}\left(1-\frac{\rho I\omega^2}{k_s}\right),\quad K_2=-\frac{L}{k_s},\quad K_3=-\frac{\omega^2L^2}{EI} \tag{14}$$

and the prime denotes $(1/L)/(d/dx)$.

Eq. (13) is in the form

$$W''''-\beta W''-\alpha^4W=P(x), \tag{15}$$

where

$$P(x)=K_1F(x)+K_2F''(x)+K_3\sum_{r=1}^{r=R}J_rW'(b_r)\delta'(x-b_r). \tag{16}$$

The dynamic Green's function is utilized to find the solution for Eq. (15). Hence, if $G(x,u)$ is the dynamic Green's function for the stated problem the solution of Eq. (15) takes the form [39]

$$W(x)=\int_0^L P(u)G(x,u)\ du. \tag{17}$$

Even though integration must be carried out, one regards Eq. (17) as the solution of the stated problem. For any $P(u)$ we can work out the integration in Eq. (17), at least in principle. In practice one might have to resort to numerical integration and/or one might be able to extract an asymptotic behavior from the integral. Fortunately, the functional form of $P(u)$ given by Eqs. (12) and (16) facilitates performing the integration. Utilizing the properties of the δ-function and performing the integration one gets

$$W(x)=F_0G(x,x_0)+\omega^2\sum_{r=1}^{r=R}m_rW(b_r)G(x,b_r)-\sum_{i=1}^{i=N}k_iW(a_i)G(x,a_i)+K_3\sum_{r=1}^{r=R}J_rW'(b_r)\bar{G}_u(x,b_r), \tag{18}$$

where $G(x,u)$ is the solution of

$$G''''(x,u)-\beta G''(x,u)-\alpha^4G(x,u)=K_1\delta(x-u)+K_2\delta''(x-u), \tag{19}$$

while $\bar{G}(x,u)$ is the solution of

$$G''''(x,u) - \beta G''(x,u) - \alpha^4 G(x,u) = \delta(x-u) \tag{20}$$

and $\bar{G}_u(x,u)$ is the derivative of $\bar{G}(x,u)$ with respect to u.

The functions $G(x,u)$ and $\bar{G}(x,u)$ can be derived using the conventional approach [40–43] or using the Laplace transformation [44]. The Green's function is a two-point function of position and is independent of the forcing term in the inhomogeneous differential equation representing the boundary value problem but depends only upon differential equation being examined and the boundary conditions which are imposed, see books by Roach [39,45]. Furthermore, it is a symmetric function that satisfies the Maxwell–Rayleigh reciprocity law. It is worth mentioning that, in principle, the Green's function for any system can be determined experimentally utilizing its physical meaning: it is the frequency response function of the system.

To this end, one evaluates the $W(x)$ from Eq. (18) at all points of spring and mass attachments, i.e. at $x = a_i$ and b_r, for $i = 1$ to N and $r = 1$ to R. In addition, one evaluates $W'(x)$ at $x = b_r$, for $r = 1$ to R. This gives $N+2R$ equations with $N+2R$ unknowns being:

$$W(a_1), W(a_2), \ldots, W(a_N), W(b_1), W(b_2), \ldots, W(b_R), W'(b_1), W'(b_2), \ldots, W'(b_R).$$

These equations can be written concisely in matrix form as

$$\mathbf{D}\,\underline{\mathbf{z}} = \underline{\mathbf{c}}, \tag{21}$$

where the vectors $\underline{\mathbf{z}}$ and $\underline{\mathbf{c}}$ are given by

$$\underline{\mathbf{z}} = \left\{W(a_1), W(a_2), \ldots W(a_N), W(b_1), W(b_2), \ldots W(b_R), W'(b_1), W'(b_2), \ldots W'(b_R)\right\}^{\mathrm{T}}, \tag{22}$$

$$\underline{\mathbf{c}} = F_0\left\{G(a_1,x_0), \ldots G(a_N,x_0), G(b_1,x_0), \ldots G(b_R,x_0), G_x(b_1,x_0), \ldots G_x(b_R,x_0)\right\}^{\mathrm{T}} \tag{23}$$

and $\mathbf{D} = \mathbf{I}+\mathbf{B}$ where $\mathbf{I}$ is the identity matrix and the matrix $\mathbf{B}$ is given by

$$\mathbf{B} = \begin{bmatrix} \mathbf{B}_{11} & \mathbf{B}_{12} \\ \mathbf{B}_{21} & \mathbf{B}_{22} \end{bmatrix}, \tag{24}$$

where

$$\begin{aligned} \mathbf{B}_{11} &= [G(d_{\ell'}, d_\ell)]\mathbf{V}, && \ell', \ell = 1,2,\ldots,N+R \\ \mathbf{B}_{12} &= \left[\bar{G}_u(d_\ell, b_r)\right]\mathbf{V}, && \ell = 1,2,\ldots,N+R,\ \ r = 1,2,\ldots,R \\ \mathbf{B}_{21} &= [G(b_r, d_\ell)]\mathbf{V}, && r = 1,2,\ldots,R,\ \ \ell = 1,2,\ldots,N+R \\ \mathbf{B}_{22} &= \left[\bar{G}_{ux}(b_r, b_{r'})\right]\mathbf{V}, && r = 1,2,\ldots,R,\ \ r' = 1,2,\ldots,R \end{aligned} \tag{25}$$

$$\begin{aligned} \underline{\mathbf{d}} &= \{a_1, a_2, \ldots, a_N, b_1, b_2, \ldots, b_R\}^{\mathrm{T}}, \\ \mathbf{V} &= \mathrm{diag}\{k_1, k_2, \ldots, k_N, \xi_1, \xi_1, \ldots, \xi_R\}, \\ \mathbf{U} &= \mathrm{diag}\left\{\eta_1, \eta_2, \ldots, \eta_R\right\} \end{aligned} \tag{26}$$

and

$$\xi_r = -m_r\omega^2, \quad \eta_r = K_3 J_r. \tag{27}$$

The unknown vector $\underline{\mathbf{z}}$ can be obtained by solving the matrix Eq. (21). One next substitutes the vector $\underline{\mathbf{z}}$ into Eq. (18) to obtain the deflection at any point x on the beam span. Specifically

$$\begin{aligned} W(x) = F_0 G(x,x_0) &- \sum_{i=1}^{i=N} k_i W(a_i) G(x,a_i) \\ &- \sum_{r=1}^{r=R} \xi_r W(b_r) G(x,b_r) + \sum_{r=1}^{r=R} \eta_r W'(b_r) \bar{G}_u(x,b_r). \end{aligned} \tag{28}$$

It is worth observing that if one is interested in evaluating the natural frequencies of the beam with mass and/or spring attachments, the determinant of the matrix **D** can be set equal to zero to obtain a highly transcendental frequency equation which can be solved by appropriate techniques.

3. Numerical simulations results

A simply supported beam is considered. The parameters selected in the numerical simulations, correspond to the data used in Ref. [8]. The beam span $L = 20\,\text{m}$, modulus of elasticity $E = 19.5 \times 10^{10}\,\text{N/m}^2$, thickness $h = 0.026458\,\text{m}$, width $b = 1\,\text{m}$, density $\rho = 7700\,\text{kg/m}^3$ and the shear factor $k' = 0.85$. Therefore the mass of the beam $m_b = 4073\,\text{kg}$. The mass ratio μ is defined as the sum of the values of the added masses divided by the mass of the beam m_b. In addition, the spatial wavelength of the flexural wave is $\lambda = 2\pi/k_b$ where k_b is the lower wavenumber of the beam without attachments. In all the figures, the dashed lines correspond to the deformed shape of the beam without attachments (unloaded beam) while the solid curves correspond to the deformed shape of the beam with attachments (loaded beam).

3.1. Inertia of added masses are neglected

Fig. 2 shows the dynamic response of a beam due to an excitation force that acts at its mid-span, which is the most important case in general, with an amplitude $F_0 = 1000\,\text{N}$ and an excitation frequency $\omega = 5\,\text{Hz}$. It is observed that the smallest mass that suppresses the vibration in the right half of the beam should be attached in the left half at any peak or trough of the flexural displacement of the unloaded beam. In the figure, a single mass equals 188.9934 kg (the mass ratio $\mu = 0.046$) is attached at $b_1 = 8.6877\,\text{m}$ which corresponds to 1.25λ. Using the close connection between waves and vibrations, Fig. 3 depicts the physical implications of the added mass. This mass was treated as if causing a purely inertial reaction acting as a secondary force with magnitude that depends on the square of the deriving frequency, the transverse displacement amplitude at mass location

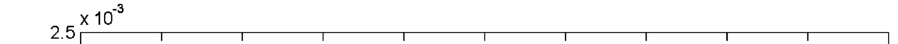

Fig. 2. Analytical displacement responses of loaded and unloaded simply supported beam when $\omega = 5\,\text{Hz}$, $F_0 = 1000\,\text{N}$, $x_0 = 10\,\text{m}$, $m_1 = 188.9934\,\text{kg}$, $J_1 = 0\,\text{kg}\,\text{m}^2$, at $b_1 = 8.6877\,\text{m}$, ——, loaded; - - - -, unloaded; ○, mass location.

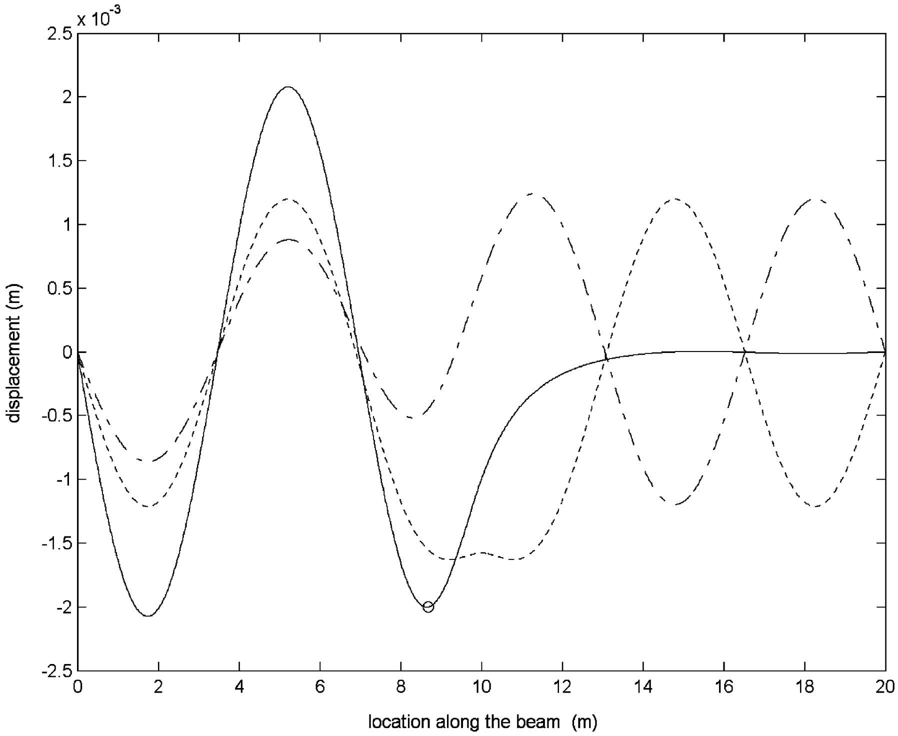

Fig. 3. Analytical displacement responses of loaded and unloaded simply supported beam when $\omega = 5\,\text{Hz}$, - - - -, unloaded with $F_0 = 1000\,\text{N}$, $x_0 = 10\,\text{m}$, -.-.-.-, unloaded; with $F_s = -370.1492\,\text{N}$, $x_s = 8.6677\,\text{m}$, ——, loaded with $F_0 = 10000\,\text{N}$, $x_0 = 10\,\text{m}$, $m_1 = 188.9934\,\text{kg}$, $J_1 = 0\,\text{kg}\,\text{m}^2$ at $b_1 = 8.6877\,\text{m}$, ○, mass location.

