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Time-varying vibration decomposition and analysis based on the Hilbert transform

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Abstract

This paper introduces a simple method for time-varying vibration decomposition based on the Hilbert transform. The non-stationary frequency of the largest component is estimated as an average function of the instantaneous frequency of the composition, and the corresponding envelope is estimated according to synchronous demodulation. The method is demonstrated using computer simulation of different types of non-stationary and nonlinear vibration.

1. Introduction

Many real-world vibration signals are non-stationary, making classic methods such as the Fourier analysis unsatisfactory since amplitude and frequency content changes over time. Known time–frequency analysis methods analyze frequency content across a short time span and then move on to another time position. The major drawback of most time–frequency transforms is that the rectangular tiling of the time–frequency plane does not match the shape of the real initial signals. Other common basic decomposition techniques, such as the Fourier decomposition or the wavelet (time-scale) decomposition [1], use basic functions that are fixed and do not necessarily match the varying nature of the signals.

An original technique, known as the empirical mode decomposition (EMD), first introduced by Huang et al. [2,3], adaptively decomposes a signal into the simplest intrinsic oscillatory modes (components). The EMD method is a powerful approach, and has become extremely popular in various areas including nonlinear and non-stationary mechanics and acoustics. The EMD method is based on an interesting spline algorithm which constructs upper and lower envelopes that are fitted to the local maxima of the initial wideband signal. The first intrinsic mode contains the highest-frequency component of the signal. The residual signal contains information of lower-frequency components.

Recently, an iterative filterbank method [4] was proposed for tracking the parameters of exponentially damped sinusoidal components of quasi-harmonic sounds. The filterbank splits the recorded signal into subbands, one per harmonic, in which time-varying parameters of multiple closely spaced sinusoids are estimated using the Steiglitz-McBride/Kalman approach.

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This paper describes a new technique, called the Hilbert vibration decomposition (HVD) method, dedicated to the same problem of decomposition of non-stationary wideband vibration. Here, however, the proposed HVD method is based on the Hilbert transform (HT) presentation itself and does not involve any additional complicated signal processing techniques. The first component separated from the initial vibration contains the varying highest amplitude. The residual signal contains information of other lower amplitude components.

2. The Hilbert transform and the analytical signal representation

The HT, as a kind of integral transformation, plays a significant role in vibration analysis. One of the common ways it can be used is a direct examination of a vibration's instantaneous attributes: frequency, phase, and amplitude. It allows rather complex signals and systems to be analyzed in the time domain.

The HT of the function $x(t)$ is defined by an integral transform:

$$H[x(t)] = \tilde{x}(t) = \pi^{-1} \int_{-\infty}^{\infty} \frac{x(\tau)}{t-\tau}\, \mathrm{d}\tau.$$

Because of the possible singularity at $t = \tau$, the integral is to be considered as a Cauchy principal value. The HT is equivalent to an interesting kind of filter, in which the amplitudes of the spectral components are left unchanged, but their phases are shifted by -0.5π.

The complex signal whose imaginary part is the HT of the real part is called the analytic signal. The term 'analytic' is used in the meaning of a complex function of the complex variable. When dealing with general modulated signals, it is often convenient to define the analytic signal $X(t) = x(t) + i\tilde{x}(t)$, where $\tilde{x}(t)$ is related to $x(t)$ by the HT. According to analytic signal theory, a real vibration process $x(t)$ measured by, say, a transducer, is only one of possible projections (the real part) of some analytic signal $X(t)$. Then the second or quadrature projection of the same signal (the imaginary part $\tilde{x}(t)$) will be conjugated according to the HT. The analytic signal has a geometrical representation in the form of a phasor rotating in the complex plane. Using the traditional representation of the analytic signal in its trigonometric or exponential form: $X(t) = |X(t)|[\cos\varphi(t) + \mathrm{i}\sin\varphi(t)] = A(t)\mathrm{e}^{\mathrm{i}\varphi(t)}$, one can determine its instantaneous amplitude (envelope, magnitude) $A(t) = |X(t)| = \sqrt{x^2(t) + \tilde{x}^2(t)} = \mathrm{e}^{\mathrm{Re}[\ln X(t)]}$, its instantaneous phase $\varphi(t) = \arctan(\tilde{x}(t)/x(t)) = \mathrm{Im}[\ln X(t)]$, and its instantaneous angular frequency $\omega(t) = \dot{\varphi}(t)$ as the first derivative of the instantaneous phase. The instantaneous frequency (IF) $\omega(t)$ measures the rate of rotation in the complex plane of the corresponding analytic signal. Naturally, for a simple monoharmonic signal, the instantaneous amplitude and frequency are constant values. In the general case, the IF of a signal is a varying function of time. Moreover, the IF of wideband analytic signals may change sign in some time intervals. This corresponds to the change of rotation of the phasor from the counterclockwise to the clockwise direction.

3. Multi-component composition of vibration signals

For more complicated vibration signals, the instantaneous amplitude and frequency are non-constant; they vary in time. While the IF is positive function, the signal itself has the same numbers of zero crossings and extrema. When the IF takes a negative value the signal has multiple extrema between successive zero crossing. It corresponds to the appearance of a complex riding wave (complicated cycle of alternating signal). Vibration signal with positive IF is considered as a monocomponent signal. A slow frequency-modulated and a narrow-band vibration signals are two typical examples of the monocomponent signal. In this case the IF will not have the fast fluctuations induced by asymmetric waveforms. With this definition, the monocomponent signal in each cycle, defined by the zero crossing, involves only one mode of oscillation; no complex riding waves are allowed.

Assume now the original signal $x(t)$ can be expressed as sum of different monocomponents with slow varying instantaneous amplitude and frequency, so that

$$x(t) = \sum_{l} a_l(t) \cos\left(\int \omega_l(t)\, \mathrm{d}t \right), \tag{1}$$

where $a_l(t)$ is the instantaneous amplitude and $\omega_l(t)$ is the IF of the l component. Here, l indicates the different components having different oscillatory frequencies and amplitudes. By construction, each monocomponent $a_l(t)\cos(\int \omega_l(t)\,\mathrm{d}t)$ is an intrinsic mode of $x(t)$ with a simple oscillatory waveform described by the envelopes $a_l(t)$ and the instantaneous frequencies $\omega_l(t)$. If $l=1$, the vibration is said to be a monocomponent signal; if, however, $l \geqslant 2$, then the vibration with a wide band spectrum is referred to as a multi-component signal [7].