and the value of the mass. Specifically $F_s = m\omega^2 W(b_1)$. In the part of the beam to the left of the excitation force, this secondary force creates a flexural wave that is out of phase of the flexural wave generated by the external (primary force). But these two waves are in phase in the part of the beam to the left of the excitation force. When these two waves superimpose, the result is zero in the right part of the beam. In Fig. 3, the dot–dashed line represents the flexural wave due to the inertia force generated by the added mass. The value of this force $F_s = -374.3904\,\text{N}$ acting at location b_1. The dashed line represents the flexural wave due to the primary force $F_0 = 1000\,\text{N}$ at $x_0 = 10\,\text{m}$. As can be seen, these two waves are 180° out of phase in the right half of the beam. Therefore, mixing of the two waves results in destructive interference in that region. From the energy point of view, what the mass does is to inhibit the energy injected into the beam from propagating towards the right side of the beam and to confine it to the left side. This results in increased displacement amplitude in one side of the beam while the amplitude in the other side is diminished. This is not similar to Anderson mode localization phenomenon [1] in which the energy injected into the system cannot propagate very far but instead is confined to the region close to the excitation source.

Obviously the minimum value of the mass required to confine vibration should be located at any peak or trough of the flexural wave since at any of these positions the magnitude of the displacement is a maximum. Therefore to find this value, we first select its location b_1 and then equate the response of the beam due to the primary force at any peak or trough in the right part of the beam, say x_p, to the corresponding response due to the secondary force. Specifically, setting $R = 1$, and $k_r = J_r = 0$, Eq. (28) is simplified to

$$W(x) = F_0 G(x, x_0) + m_1\omega^2(b_1)G(x, b_1). \tag{29}$$

Accordingly Eq. (21) is reduced to a scalar algebraic equation that can be solved for $W(b_1)$. Hence Eq. (29) becomes

$$W(x) = F_0 G(x, x_0) + \frac{m_1\omega^2 F_0 G(b_1, x_0)}{1 - m_1\omega^2 G(b_1, b_1)} G(x, b_1). \tag{30}$$

Setting $x = x_p$, equating $W(x_p) = 0$ and then solving for the required mass lead to the following simple analytical expression:

$$m_1 = \frac{G(x_p, x_0)}{\omega^2 \left[G(b_p, x_0) G(b_1, b_1) - G(b_1, x_0) G(x_p, b_1) \right]}. \quad (31)$$

Therefore, one can always find the minimum value of the mass that can be attached to the beam for any excitation frequency as long as $m_1\omega^2 G(b_1,b_1) \neq 1$ and m_1 is positive. The extension to more than one mass and/or springs are obvious (one must solve a system of simultaneous algebraic equations).

It is to be noted that the ratio of the added mass to the beam mass is very small. Therefore, the changes in the natural frequencies could not be significant as seen from Table 2. The results in Table 2 agree very well with that extracted from Grant [14] who studied vibration of Timoshenko beam carrying a concentrated mass

Table 2
The first six natural frequencies of the unloaded and loaded beam with attached mass $m_1 = 188.9934$ kg at $b_1 = 8.6877$ m (mass ratio $\mu = 0.464$)

Natural frequency (Hz)	Loaded	Unloaded
ω_1	0.1446335	0.150938
ω_2	0.5996017	0.603774
ω_3	1.3210999	1.358405
ω_4	2.3640625	2.414892
ω_5	3.7733779	3.773168
ω_6	5.2571589	5.433185

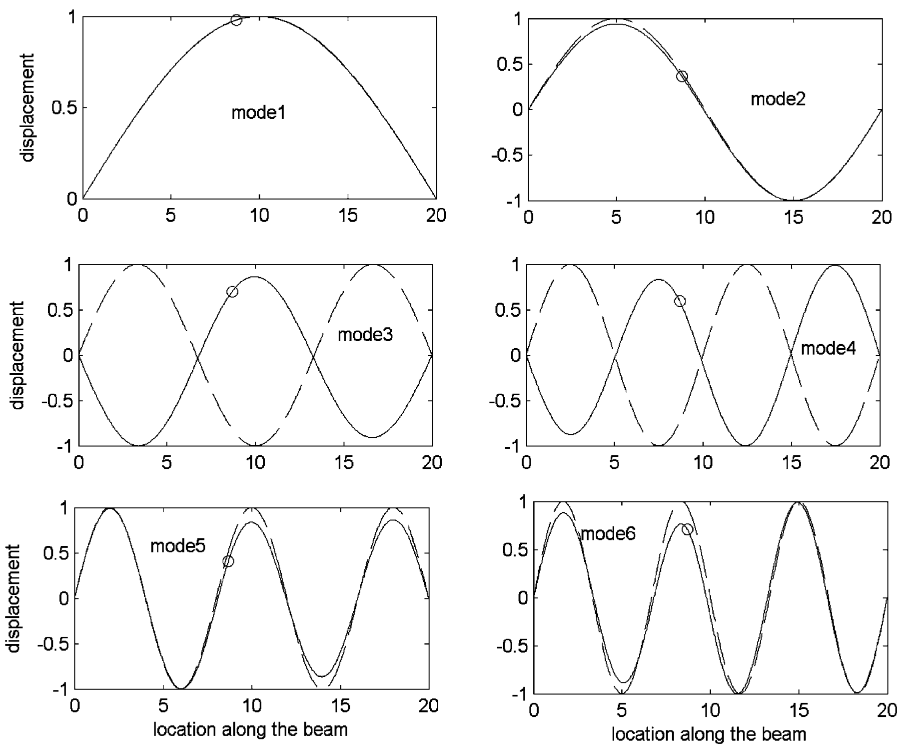

Fig. 4. Mode shapes for loaded and unloaded simply supported beam when, $m_1 = 188.9934$ kg, $J_1 = 0\,\mathrm{kg\,m^2}$ at $b_1 = 8.6877$ m, ——, loaded; - - - -, unloaded; ○, mass location.

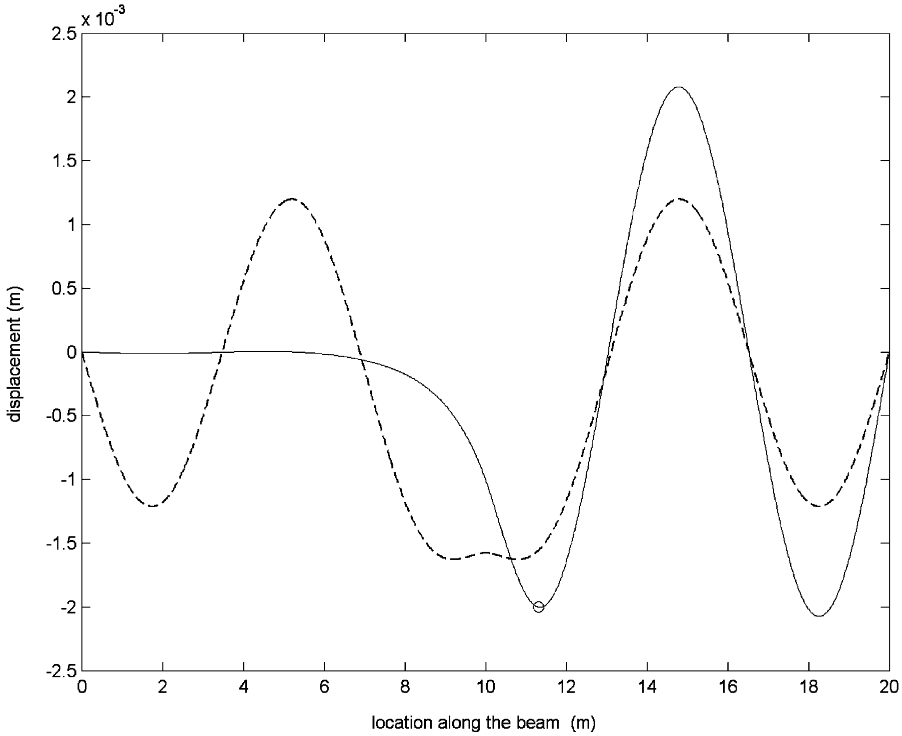

Fig. 5. Analytical displacement responses of loaded and unloaded simply supported beam when $\omega = 5\,\text{Hz}$, $F_0 = 1000\,\text{N}$, $x_0 = 10\,\text{m}$, $m_1 = 188.9934\,\text{kg}$, $J_1 = 0\,\text{kg}\,\text{m}^2$ at $b_1 = 11.3123\,\text{m}$, ——, loaded; - - - -, unloaded; ○, mass location.

at arbitrary position using Laplace transformation. Fig. 4 depicts the corresponding mode shape of the loaded beam compared to the unloaded one. It is seen that the third and the fourth modes of the loaded beam are 180° out of phase of the corresponding unloaded one and the locations of the nodes are displaced a little bit. In addition the modes of the loaded beam are extended through out the whole beam (not localized) as that of the unloaded beam. Therefore, we cannot ascribe the vibration confinement that occurred due to the added mass to Anderson's localization phenomenon.

If one is interested in controlling the vibration in the left half of the beam, a mass of the same value should be attached in the right half instead, as shown in Fig. 5. Instead, one can use a set consisting of two masses each of 93 kg, or a set consisting of three masses of 61 kg each. In these alternative choices, masses are attached in the right half of the beam at peaks or troughs of the flexural wave of the unloaded beam.

A final added note is concerned with the sensitivity of the dynamic response of the loaded beam to small differences in system's parameters. It was found that small deviations in the position of the added mass and/or its location can cause small changes in the dynamic behavior of the loaded beam compared with the case when the optimal mass is located at a peak or a trough. But small deviations in the excitation frequency (detuning), while fixing the value of the added mass and its location, can cause drastic change in the dynamic behavior of the loaded beam and break down the vibration confinement. Therefore, confinement is much more sensitive to the deviations in the excitation frequency than the deviations in the value of the added mass or its location.