For such composition, it is desirable to express it by a sum of proper simple components. However, almost all of the global conditions discussed earlier did not provide a scheme as to how to obtain the decomposition given in Eq. (1). Only Huang et al. [2,3] introduced the empirical mode decomposition (EMD) method for generating intrinsic modes that are monocomponents. The heart of the EMD method is to identify the innate undulations belonging to different time scales and sift them out to obtain one intrinsic mode at a time. This can be achieved by making use of the envelopes defined by the local maxima and minima to discern waves riding on top of the others [2,5].

In our paper we proposed another technique, called the HVD method, dedicated to the same problem of decomposition of non-stationary wideband vibration. Here, however, the proposed HVD method is based on the HT presentation itself and does not involve any additional complicated signal processing procedure.

4. Separation of vibration components by using the Hilbert transform

To better understand the meaning of the proposed HVD, we examine some mathematical issues. The principle of the proposed HVD method is to decompose an initial vibration $x(t)$ (Eq. (1)) into a sum of components with slow varying instantaneous amplitude and frequency. Such identification of every inherent component belonging to different time scales can be made on the base of time domain analysis of the IF of the initial signal.

4.1. Estimation of the instantaneous frequency of the largest energy vibration component

It is natural that each of the inherent components must have physical and mathematical significance. First, let us consider a vibration signal composed of two quasi-harmonics, each with a slow variable amplitude and frequency in the time domain as the simplest example of the multi-component signal. In this case, the signal can be modelled as a weighted sum of monocomponent signals, each with its own IF [6] and amplitude function: i.e.,

$$a_1(t)\mathrm{e}^{\mathrm{i}\int_0^t \omega_1(t)\,\mathrm{d}t} + a_2(t)\mathrm{e}^{\mathrm{i}\int_0^t \omega_2(t)\,\mathrm{d}t},$$

with $a_1(t)$, $a_2(t)$, $\omega_1(t)$ and $\omega_2(t)$ being unknown slow functions in the time domain. The envelope $a(t)$ and the IF $\omega(t)$ of the double-component vibration signal are

$$a(t) = \left[a_1^2 + a_2^2 + 2a_1a_2\cos\left(\int(\omega_2-\omega_1)\,\mathrm{d}t\right)\right]^{1/2},$$
$$\omega(t) = \omega_1 + \frac{(\omega_2-\omega_1)\left[a_2^2 + 2a_1a_2\cos\left(\int(\omega_2-\omega_1)\,\mathrm{d}t\right)\right]}{a^2(t)}. \tag{2}$$

The signal envelope $a(t)$ consists of two different parts, i.e., a slow varying part including the sum of the component amplitudes squared and a rapidly varying (oscillating) part. The IF of the two tones considered in Eq. (2) is generally time varying and exhibits asymmetrical deviations about the frequency ω_1. For the two tones, not only are there time-varying deviations in the IF, but also these deviations always force the IF beyond the frequency range of the signal components. The IF $\omega(t)$ (Eq. (2)) in principal also consists of two different parts, i.e., a slow varying frequency of the first component ω_1 and a rapidly varying asymmetrical oscillating part. It appears, then, that in general, the IF of a signal and the average frequency at each time of the signal are different quantities [6,7]. However, the rapidly varying asymmetrical oscillating part of the IF has an important feature. If now we integrate the oscillating part with the integration limits corresponding to the full period of the difference frequency $[0 \quad T = 2\pi/(\omega_2 - \omega_1)]$, assuming that $a_1 > a_2$,

$$\int_0^T \frac{(\omega_2 - \omega_1)\left[a_2^2 + 2a_1 a_2 \cos(\int(\omega_2 - \omega_1)\,\mathrm{d}t)\right]}{a^2(t)}\,\mathrm{d}t = 0, \tag{3}$$

we get the definite integral equal to zero. This obtained new result is a central condition allowing further HVD.

The IF $\omega(t)$ (Eq. (2)) of the composition consists of two different parts, i.e., a slow varying frequency of the first component $\omega_1(t)$ and a rapidly varying asymmetrical oscillating part. Because the average value of the second rapid part is equal to zero, the remaining average value of the IF is equal only to the frequency of the first part, namely the frequency of largest harmonics $\langle\omega(t)\rangle = \int_0^T \omega(t) = \omega_1(t)$.

This interesting property of the IF offers the simplest way to estimate the frequency of the largest vibration component. An averaging, or smoothing or low-pass filtering of the IF of the vibration composition will cut down the asymmetrical oscillations and leave only the slow varying frequency of the main vibration component. Thus, the IF is a useful function enabling estimation of the largest energy component frequency.

In the more general case of three and more quasi-harmonics in the composition, the IF will have a more complicated form, but again the averaging or the low-pass filtering will extract only the IF of the largest energy component.

A low-pass filter will eliminate all high-frequency components outside of the cutoff frequency and allow through, without modification all the low-frequency components. It is desirable to choose the cutoff frequency value as small as possible considering shape factor accuracy and stability of the filter.

4.2. The envelope detection

As in radio-signal demodulation, for the proposed HVD method we chose a technique i.e. well known by many names, including synchronous detection, in-phase/quadrature demodulation, coherent demodulation, auto-correlation, signal mixing and frequency shifting, lock-in amplifier detection, and phase sensitive detection. In essence, the technique extracts the amplitude details about a vibration component with a known frequency by multiplying the initial vibration composition by two reference signals exactly 90° out of phase with one another. For output we get two projections, the in-phase and the HT (quadrature) phase output. The amplitude of the vibration can be obtained by taking the square root of the sum of the squares of these projections.

In this case, a single vibration component $x_{l=r}(t) = A_{l=r}(t)\cos(\int \omega_{l=r}(t)\,\mathrm{d}t)$ on exactly the same frequency as the reference vibration $\cos(\int \omega_r(t)\,\mathrm{d}t)$ is mixed with the other l vibration components. The in-phase signal part $x_{l=r}(t)$ is given, as

$$\begin{aligned}
x_{l=r}(t) &= \sum_l \left[A_l(t)\cos\left(\int \omega_l(t)\,\mathrm{d}t + \varphi_l(t)\right)\right]\cos\left(\int \omega_r(t)\,\mathrm{d}t\right) \\
&= \frac{1}{2}A_l(t)\left[\cos\left(\int(\omega_l(t) - \omega_r(t))\,\mathrm{d}t + \varphi_l(t)\right) + \cos\left(\int(\omega_l(t) + \omega_r(t))\,\mathrm{d}t + \varphi_l(t)\right)\right] \\
&= \frac{1}{2}A_l(t)\left[\cos(\varphi_l(t)) + \cos\left(\int(\omega_l(t) + \omega_r(t))\,\mathrm{d}t + \varphi_l(t)\right)\right],
\end{aligned}$$

where $A_l(t)$, $\omega_l(t)$ and $\varphi_l(t)$ are, respectively, the amplitude, the IF and the phase angle of the l-vibration component, and $\omega_r(t)$ is the IF of the r-reference largest vibration.

The second phase-shifted quadrature part $\tilde{x}_{l=r}(t)$ is given by the analogical formula

$$\tilde{x}_{l=r}(t) = \frac{1}{2}A_l(t)\left[\sin(\varphi_l(t)) - \sin\left(\int(\omega_l(t) + \omega_r(t))\mathrm{d}t + \varphi_l(t)\right)\right].$$

Each of the obtained parts consists of two different functions. One is a slow varying function, which includes the amplitude and the phase, and the other is a fast varying (oscillating) part, which includes the double frequency harmonics. In such a case, it is possible to remove the oscillating part again by using low-pass filtration. Vibration components that are not of the exact same frequency as the reference ($\omega_l \neq \omega_r$) will not yield this slow varying function. Thus, only the slow part will be retained, and both the vibration amplitude and phase can be calculated.

$$\langle x_{l=r}(t)\rangle = \begin{cases} \frac{1}{2}A_l(t)\cos\varphi_l(t), & \text{if } \omega_l=\omega_r, \\ 0, & \text{if } \omega_l\neq\omega_r, \end{cases} \qquad \langle \tilde{x}_{l=r}(t)\rangle = \begin{cases} \frac{1}{2}A_l(t)\sin\varphi_l(t), & \text{if } \omega_l=\omega_r, \\ 0, & \text{if } \omega_l\neq\omega_r \end{cases}$$

$$A_{l=r}(t) = 2\sqrt{\langle x_{l=r}(t)\rangle^2 + \langle \tilde{x}_{l=r}(t)\rangle^2}, \qquad \varphi_{l=r}(t) = \arctan\frac{\langle \tilde{x}_{l=r}(t)\rangle}{\langle x_{l=r}(t)\rangle}.$$

No matter what the instantaneous phase, the resultant envelope $A_{l=r}(t)$ always represents the vibration component envelope. The synchronous detection technique is capable of measuring even small varying vibrations that are obscured by large amounts of other components. It is desirable to choose the cutoff frequency value as small as possible, but not less than the frequency value of the lowest vibration component.

4.3. Subtraction of the largest component

During the first step of the iteration for the proposed HVD method, we found the largest vibration component $x_1(t) = a_1(t)\cos\left(\int\omega_1(t)\,\mathrm{d}t\right)$. Using the idea of signal sifting [2], we subtract the largest component from the initial composition $x_{l-1}(t) = x(t) - x_1(t)$, thus obtaining the new vibration composition $x_{l-1}(t)$ that again should be decomposed during the next iteration. As a result, we separate the initial composition to several slow varying vibration components. A criterion for stopping the sifting process can be the limit value of the standard deviation difference, computed from the two consecutive iteration results.

The number of iterations necessary to provide a good approximation to a vibration composition depends on how rapidly the initial vibration changes.

4.4. Description of the scheme and suggested types of signals

The key factor accompanying precise decomposition lies in using appropriate methods to extract the IF and envelope of the initial vibration composition. The proposed decomposition is an iterative method, and every iteration step includes the following three procedures: (a) estimation of the IF of the largest component, (b) detection of the corresponding envelope of the largest component, and (c) subtraction of the largest component from the composition.

On each iteration step, the corresponding slow varying vibration component is, extracted using the low pass filtering of the IF and envelope. A cutoff frequency that divides the pass band and the stop band of the low-pass filter will control the frequency resolution of the HVD method. It is desirable to choose the cutoff frequency value as small as possible, considering shape factor accuracy and stability of the filter. On each iteration step after subtracting the previous frequency, the current frequency becomes the next in terms of lowest frequency, and again it should be more than the cutoff frequency value. This means that the difference between close frequencies should be more than the cutoff frequency value.

The proposed HVD is based on the following assumptions: (1) the underlying vibration is formed by the superposition of quasi-harmonics functions, (2) the envelopes of each vibration component differ, and (3) the total length of the vibration data includes several longest periods of the corresponding slowest component.

The proposed method is dedicated primarily to quasi and almost periodic oscillating-like signal decomposition. Such a vibration type could be, for example, modulated vibration, non-stationary vibration similar to rotor start-up or shutdown vibration, nonlinear dynamic system vibration.

The HVD method cannot separate other types of motion, such as random, impulse, short or non-oscillating (aperiodic) signals.

The HVD method, as opposed to the other known decomposition methods, is extremely simple and fast in calculation. One additional feature that can be very useful is its ability to detect vibration components with desirable or specified frequencies, for example, those with only odd high harmonics of the main vibration component.

5. Simulation examples

5.1. Non-stationary vibration decomposition

To get an idea of the proposed HVD method, it is instructive to find the components for the non-stationary square wave function with varying amplitude and period:

$$x(k) = (1 + 0.003k) \times \mathrm{sgn}[\sin((0.02 + 3 \times 10^{-5}k)k)], \quad k = [0.2048].$$

This signal is generated by linear modulating the amplitude (from 1 till 7) of a carrier square signal. The increasing modulated frequency starts from 0.02 and ends with 0.07 rad/s. Since the lowest signal frequency is equal to 0.02 we will chose the cutoff frequency of the lowpass filter equal to 0.02. The first five decomposed terms are plotted singly in Fig. 1 and as a sum together with the initial wave in Fig. 2. The results of the HVD describe the non-stationary square wave in the time domain with a high degree of accuracy. This example illustrates the fact that the proposed decomposition method enables construction of perfect match between a complicated non-stationary signal and the composition of the small number of time-varying elementary oscillating components.