Fig. 6 depicts the dynamic response of the beam for excitation frequency $\omega = 60\,\text{Hz}$. The excitation force acts at a mid-span with an amplitude $F_0 = 1000\,\text{N}$. A single mass of value 10.85 kg is attached at 8.5315 m which corresponds to 4.25λ, is needed to suppress the vibration in the right half of the beam. Instead, one can use two masses each of 5.3 kg, or a set consisting of three masses of 3.5 kg each. Alternatively, a set consisting of four masses each equals 2.62 kg can be used. These masses are attached in the left half of the beam at peaks or troughs of the flexural wave of the unloaded beam. Another interesting observation is that a similar effect

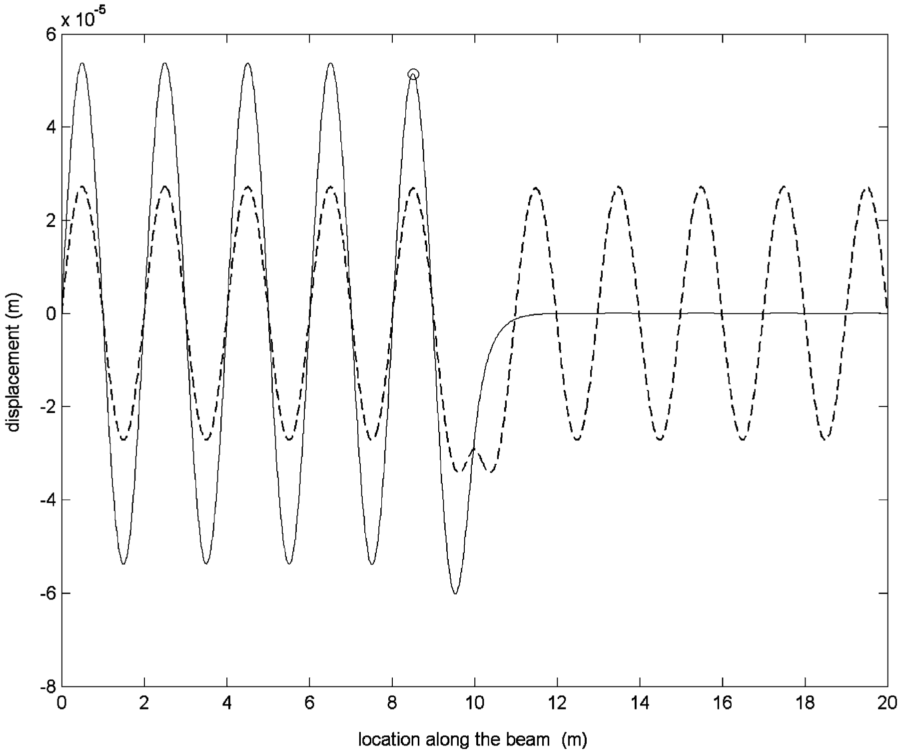

Fig. 6. Analytical displacement responses of loaded and unloaded simply supported beam when $\omega = 60\,\text{Hz}$, $F_0 = 1000\,\text{N}$, $x_0 = 10\,\text{m}$, $m_1 = 10.85\,\text{kg}$, $J_1 = 0\,\text{kg}\,\text{m}^2$, at $b_1 = 8.5315\,\text{m}$, ——, loaded; - - - -, unloaded; ○, mass location.

can be obtained using a spring instead of a mass. In Fig. 7, the excitation frequency is $\omega = 60\,\text{Hz}$, a spring of stiffness 26 MN/m is attached at $a_1 = 10.0371\,\text{m}$, which corresponds approximately to 5λ from the left end of the beam, has an effect similar to that of a mass in controlling the vibration propagation. Other alternative, which gives a similar effect, can be obtained using a combination of a spring and a mass. In Fig. 8, the excitation frequency $\omega = 60\,\text{Hz}$, a spring of stiffness 4 MN/m is attached at $a_1 = 10.539\,\text{m}$, which corresponds to 5.25λ from the right end of the beam and a mass equals 7.75 kg is attached at $b_1 = 8.5315\,\text{m}$ which corresponds to 4.25λ.

With regards to the change of the location of the excitation force, Fig. 9 indicates the transverse displacements when the force with excitation frequency $\omega = 100\,\text{Hz}$ acts at $x_0 = 6\,\text{m}$ from the left end of the beam. For the unloaded beam the vibrational amplitude to the right of the force is more pronounced than that to the left of the force. However, the attachment of 276 kg mass can shift all the vibrational energy to the left of the force and leave the right region at rest position despite the fact that the unloaded beam experiences substantial deflection within that region. The added mass appears to be relatively large however the mass ratio is 0.0677, which is acceptable for engineering applications. If the beam were modeled as an Euler–Bernoulli beam, the attached mass would be 299 kg. Therefore for a Timoshenko formulation one uses less mass to suppress the vibration. Fig. 10 shows that one can attach a spring of stiffness 71.75 MN/m at $a_1 = 6.6111\,\text{m}$, which corresponds to 4.25λ from the left end of the beam, to give a similar effect as of a mass in controlling the propagation of vibration.

It is interesting to note that if one requires to confine the vibration near the mechanical disturbance and to prevent it from transmitting towards other parts of the structure, he can use two springs as shown in Fig. 11. The excitation force is 1000 N acts at the mid-span of the beam and the excitation frequency $\omega = 50\,\text{Hz}$. It is needed to suppress the vibration on both sides of the beam. This can be achieved by attaching two springs with the same stiffness $k_1 = k_2 = 28\,\text{MN/m}$ at $a_1 = 8.2455\,\text{m}$ and $a_2 = 11.7545\,\text{m}$. These locations correspond to 3.75λ from the left support and 3.75λ from the right support respectively.

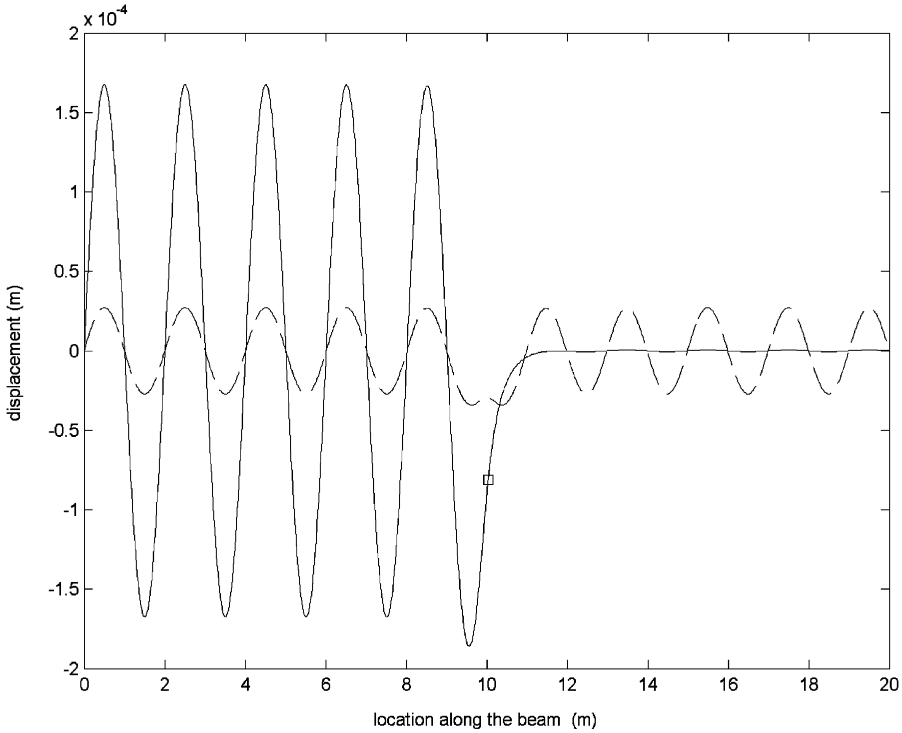

Fig. 7. Analytical displacement responses of loaded and unloaded simply supported beam when $\omega = 60\,\text{Hz}$, $F_0 = 1000\,\text{N}$, $x_0 = 10\,\text{m}$, $k = 26\,\text{MN/m}$, at $a_1 = 10.0371\,\text{m}$, ——, loaded; - - - -, unloaded; □, spring location.

One particular consequence of the above results is that there are many alternatives that can be used to suppress vibration in a certain part of a beam structure. This is beneficial to practical implementation because one can choose the alternative that suits certain application.

3.2. Inertia of added masses are not neglected

A point mass is, of course, a mathematical fiction which cannot be realized physically. Therefore, a finite mass of a suitable size is added instead. To investigate the effect of the mass moment of inertia of this added mass on the vibration suppression, Fig. 12 depicts the dynamic response of a simply supported beam with excitation force acts at a mid-span with an amplitude $F_0 = 1000\,\text{N}$ and an excitation frequency $\omega = 60\,\text{Hz}$. A single mass of a 7.9 kg with a mass moment of inertia $J_1 = 0.9\,\text{kg}\,\text{m}^2$ attached at 8.5315 m, which corresponds to 4.25λ, is needed to suppress the vibration in the right half of the beam. This is compared to a mass of value 10.85 kg if one neglects the moment of inertia of the attached mass as given in Fig. 6. Therefore, the value of the added mass required to suppress vibration is decreased when its mass moment of inertia was taken into account. Table 3 summarizes the results for the Timoshenko beam model with and without neglecting the effect of moment of inertia of the added masses compared to the results obtained from using Euler–Bernoulli beam model for different excitation frequencies. This table shows that the results from the Timoshenko beam theory are markedly different from that of the Euler–Bernoulli theory. These differences become more pronounced at high frequencies. Furthermore, the analyst should be aware that the aspect ratio of the beam also could cause many differences between both theories.

3.3. A bi-harmonic excitation force

Fig. 13 shows the dynamic response of a beam subjected to a bi-harmonic force that acts at the mid-span of the beam; $x_0 = 10\,\text{m}$. The excitation amplitudes are $F_{01} = F_{02} = 1000\,\text{N}$ with excitation frequencies

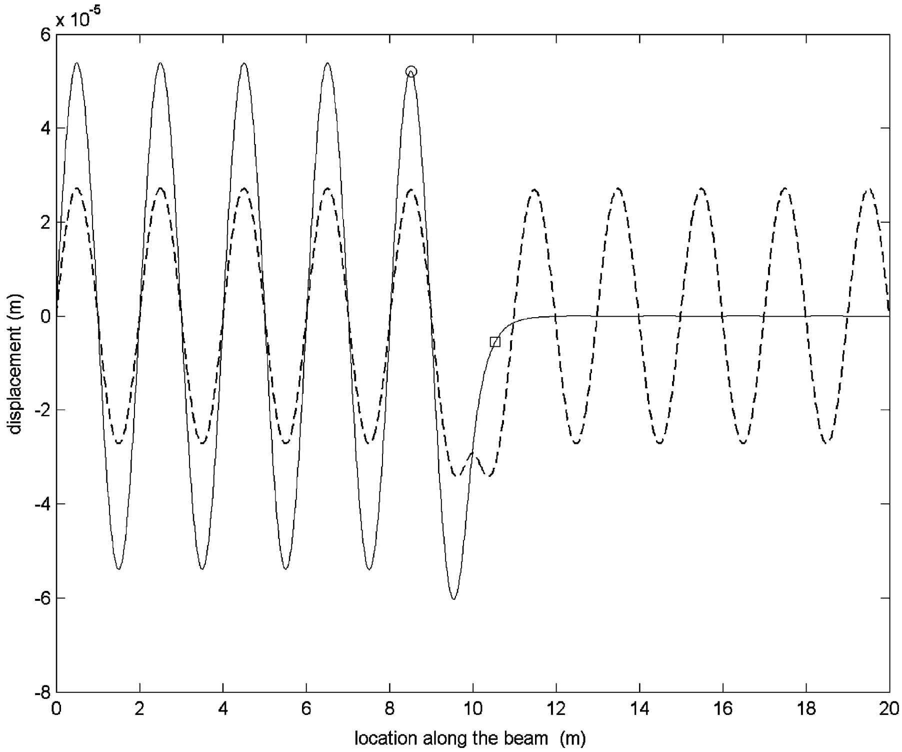

Fig. 8. Analytical displacement responses of loaded and unloaded simply supported beam when $\omega = 60\,\text{Hz}$, $F_0 = 1000\,\text{N}$, $x_0 = 10\,\text{m}$, $k = 4$ MN/m at $a_1 = 10.5390\,\text{m}$, $m = 7.75\,\text{kg}$, $J_1 = 0\,\text{kg}\,\text{m}^2$ at $b_1 = 8.5315\,\text{m}$, ——, loaded; - - - -, unloaded; ○, mass location; □, spring location.

$\omega_1 = 60\,\text{Hz}$ and $\omega_2 = 100\,\text{Hz}$. In the figure, $m_1 = 10.5\,\text{kg}$ is attached at $b_1 = 8.5315\,\text{m}$ which corresponds to $4.25\lambda_1$, and $m_2 = 38.5\,\text{kg}$ is attached at $b_2 = 6.6111\,\text{m}$ which corresponds to $4.25\lambda_2$, where λ_1 and λ_2 are the flexural wavelengths corresponding to the excitation frequencies ω_1 and ω_2, respectively. The mass moments of inertia of the added masses $J_1 = J_2 = 0.25\,\text{kg}\,\text{m}^2$. This situation can be understood physically as the system is linear and the superposition principle holds. From the results presented one concludes that the analysis can be extended in a relatively straightforward way to tackle a beam subjected to a periodic excitation.

3.4. Suppression of vibration near resonance

The vibration reduction along the entire beam can be achieved near resonance conditions because in this case the motion of the beam is a simple spatial waveform that can be cancelled by a single opposing wave created by a single attachment. Fig. 14 depicts the displacement responses of loaded and unloaded beams. The excitation frequency $\omega = 25.49\,\text{Hz}$ which is near the 13th natural frequency of the beam without attachments. If a 40 kg mass was attached at $b_1 = 8.4667\,\text{m}$ which corresponds to 2.75λ, the vibration along the entire beam would be cancelled. The mass ratio is 0.0098. Also the added mass can be attached at any peaks or troughs of the flexural wave. This effect is similar to that of the dynamic vibration absorber, which is used to quench the vibration of the harmonically excited single degree of freedom system. Physically the inertia force of the added mass creates anti-wave so that when the original flexural wave and its anti-dote are added together the result is no vibration. Comparable phenomena are encountered in active noise cancellation by introducing annihilating signal (anti-sound) [46].

3.5. Dual excitation forces

For a beam excited by two excitation forces acting at different locations, one can control the vibration in the middle or in any part of the beam depending on the applications. Fig. 15 shows a simply supported beam

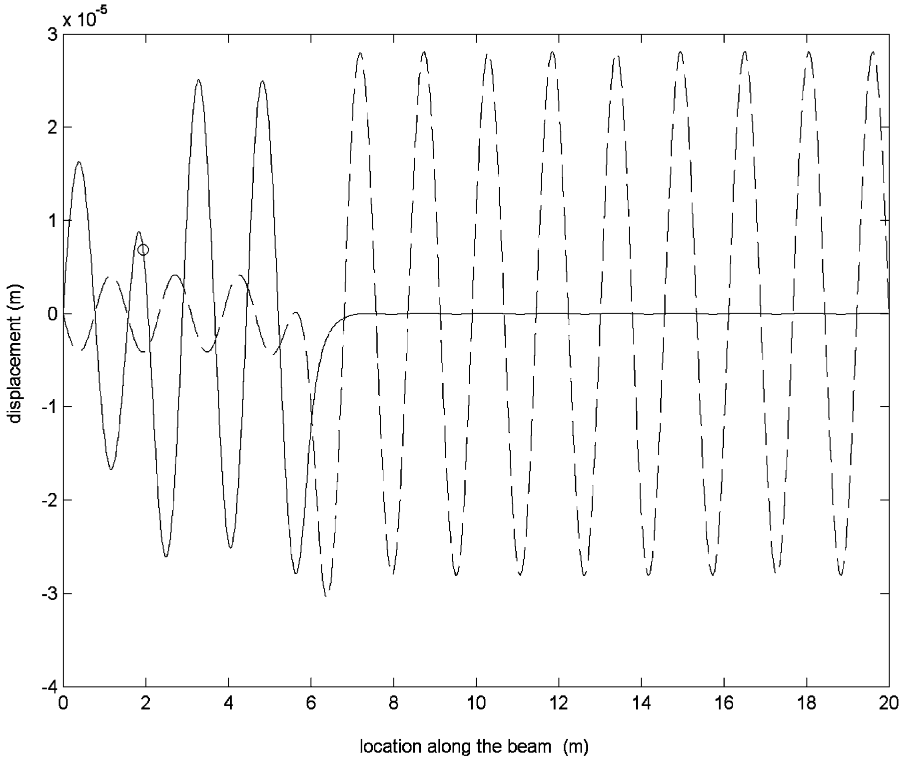

Fig. 9. Analytical displacement responses of loaded and unloaded simply supported beam when $\omega = 100\,\text{Hz}$, $F_0 = 1000\,\text{N}$, $x_0 = 6\,\text{m}$, $m_1 = 276\,\text{kg}$, $J = 0\,\text{kg}\,\text{m}^2$ at $b_1 = 1.9444\,\text{m}$, ——, loaded; ----, unloaded; ○, mass location.

subjected to two excitation forces acting at $x_{01} = 6\,\text{m}$ and $x_{02} = 14\,\text{m}$ from the left end support. The value of each force is $F_{01} = F_{02} = 1000\,\text{N}$ with frequency $\omega_1 = \omega_2 = 100\,\text{Hz}$. To suppress the vibration in the part of the beam between the two forces one can attach two masses $m_1 = m_2 = 260\,\text{kg}$ at $b_1 = 1.9444\,\text{m}$ and $b_2 = 14.944\,\text{m}$, which corresponds to 1.25λ from the left support and 3.25λ from the right support, respectively. The mass ratio is 0.1276, which is still reasonable for engineering applications.

Fig. 16 depicts the same beam but for excitation frequency $\omega = 60\,\text{Hz}$. The response of the unloaded beam shows that the vibration is diminished between the two applied forces. However, in certain application one needs to suppress the vibration at the left part of the force that acts at 6 m. Then one should attach an 80 kg mass at 2.25λ from the left support and another 20 kg mass at 4.25λ from the right support. These distances correspond to 4.5167 m and 11.4685 m from the left support, respectively. On the other hand, if one is interested in suppressing the vibration to the right of the force that acts at $x_{02} = 14\,\text{m}$, a 20 kg mass should be attached at $b_1 = 11.4685\,\text{m}$ and another 80.5 kg mass at $b_2 = 15.483\,\text{m}$ as shown in Fig. 17.

3.6. Different end conditions

The formulation presented is applicable to any beam structure with conventional and non-conventional boundary conditions. This is because the boundary conditions are impeded in the derived Green's function of a specific beam. For example, to suppress the vibration in a cantilevered beam, Fig. 18 shows a 4 m long beam excited at its tip with an excitation force $F_0 = 1000\,\text{N}$ and excitation frequency $\omega = 23.887\,\text{Hz}$. This excitation frequency is close to the third resonance frequency of the unloaded beam. The mass required to suppress the vibration along that beam and makes it nearly stationary is found to be equal to 20 kg and to be located at $b_1 = 2.775\,\text{m}$ from the fixed end. The mass ratio is 0.0245. A similar effect can be obtained by using a spring instead of a mass. The required spring stiffness is 0.6 MN/m and to be attached at $a_1 = 2.775\,\text{m}$ as depicted in Fig. 19.

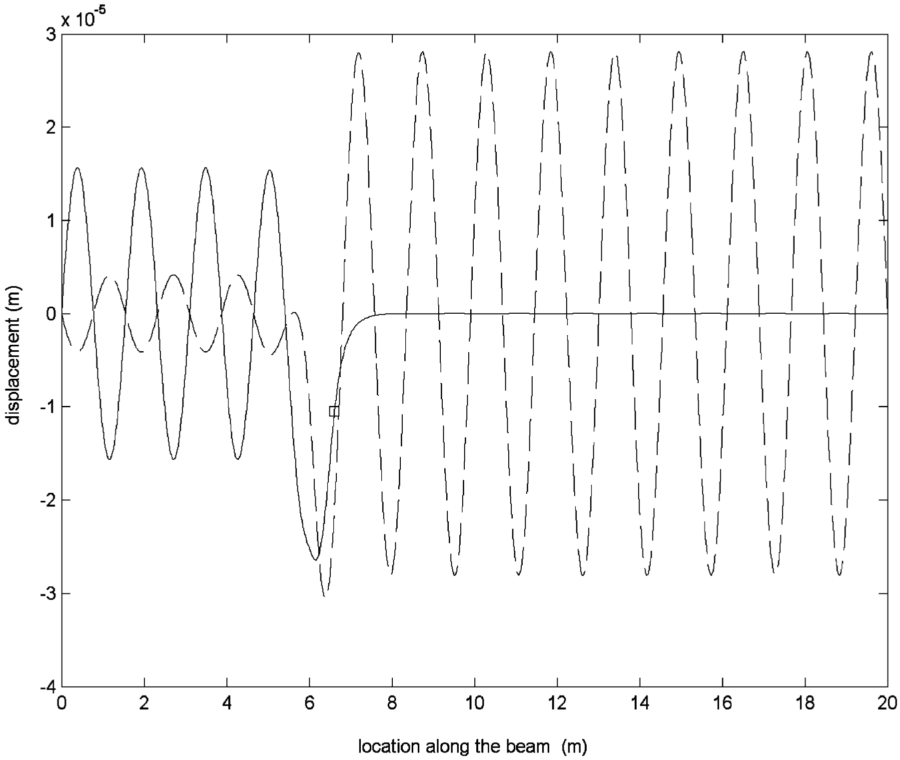

Fig. 10. Analytical displacement responses of loaded and unloaded simply supported beam when $\omega = 100\,\text{Hz}$, $F_0 = 1000\,\text{N}$, $x_0 = 6\,\text{m}$, $k = 71.75\,\text{N/m}$ at $a_1 = 6.6111\,\text{m}$, ——, loaded; - - - -, unloaded; □, spring location.

Fig. 20 illustrates the deformation fields of loaded and unloaded clamped–clamped beam. The excitation frequency $\omega = 4.55\,\text{Hz}$ which is near the fourth natural frequency of unloaded beam. The excitation force is $F_0 = 1000\,\text{N}$ acts at mid-span of the beam. A spring of stiffness $k = 8\,\text{MN/m}$ attached at $a_1 = 5.4946\,\text{m}$, which corresponds to 0.75λ, is found to cancel the vibration along the entire beam.