5.2. Forced and free vibration decomposition

To illustrate potentials of the proposed decomposition method, consider the case of non-stationary vibration (involving a steady-state and a transient component) generated by chirp force. The following

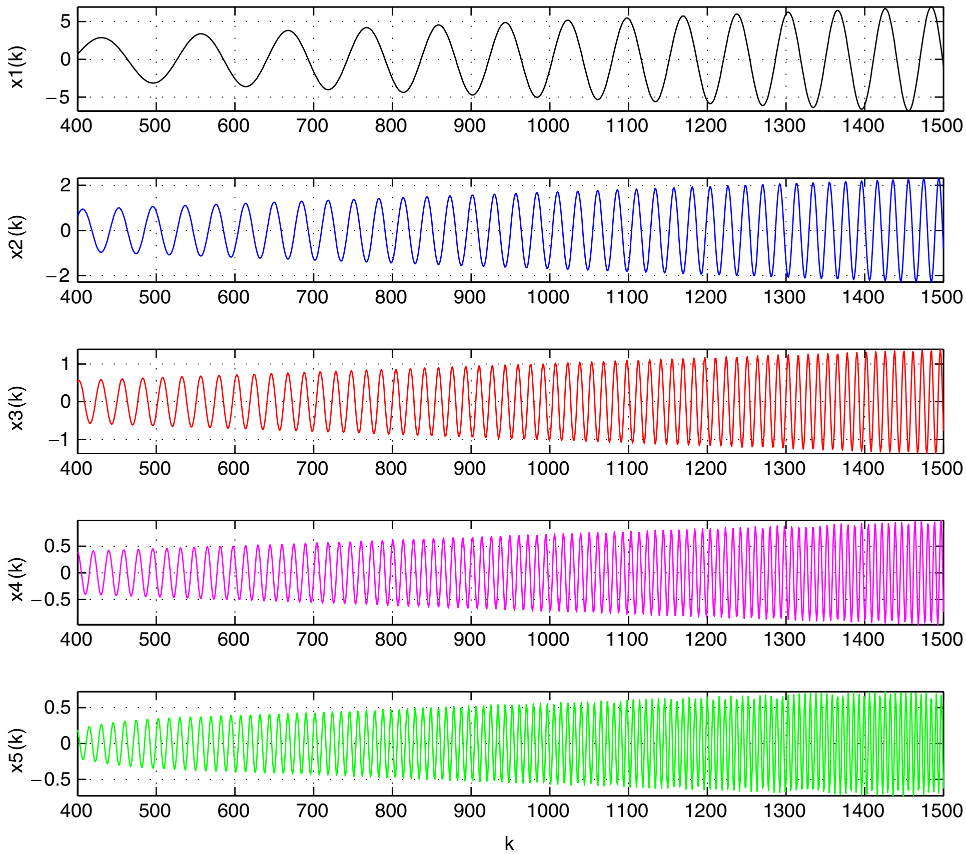

Fig. 1. The first five components of the non-stationary square wave.

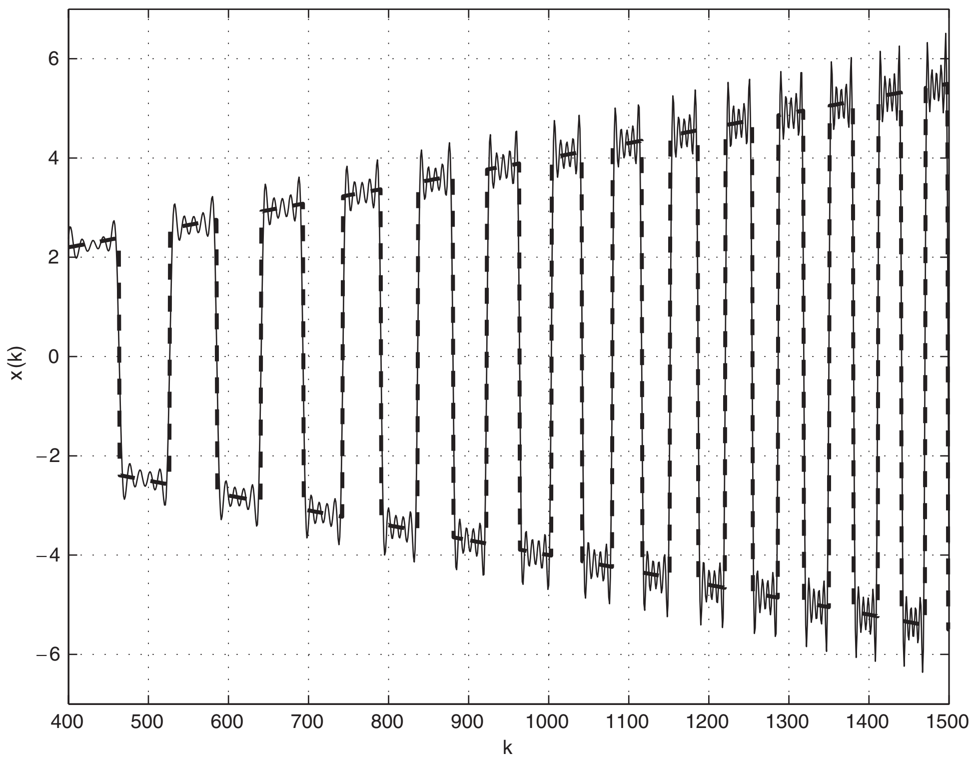

Fig. 2. The initial non-stationary square wave (- - -) and sum of the first five components (——).

numerical example includes the linear dynamic system excited by sine wave whose frequency increases over time at a linear rate $\ddot{x} + 0.07\dot{x} + x = \cos(6.5 \times 10^{-5} t^2)$.

Fig. 3(a) shows the initial forced vibration resulting from the application of an external periodic force to a linear vibration system. It is known that the total general response of the system with the external force is the sum (superposition) of the steady state and particular (homogeneous) solutions. Because of this the general response has a typical beating form (Fig. 3(a)). The natural frequency of the system is equal to 1 rad/s, and the frequency of excitation only increases over time, so again we can choose the cutoff frequency of the lowpass filtering equal to 0.2.

After application of the proposed decomposition method we will receive two main terms of the motion. The first steady-state term (Fig. 3(b)), separated from the solution by means of the HVD method, does not decay over time, while the second term, the transient component (free vibration), does decay (Fig. 3(c)). After the free vibration part of the solution is damped out, the system will oscillate according to excitation so long as the driving force is applied. The IF of the transient component (Fig. 4(a), dashed) remains constant close to the resonance frequency $1/2\pi \approx 0.16$ Hz, while the IF of the transient solution frequency increases at a linear rate (Fig. 4(a), bold). Fig. 4(b) also shows a pure exponential type of decay typical for free vibration in linear systems, whereas the steady state component has a more complicated envelope, depending on the frequency response function. Fig. 5 includes the corresponding three-dimensional (3D) plot of the IF and the envelope of every component and in this case consists of two functions. This is seen in Fig. 5, where all decomposed waveform components are plotted in the time–frequency–amplitude domain.

5.3. Nonlinear equation identification and force characteristics restoration

The time domain HT identification methods, namely FREEVIB and FORCEVIB, were proposed as non-parametric methods for identification of instantaneous modal parameters, including determination of

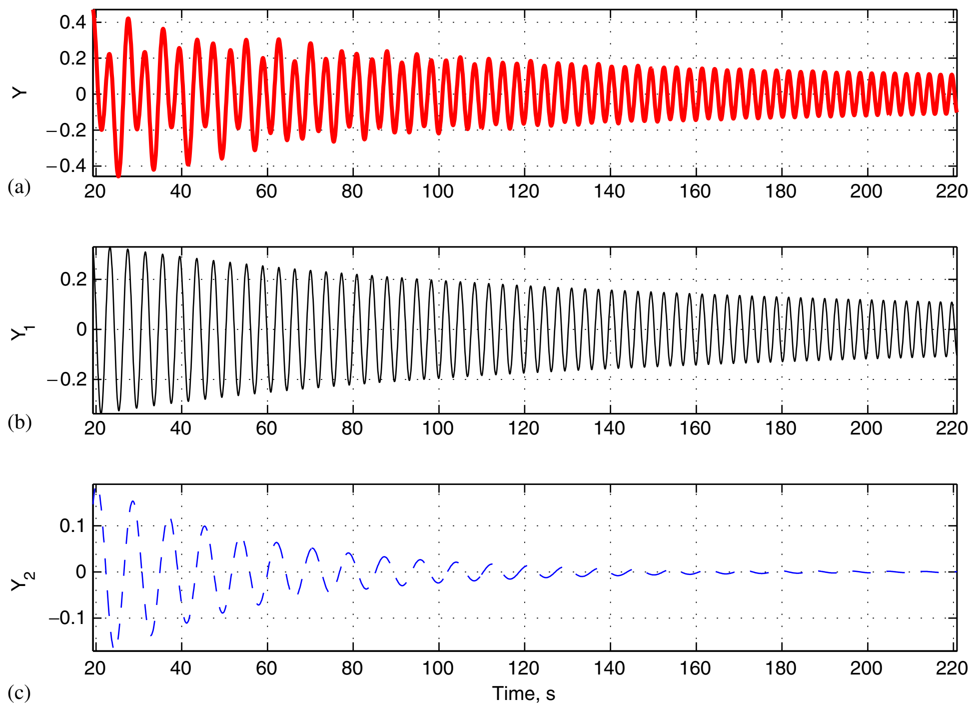

Fig. 3. The non-stationary vibration solution (a), and the separated vibration components: the steady state (b) and the transient (c).

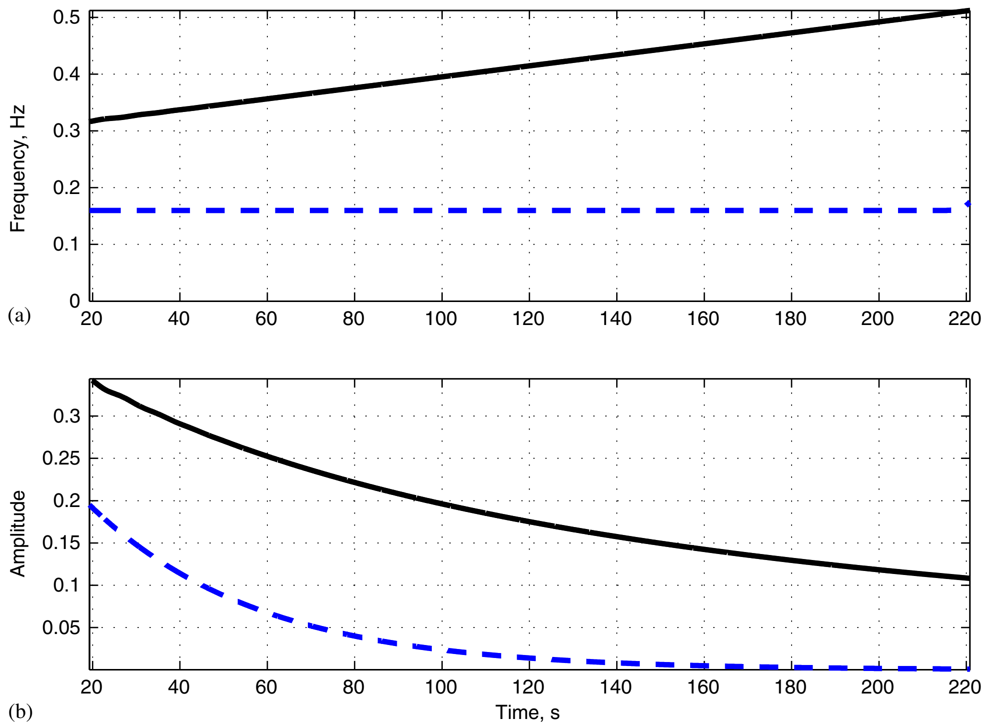

Fig. 4. The instantaneous frequency (a) and the envelope (b) of the non-stationary vibration solution: the steady state component (——), the transient component (- - -).

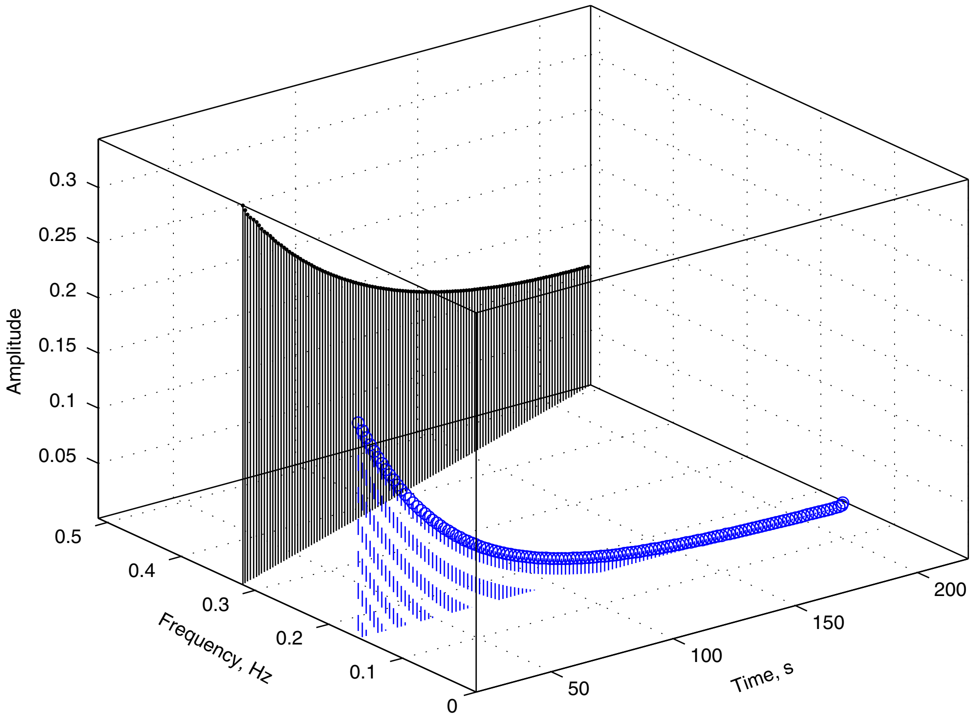

Fig. 5. 3D plot of the instantaneous frequency and the envelope of each component of the non-stationary vibration solution.

system backbone, damping curves, and static force characteristics [8]. These HT methods are suggested for identification of sdof linear and nonlinear systems under free or forced vibration conditions.