In Fig. 21, the same clamped–clamped beam is excited by a force $F_0 = 1000\,\text{N}$ that acts at its mid-span. The excitation frequency $\omega = 10\,\text{Hz}$. It is observed that a spring of stiffness $k = 1.95\,\text{MN/m}$ attached at $a_1 = 8.6\,\text{m}$, which corresponds to 1.75λ would cancel out the vibration in the left part of the beam. If the spring cannot be used due to some practical constraints, a mass equals 354 kg would be attached at $b_1 = 8.6\,\text{m}$ to give the same effect as that of the spring as shown in Fig. 22. The mass ratio is 0.0869.

3.7. Imposing a node at certain location

By properly selecting values of the attachments' parameters and their locations, one could eliminate motion at a specific point on the beam, i.e. creating a node at this location. This is important for some design applications where one could place instrument that sensitive to vibration at or near a purposely created node. For the sake of simplicity we assume no springs are attached to the beam. In addition we assume only one point mass is to be attached at distance b_1. Therefore, Eq. (30) holds. To introduce a node at a desired location x_n along the beam we require that

$$W(x_n) = F_0 G(x_n, x_0) + \frac{m_1 \omega^2 F_0 G(b_1, x_0)}{1 - m_1 \omega^2 G(b_1, b_1)} G(x_n, b_1) = 0. \tag{32}$$

Solving for the required mass

$$m_1 = \frac{G(x_n, x_0)}{\omega^2 [G(b_1, b_1) G(x_n, x_0) - G(b_1, x_0) G(x_n, b_1)]}. \tag{33}$$

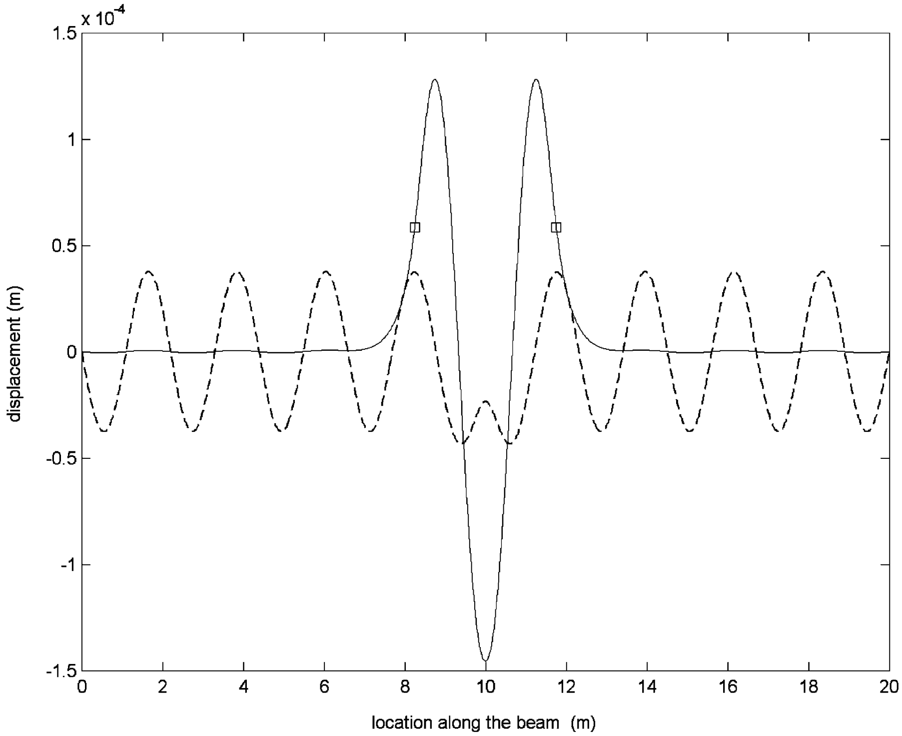

Fig. 11. Analytical displacement responses of loaded and unloaded simply supported beam when $\omega = 50\,\text{Hz}$, $F_0 = 1000\,\text{N}$, $x_0 = 10\,\text{m}$, $k_1 = k_2 = 28$ MN/m at $a_1 = 8.2455\,\text{m}$, $a_2 = 11.7545\,\text{m}$, ——, loaded; - - - -, unloaded; □, spring location.

Therefore, a node can always be introduced anywhere along the structure for any excitation frequency as long as $m_1\omega^2 G(b_1,b_1)\neq 1$ and m_1 is positive in addition to the mass ratio should be small to have a meaningful result. Eq. (33) indicates that the value of the required mass does not depend on the amplitude of the excitation force. More recently, Cha [47] has used simple oscillators to introduce a node at any location of a harmonically oscillating Euler–Bernoulli beam. His analysis is based on the assumed mode method.

Fig. 23 illustrates the above results. A simply supported beam is excited by a harmonic force of amplitude $F_0 = 1000\,\text{N}$ with excitation frequency $\omega = 5\,\text{Hz}$ that acts at the mid-span of the beam; $x_0 = 10\,\text{m}$. The required node, say, is to be introduced at the point of application of the external force i.e. $x_n = 10\,\text{m}$, and the selected location of the attached mass is $b_1 = 1.25\lambda$ which corresponds to 8.6877 m. Then value of the mass would be $m_1 = 369.7794\,\text{kg}$. The mass ratio is 0.0908. One may suspect the results shown in Fig. 23 arguing that since the force is being applied at a single point, then if the beam has a node at this point, then the forced vibration will be zero because no work can be done by the excitation force. But this suspicion can be proven unfounded because a similar situation occurs in the undamped dynamic vibration absorber theory. The excitation force that acts on the main mass (primary mass) results in vibration of both main mass and absorber mass (auxiliary mass). However, when tuning is satisfied, the main mass becomes stationary (no work is injected into the system) but the absorber mass is being vibrated. Ginsberg [48] has a good discussion on this subject. Consider again the aforementioned simply supported beam, now if it is desired that the node be imposed at $x_n = 11\,\text{m}$, then the value of the attached mass would be $m_1 = 234.1015\,\text{kg}$ ($\mu = 0.0575$) as illustrated in Fig. 24. Consider now a cantilevered beam of length 4 m that is subjected to a concentrated harmonic force of amplitude $F_0 = 1000\,\text{N}$ and frequency $\omega = 20\,\text{Hz}$ acts at $x_0 = 4\,\text{m}$. It is wished that a node be imposed at $x_n = 3.5\,\text{m}$. If one selects the position of the added mass to be $b_1 = 3\,\text{m}$, then its value will be 149.9417 kg.

The mass ratio is $\mu = 0.184$. Fig. 25 illustrates the lateral displacement of the loaded beam compared to that of the unloaded one.

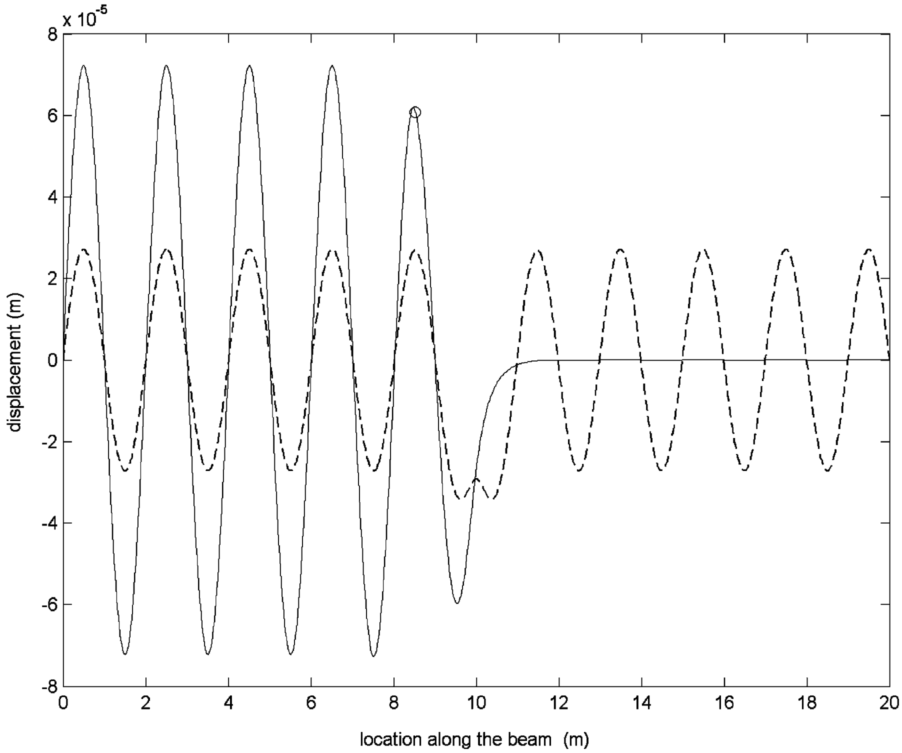

Fig. 12. Analytical displacement responses of loaded and unloaded simply supported beam when $\omega = 60\,\text{Hz}$, $F_0 = 1000\,\text{N}$, $x_0 = 10\,\text{m}$, $m_1 = 7.9\,\text{kg}$, $J_1 = 0.9\,\text{kg}\,\text{m}^2$ at $b_1 = 8.5315\,\text{m}$; ——; loaded; - - - -, unloaded; ○, mass location.

Table 3
Comparison between Timoshenko and Euler–Bernoulli models

No. of masses or springs	Frequency	Euler–Bernoulli formulation	Timoshenko formulation	Timoshenko formulation	
	(Hz)	m (kg)	m (kg), $J = 0$	m (kg)	J (kg m^2)
One mass	20	420	420	363	0.9
	60	13.5	10.85	7.9	0.9
	100	55.6	49.5	38.5	0.25
Two masses	20	145	145	115	0.9
	60	6.55	5.3	3.72	0.9
	100	24.5	21.75	14.6	0.25
Three masses	20	85	85	63	0.9
	60	4.3	3.5	1.88	0.9
	100	15.7	14.1	7.63	0.25
Four masses	20	61.5	61.5	41.75	0.9
	60	3.24	2.62	0.95	0.9
	100	11.55	10.41	4.51	0.25

4. Conclusions

The problem of achieving vibration suppression or confinement in a beam structure subjected to an external excitation force has been examined analytically. This is achieved by adding translational springs and/or finite masses having mass moment of inertia at different locations on the beam. For a simply supported beam with

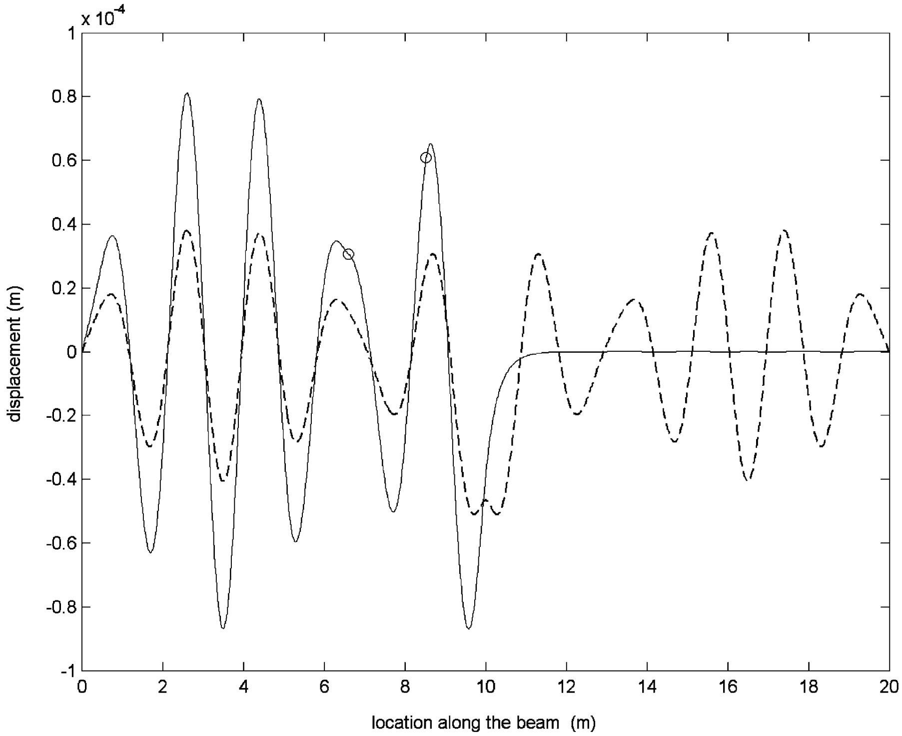

Fig. 13. Analytical displacement responses of loaded and unloaded simply supported beam when $\omega_1 = 20\,\text{Hz}$, $\omega_2 = 60\,\text{Hz}$, $F_{01} = F_{02} = 1000\,\text{N}$, $x_0 = 10\,\text{m}$, $m_1 = 400\,\text{kg}$, $m_2 = 10.5\,\text{kg}$, $J_1 = J_2 = 0.25\,\text{kg}\,\text{m}^2$ at $b_1 = 4.3445\,\text{m}$, $b_2 = 8.5315\,\text{m}$, ——, loaded; - - - -, unloaded; ○, mass location.

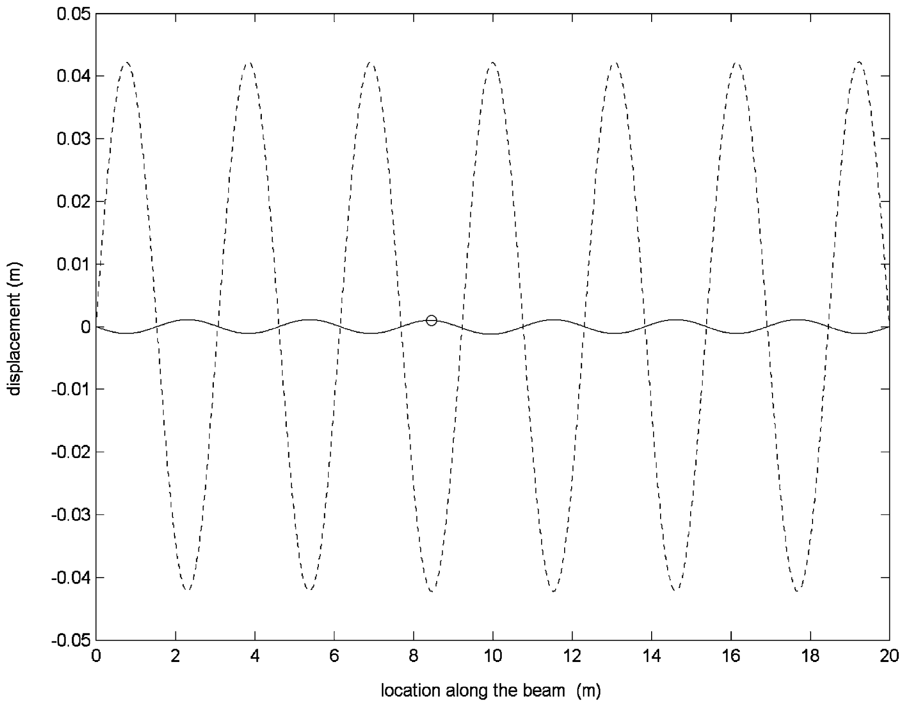

Fig. 14. Analytical displacement responses of loaded and unloaded simply supported beam when $\omega = 25.49\,\text{Hz}$, $F_0 = 1000\,\text{N}$, $x_0 = 10\,\text{m}$, $m_1 = 40\,\text{kg}$, $J_1 = 0\,\text{kg}\,\text{m}^2$ at $b_1 = 8.4667$, ——, loaded; - - - -, unloaded; ○, mass location.

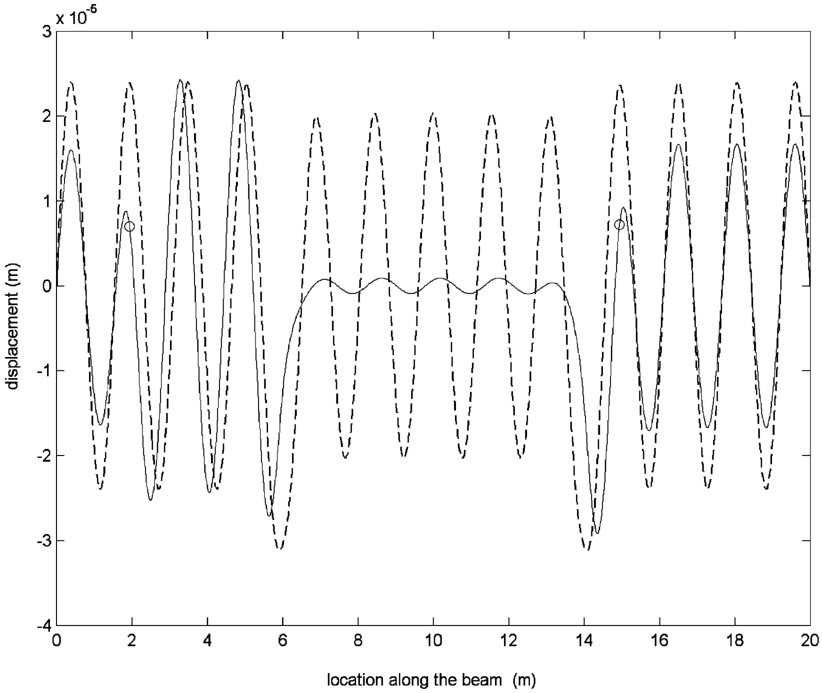

Fig. 15. Analytical displacement responses of loaded and unloaded simply supported beam when $F_{01} = F_{02} = 1000\,\mathrm{N}$, $\omega_1 = \omega_2 = 100\,\mathrm{Hz}$, $x_{01} = 6\,\mathrm{m}$, $x_{02} = 14\,\mathrm{m}$, $m_1 = m_2 = 260\,\mathrm{kg}$, $J_1 = J_2 = 0\,\mathrm{kg\,m^2}$ at $b_1 = 1.944\,\mathrm{m}$ and $b_2 = 14.944\,\mathrm{m}$, ——, loaded; - - - -, unloaded; ○, mass location.

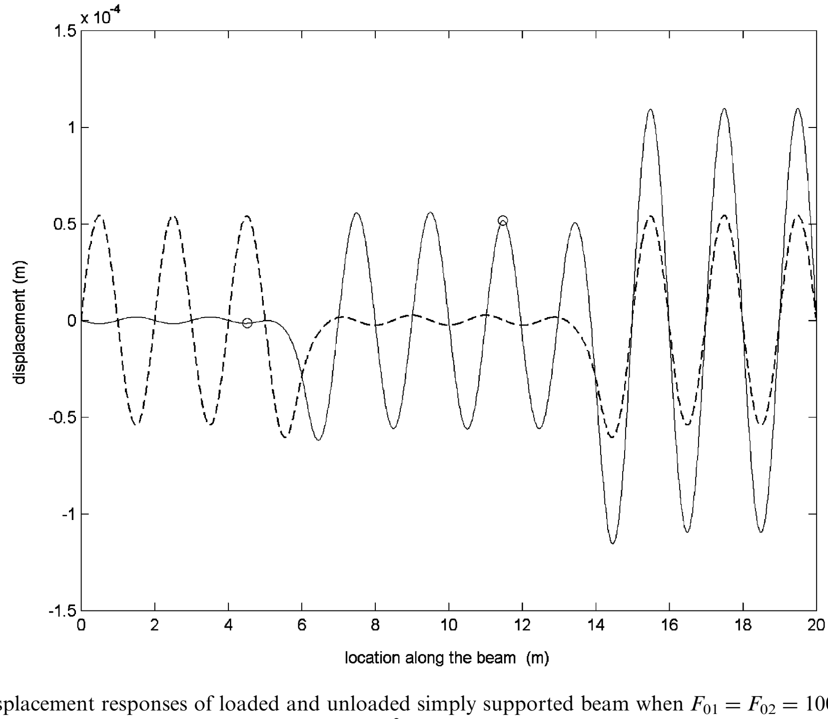

Fig. 16. Analytical displacement responses of loaded and unloaded simply supported beam when $F_{01} = F_{02} = 1000\,\mathrm{N}$, $\omega_1 = \omega_2 = 60\,\mathrm{Hz}$, $x_{01} = 6\,\mathrm{m}$, $x_{02} = 14\,\mathrm{m}$, $m_1 = 80.5\,\mathrm{kg}$, $m_2 = 20\,\mathrm{kg}$, $J_1 = J_2 = 0\,\mathrm{kg\,m^2}$ at $b_1 = 4.5167\,\mathrm{m}$ and $b_2 = 11.4685\,\mathrm{m}$, ——, loaded; - - - -, unloaded; ○, mass location.