A second-order conservative system with a nonlinear restoring force $k(x)$, a nonlinear damping force $h(\dot{x})\dot{x}$, and a solution $x(t) = A(t)\cos[\omega(t)t]$ can be represented in a general form $\ddot{x} + h(\dot{x})\dot{x} + k(x) = 0$. In the first stage of the identification technique, the envelope $A(t)$ and the IF $\omega(t)$ are extracted from the vibration and excitation signals on the base of the HT signal processing. In the next stage, by applying the multiplication property of the HT for overlapping functions to the equation of motion, the instantaneous undamped natural frequency and the instantaneous damping coefficient are calculated according to formulas [8]:

$$\omega_0^2(t) = \omega^2 - \frac{\ddot{A}}{A} + \frac{2\dot{A}^2}{A^2} + \frac{\dot{A}\dot{\omega}}{A\omega}, \quad h_0(t) = -\frac{\dot{A}}{A} - \frac{\dot{\omega}}{2\omega},$$

where $A(t)$ and $\omega(t)$ are the envelope of the IF of the vibration.

In the last stage of lowpass filtering the set of duplet modal parameters (the instantaneous natural frequency $\langle\omega_0^2(t)\rangle$ and instantaneous damping $\langle h(t)\rangle$) of each natural mode of vibration are defined. In this final stage, the nonlinear elastic and the damping static force characteristics are calculated according to the scaling and decomposition technique [8]:

$$k(x) \approx \begin{cases} \langle\omega_0^2(t)\rangle A(t), & x > 0, \\ -\langle\omega_0^2(t)\rangle A(t), & x < 0, \end{cases} \qquad h(\dot{x})\dot{x} \approx \begin{cases} \langle h_0(t)\rangle a_{\dot{x}}(t), & \dot{x} > 0, \\ -\langle h_0(t)\rangle a_{\dot{x}}(t), & \dot{x} < 0, \end{cases} \tag{4}$$

where $A(t)$ and $a_{\dot{x}}(t)$ are the envelope of the displacement and the velocity of the vibration motion, respectively.

The initial static force characteristics and those identified by the HT are very close to each other; nevertheless, the identified static force characteristics have a small "natural linearization" deviation from the initial characteristics to the "linear" direction [9]. In other words, they are slightly less nonlinear than the initial force characteristics. This means that the averaging HT identification based on averaging (filtering) restores only the main first term of the motion. This approximation can be recommended mostly for identification of nonlinear systems in the case of noisy experimental conditions.

The proposed HVD method provides a new way for more precise identification of nonlinear systems. Actually, a real solution for nonlinear systems contains the principal and multiple high-frequency sub-harmonics. The vibration decomposition divides the real multi-component motion into a number of separated sub-harmonics $x(t) = \sum_l x_l(t)$. Applying the HT identification methods separately to every component yields the corresponding part of nonlinear elastic and the damping static force characteristics associated with every component. Summarizing all partial static force characteristics equation (4) will provide the final combined static force characteristics of the nonlinear system

$$k(x) = \lim_{l\to\infty}\begin{cases}\Sigma_l\omega_{0l}^2(t)A_l(t), & x>0,\\ -\Sigma_l\omega_{0l}^2(t)A_l(t), & x<0,\end{cases} \qquad h(\dot{x})\dot{x} = \lim_{l\to\infty}\begin{cases}\Sigma_l h_{0l}(t)a_{\dot{x}l}(t), & \dot{x}>0,\\ -\Sigma_l h_{0l}(t)a_{\dot{x}l}(t), & \dot{x}<0,\end{cases} \tag{5}$$

The accuracy of the last expression depends on the total number l of the considered vibration components, and it is significant that Eq. (5) theoretically defines the exact solution for the identified static force characteristics.

Let us now examine the identification results for the nonlinear vibration model, which contains a nonlinear cubic elastic component inherent in the Duffing equation with hardening stiffness $\ddot{x} + 1.8\dot{x} + (2\pi 8)^2 x + 400x^3 = 0$, $x_0 = 10$. The simulated free vibration is shown in Fig. 6(a), where the smallest natural frequency equal to 8 Hz is corresponded to the smallest amplitude.

The first three resultant components according to the proposed decomposition method with the cutoff frequency of the lowpass filter equal to 10 Hz are plotted in Fig. 7. Each of the obtained separate components shows a typical free vibration motion. Plotted together in Fig. 6(b), these components illustrate the high amplitude level of the first principal harmonics and the smaller amplitude level of the next odd high harmonics. The set of instantaneous frequencies of every component, shown in Fig. 8, like the popular "spectrogram" function that uses a short-time Fourier transform, shows the frequency distribution in the