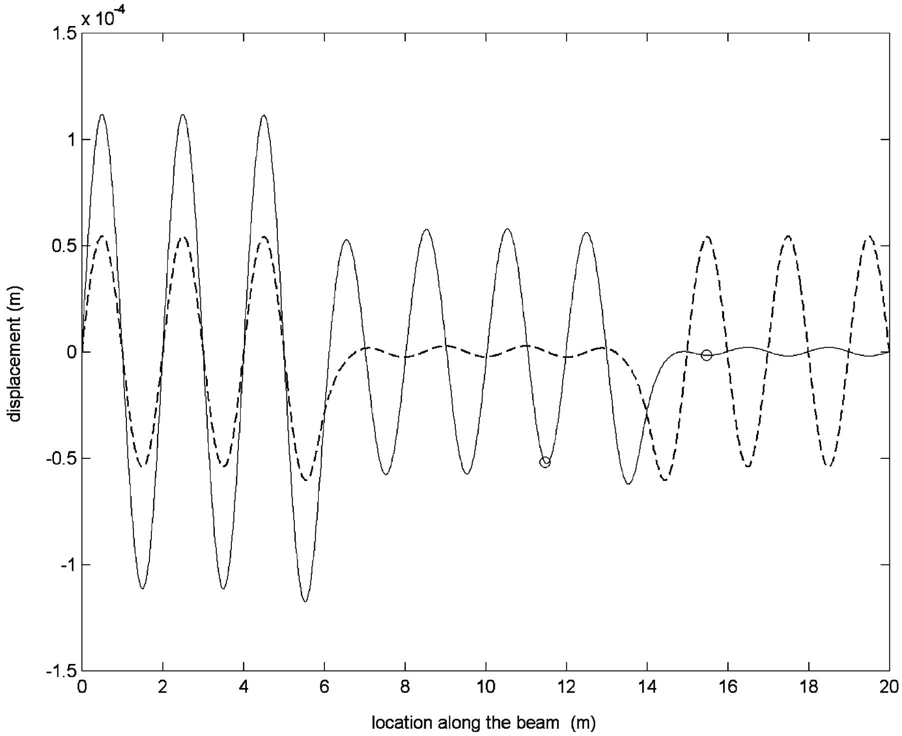

Fig. 17. Analytical displacement responses of loaded and unloaded simply supported beam when $F_{01} = F_{02} = 1000\,\text{N}$, $\omega_1 = \omega_2 = 60\,\text{Hz}$, $x_{01} = 6\,\text{m}$, $x_{02} = 14\,\text{m}$, $m_1 = 20\,\text{kg}$, $m_2 = 80.5\,\text{kg}$, $J_1 = J_2 = 0\,\text{kg}\,\text{m}^2$ at $b_1 = 11.4685\,\text{m}$ and $b_2 = 15.483\,\text{m}$, ——, loaded; - - - -, unloaded; ○, mass location.

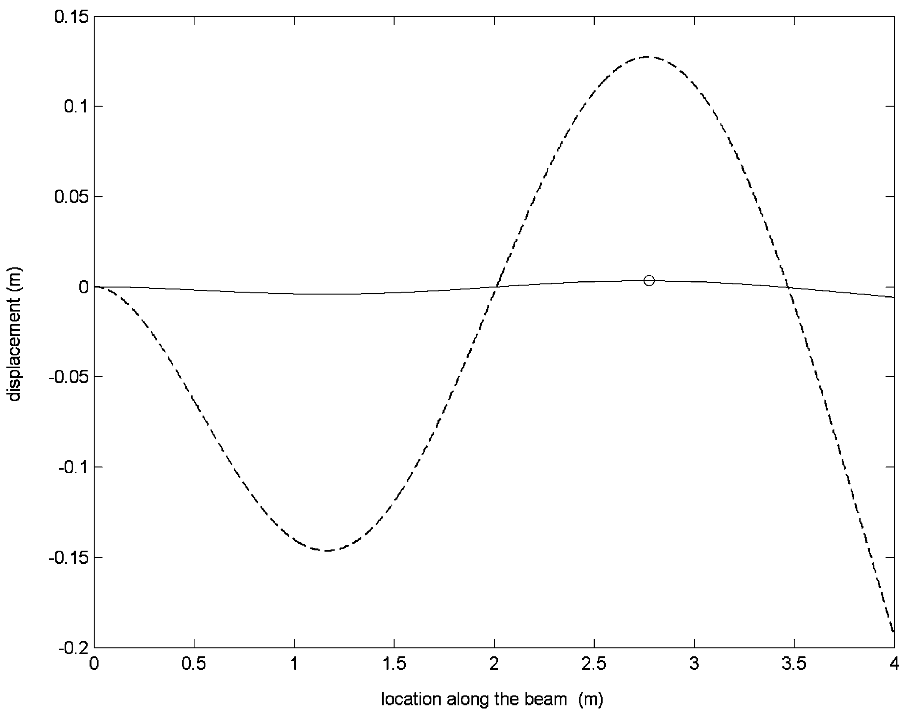

Fig. 18. Analytical displacement responses of loaded and unloaded cantilever beam when $\omega = 23.5887\,\text{Hz}$, $F_0 = 1000\,\text{N}$, $x_0 = 4\,\text{m}$, $m_1 = 20\,\text{kg}$, $J_1 = 0\,\text{kg}\,\text{m}^2$ at $b_1 = 2.775\,\text{m}$, ——, loaded; - - - -, unloaded; ○, mass location.

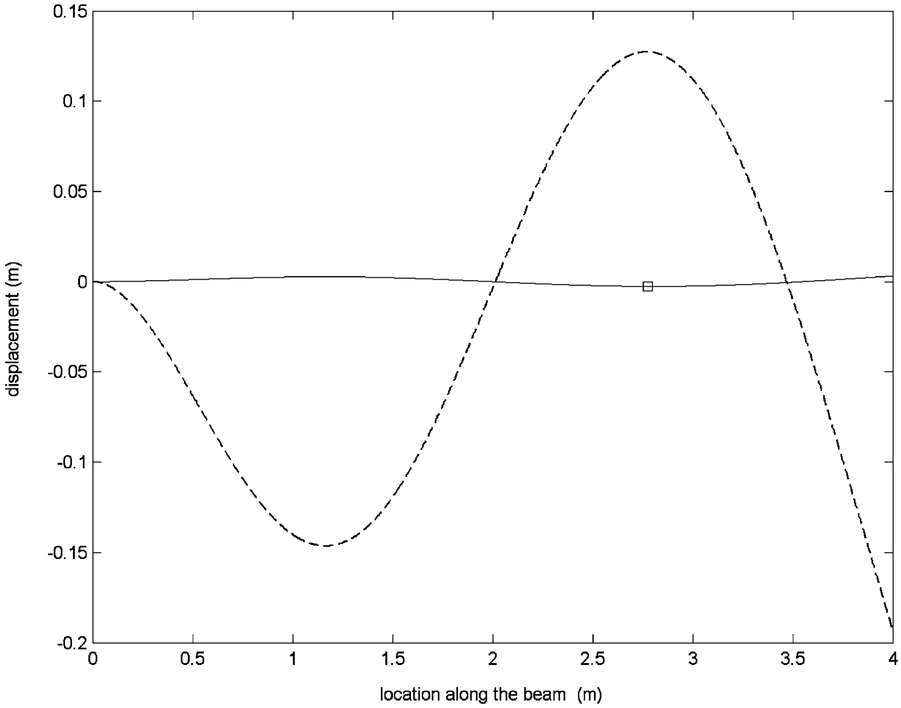

Fig. 19. Analytical displacement responses of loaded and unloaded cantilever beam when $\omega = 23.5887\,\text{Hz}$, $F_0 = 1000\,\text{N}$, $x_0 = 4\,\text{m}$, $k_1 = 20\,\text{kg}$, at $a_1 = 2.775\,\text{m}$, ——, loaded; - - - -, unloaded; □, spring location.

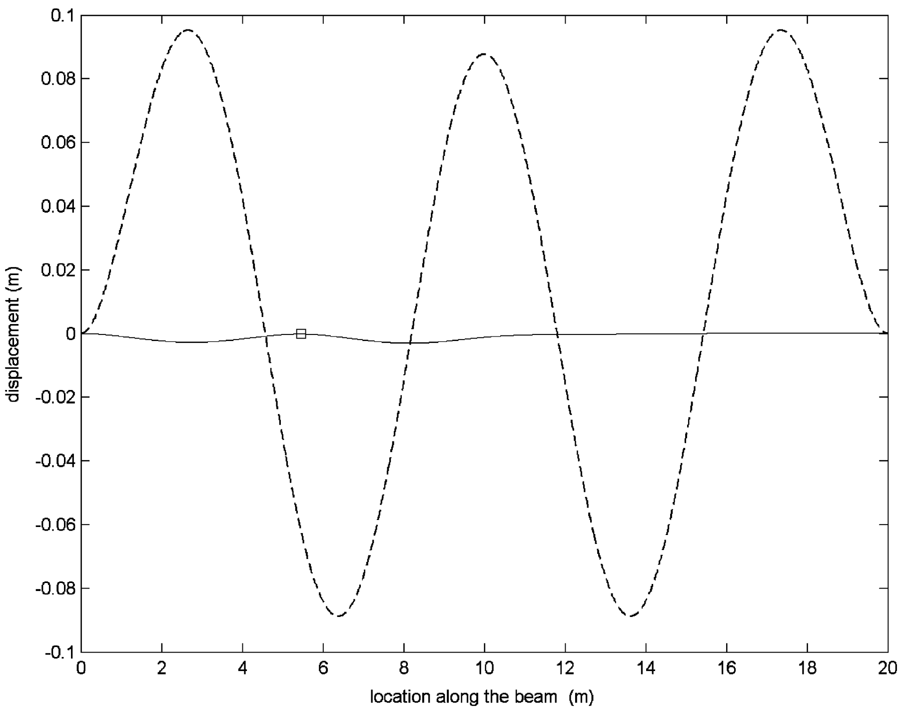

Fig. 20. Analytical displacement responses of loaded and unloaded clamped–clamped beam when $\omega = 10\,\text{Hz}$, $F_0 = 1000\,\text{N}$, $x_0 = 10\,\text{m}$, $k_1 = 1.95\,\text{MN/m}$ at $a_1 = 8.6\,\text{m}$, ——, loaded; - - - -, unloaded; □, spring location.

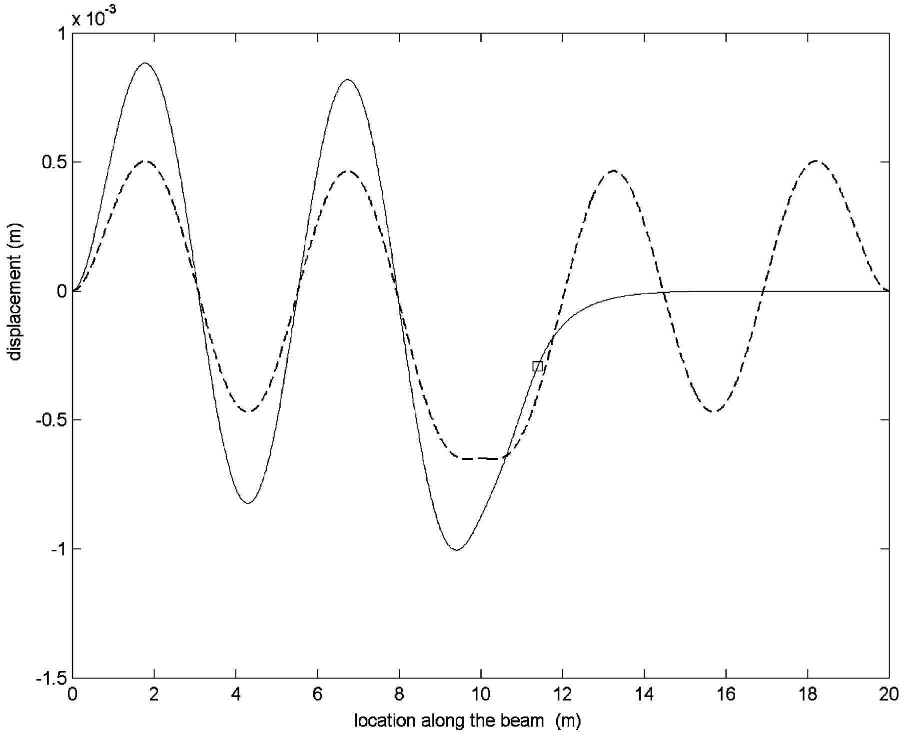

Fig. 21. Analytical displacement responses of loaded and unloaded clamped–clamped beam when $\omega = 10\,\text{Hz}$, $F_0 = 1000\,\text{N}$, $x_0 = 10\,\text{m}$, $k_1 = 1.95\,\text{MN/m}$ at $a_1 = 11.3992\,\text{m}$, ——, loaded; - - - -, unloaded; □, spring location.