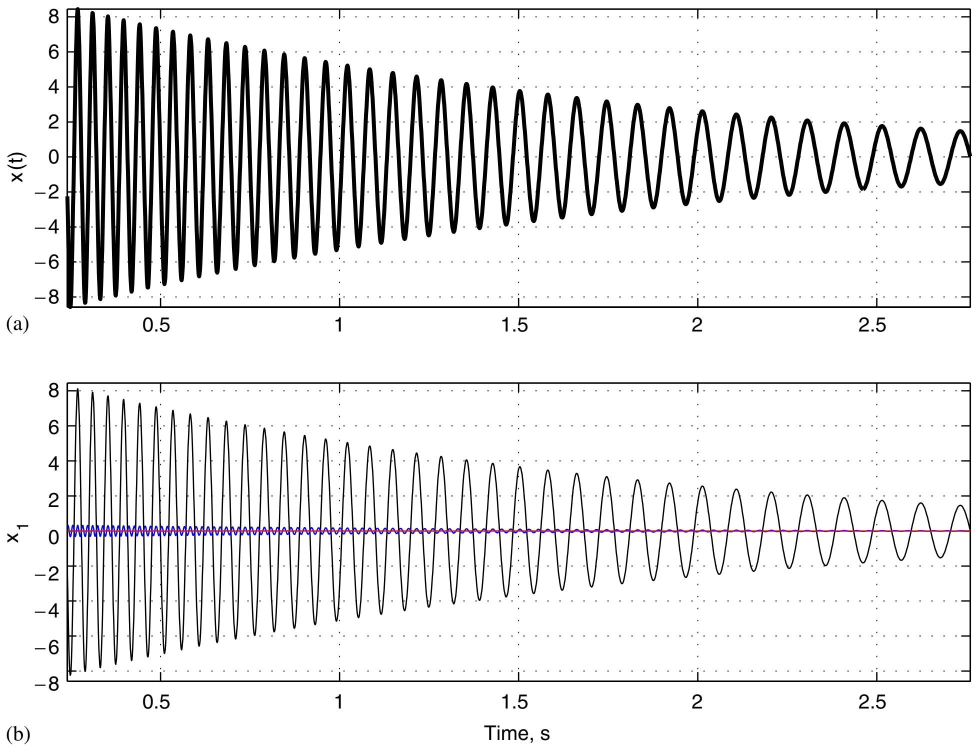

Fig. 6. The solution (a) and the superimposed vibration components (b) of the Duffing equation.

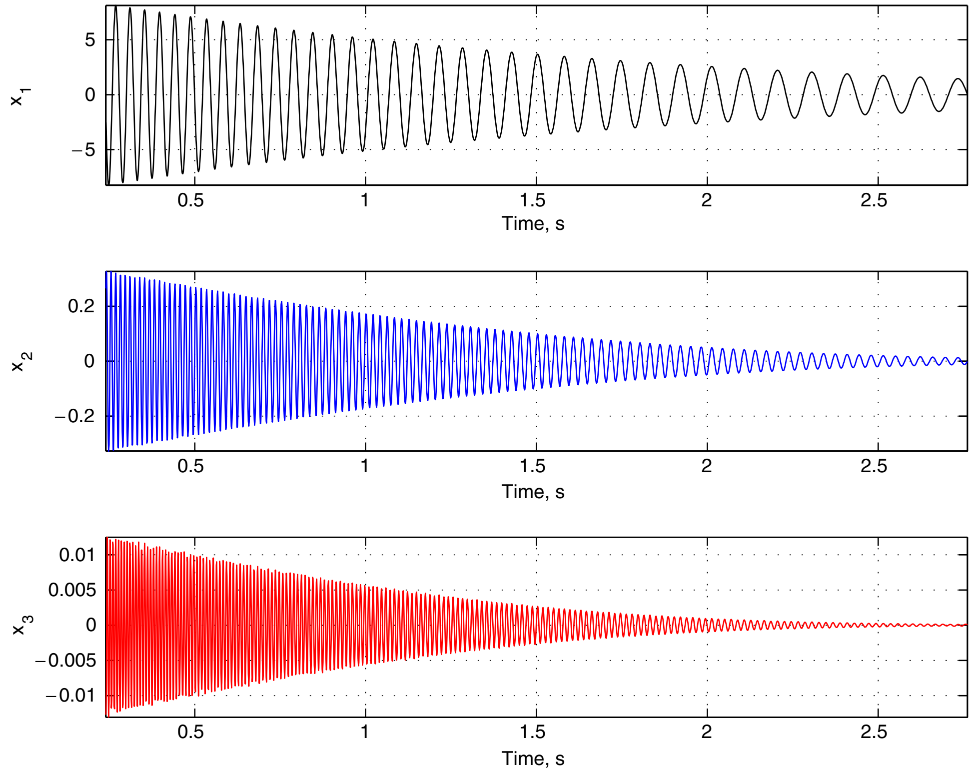

Fig. 7. The first three separated vibration components of the Duffing equation.

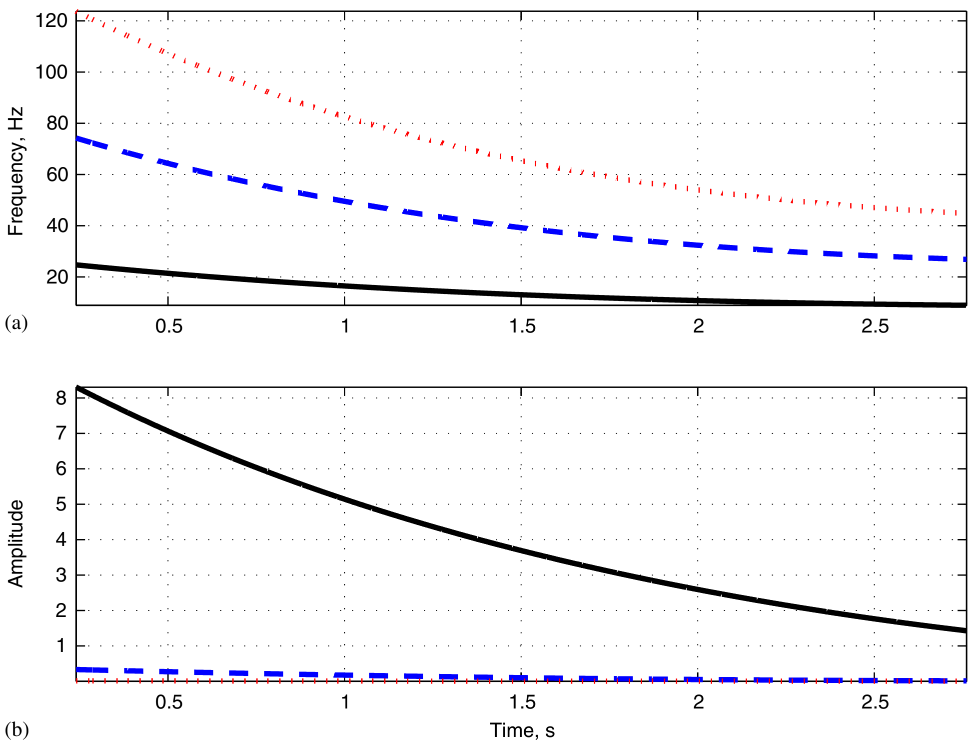

Fig. 8. The instantaneous frequency (a) and the envelope (b) of the first three components of the Duffing equation.