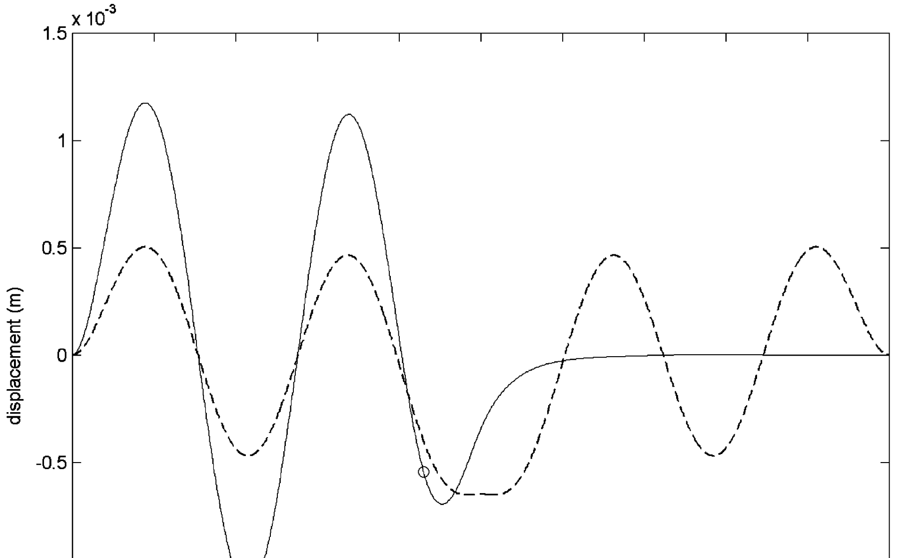

Fig. 22. Analytical displacement responses of loaded and unloaded clamped–clamped beam when $\omega = 10\,\text{Hz}$, $F_0 = 1000\,\text{N}$, $x_0 = 10\,\text{m}$, $m_1 = 354\,\text{kg}$, $J_1 = 0\,\text{kg}\,\text{m}^2$ at $b_1 = 5.4946\,\text{m}$, ——, loaded; - - - -, unloaded; ○, mass location.

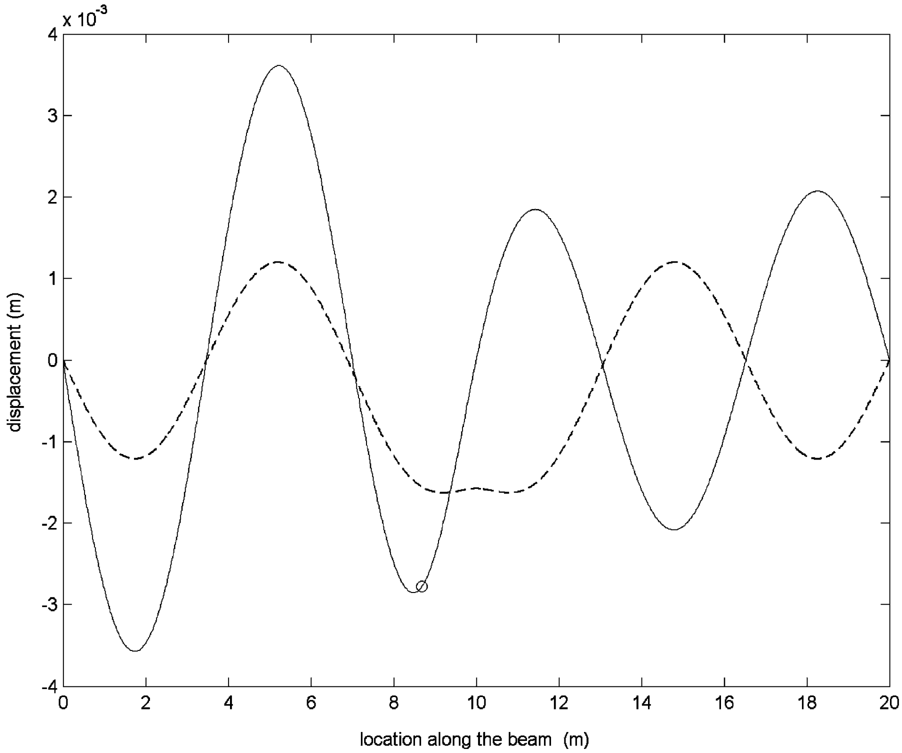

Fig. 23. Analytical displacement responses of loaded and unloaded simply supported beam when $\omega = 5\,\text{Hz}$, $F_0 = 1000\,\text{N}$, $x_0 = 10\,\text{m}$, $x_n = 10\,\text{m}$, $m_1 = 369.7794\,\text{kg}$, $J_1 = 0\,\text{kg}\,\text{m}^2$, at $b_1 = 8.6877\,\text{m}$, ——, loaded; - - - -, unloaded; ○, mass location.

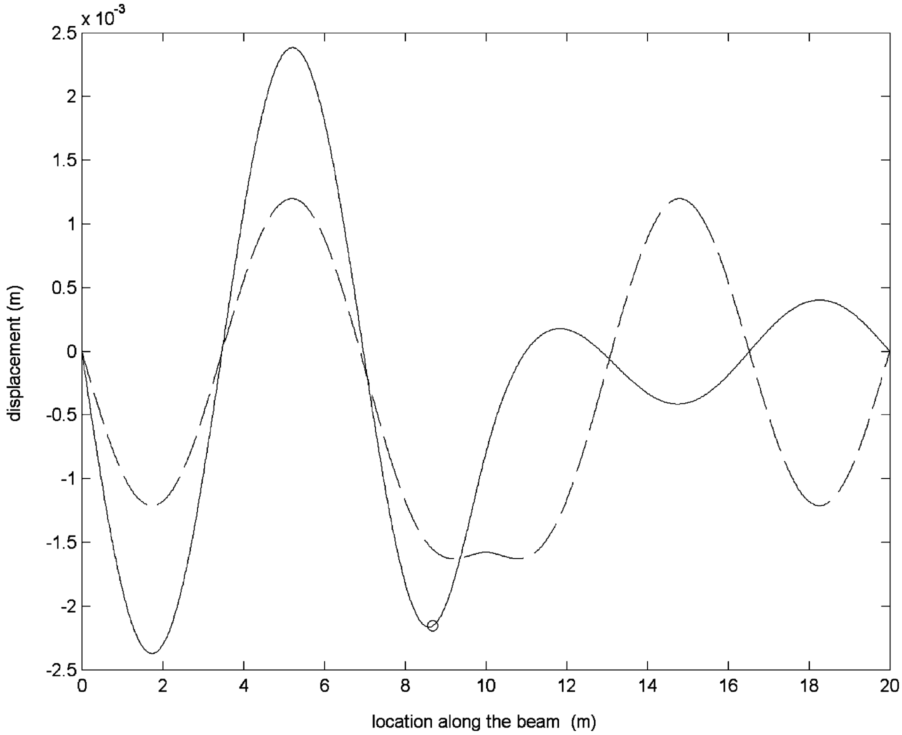

Fig. 24. Analytical displacement responses of loaded and unloaded simply supported beam when $\omega = 5\,\text{Hz}$, $F_0 = 1000\,\text{N}$, $x_0 = 10\,\text{m}$, $x_n = 11\,\text{m}$, $m_1 = 234.1015\,\text{kg}$, $J_1 = 0\,\text{kg}\,\text{m}^2$, at $b_1 = 8.6877\,\text{m}$, ——, loaded; - - - -, unloaded; ○, mass location.

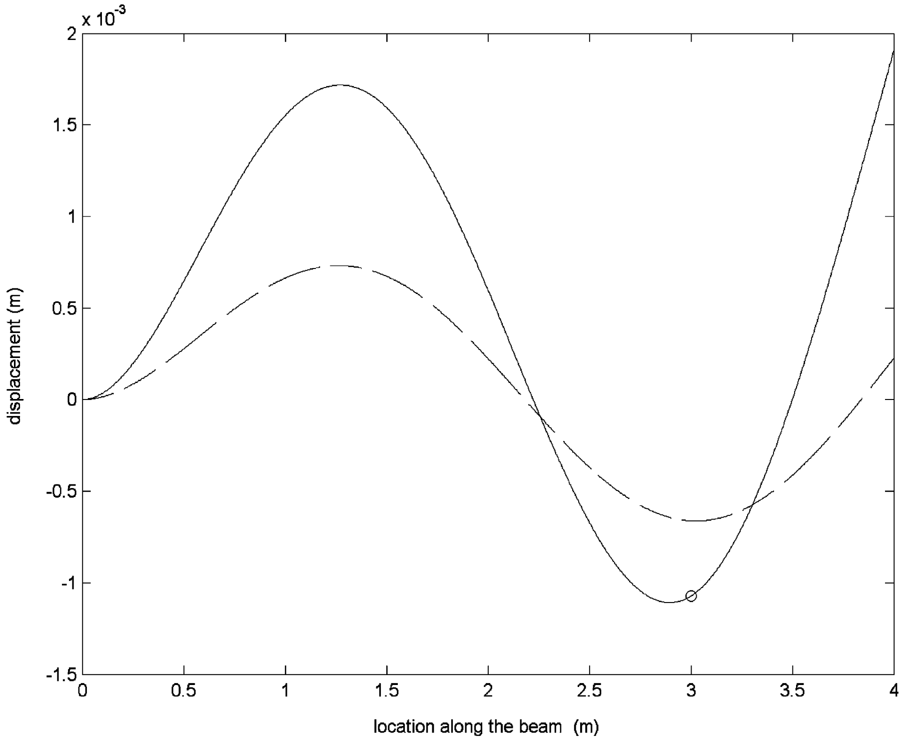

Fig. 25. Analytical displacement responses of loaded and unloaded cantilevered beam when $\omega = 20\,\text{Hz}$, $F_0 = 1000\,\text{N}$, $x_0 = 4\,\text{m}$, $x_n = 3.5\,\text{m}$, $m_1 = 149.9417\,\text{kg}$, $J_1 = 0\,\text{kg}\,\text{m}^2$, at $b_1 = 3\,\text{m}$, ——, loaded; - - - -, unloaded; ○, mass location.

an external force that acts at its mid-span, the displacement amplitude on either side of the external force can be suppressed by adjusting the value of the added mass. From the control point of view this is called a passive control method aimed to regulate the vibration displacement. Essentially the added masses cause modal cancellation, which occurs when the dominant modes involved in the vibration appear in phase on one side but out of phase on the other side of the point force. This study may be of particular use if there is an interest in eliminating unwanted vibrations from certain parts of a beam structure more than other or preventing a localized excitation from propagating into certain parts.

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