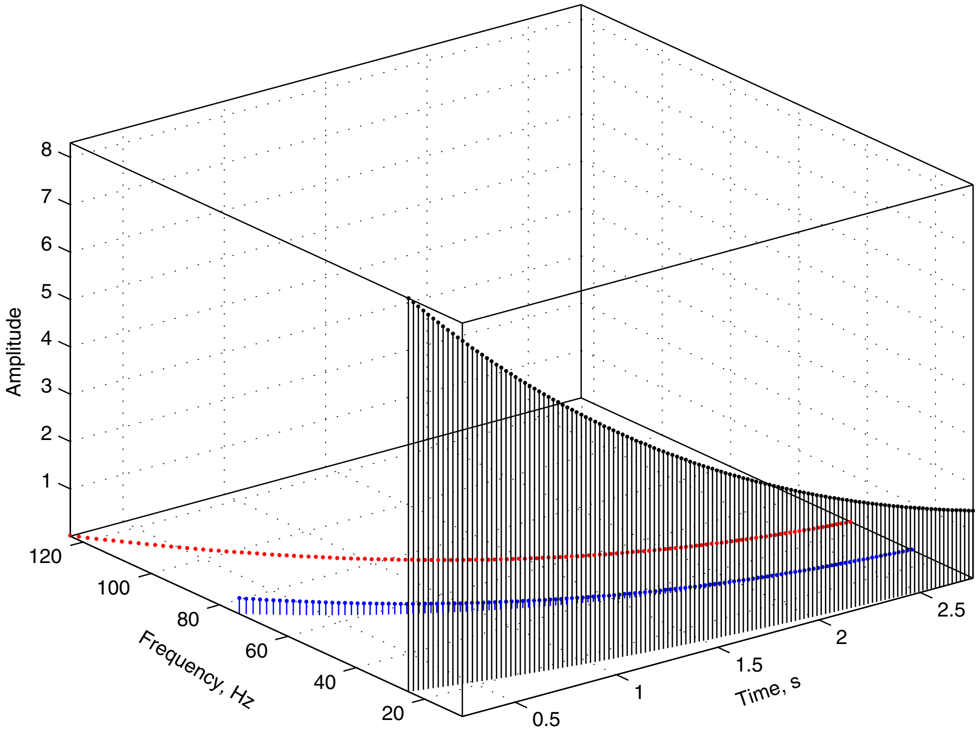

Fig. 9. 3D plot of the instantaneous frequency and the envelope of each component of the Duffing equation.

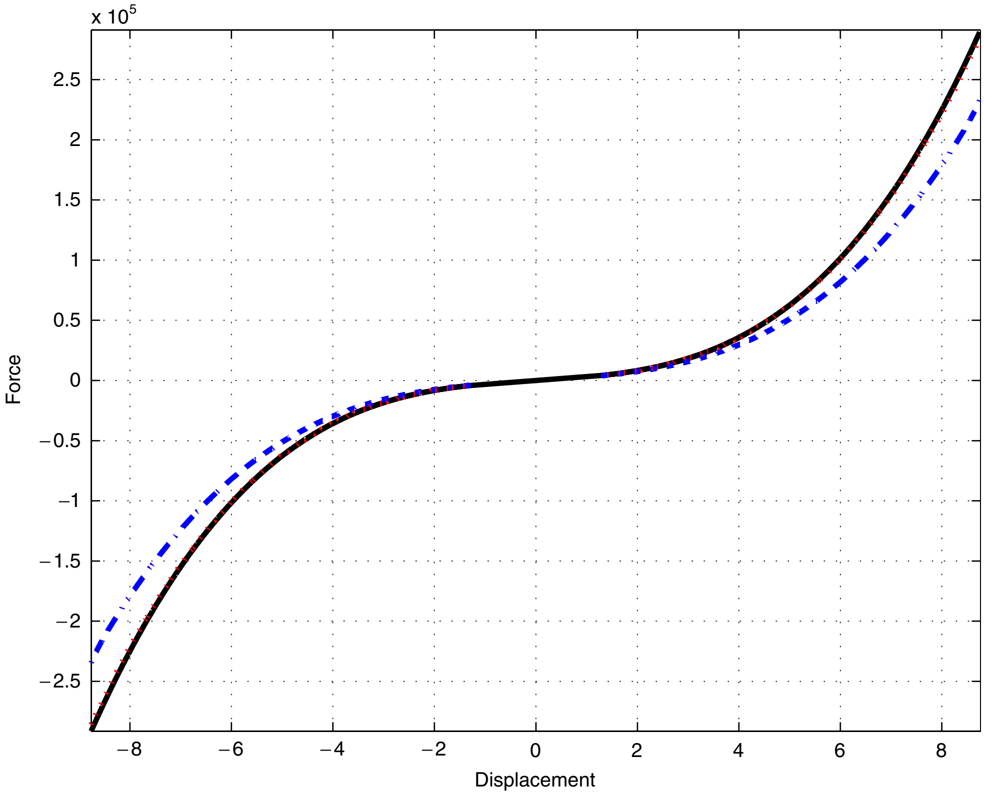

Fig. 10. The elastic static force characteristics of the Duffing equation: the initial (——), the identified first component (- · · -), sum of the first three identified components (⋯).

initial vibration. Generally, both the IF and the envelope of every vibration component are mapped together as 3D functions (Fig. 9).

The HT identification results according to the final formula (Eq. (2)) are presented in Fig. 10, where the bold line plots the initial elastic static force in the Duffing equation example $k(x) = (2\pi 8)^2 x + 400x^3$. The dash–dot line in Fig. 10 corresponds to the static force calculated only for the single first (principal) component. The identified force characteristics are in good agreement with the initial line. The dotted line in Fig. 10 corresponds to the static force for the cumulative sum of the three first components. However, the difference between the initial and the estimated force characteristics is so small that it cannot be distinguished in Fig. 10. Hence, by summarizing only the first three static force characteristics of the corresponding partial components, we can achieve the highly desirable accuracy of nonlinear system identification.

6. Conclusions

A new and extremely simple HVD method has been developed for vibration separation using the Hilbert transform. Estimation of the varying frequency of the largest energy vibration component is effected by low-pass filtration of the IF of the vibration. Synchronous envelope demodulation is performed by multiplying the composition by a sine and Hilbert projection waves, which are phase locked to the current component. A non-stationary example of separation of a transient and a forced vibration regime is described, together with an example of decomposition of a time varying vibration generated by a dynamical system obeying a nonlinear equation. For the case of analysis and identification of nonlinear vibration systems, an exact solution has been developed for restoring the initial static force characteristics.

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