Available online at www.sciencedirect.com

SCIENCE DIRECT®

Journal of Sound and Vibration 295 (2006) 584–601

JOURNAL OF SOUND AND VIBRATION

www.elsevier.com/locate/jsvi

Analytical description of the effects of system nonlinearities on output frequency responses: A case study

Z.Q. Lang[a,*], S.A. Billings[a], G.R. Tomlinson[b], R. Yue[a]

[a]Department of Automatic Control and Systems Engineering, The University of Sheffield, Mappin Street, Sheffield, S1 3JD, UK
[b]Department of Mechanical Engineering, The University of Sheffield, Mappin Street, Sheffield, S1 3JD, UK

Received 7 April 2004; received in revised form 4 January 2006; accepted 6 January 2006
Available online 11 April 2006

Abstract

The output spectrum of a linear dynamic system can be expressed as the input spectrum multiplied by the system frequency response function. This well-known relationship analytically exposes the effects of the system parameters on the output frequency response. In this paper, the extension of this relationship to the nonlinear case is investigated via a case study where an analytical relationship between the output frequency response and the nonlinear damping characteristic parameters is derived for a sdof spring damper system. The analysis is based on the frequency domain analysis of nonlinear systems, and the basic idea can be extended to general situations. Simulation studies are included to verify the theoretical analysis and demonstrate the effectiveness of the new relationship. The results provide an important basis for the analytical study and the design of nonlinear engineering systems and structures in the frequency domain.

1. Introduction

The output frequency response of engineering systems has been widely applied in many fields to investigate and study system behaviours. If the underlying system is linear the relationship between the system output frequency response and the input is well known; the output spectrum $Y(j\omega)$ is equal to the input spectrum $U(j\omega)$ multiplied by the system frequency response function (FRF) $H(j\omega)$. The simple linear frequency domain relationship $Y(j\omega) = H(j\omega)U(j\omega)$ analytically describes the effect of system properties on the output frequency response. This analytical relationship has been applied in control engineering for systems analysis and controller design, in electronics and communications for the synthesis of analogue and digital filters, and in mechanical and civil engineering for the analysis of vibrations.

Nonlinear systems have been widely studied by many authors and significant progress towards understanding these systems has been made. Many of these studies have been based in the time domain with results relating to the Volterra series [1–3], NARMAX (Nonlinear AutoRegresive Moving Average with eXogenous input) models [4,5], neural networks and fuzzy systems [6], and classical nonlinear models such as the Duffing equation [7,8] and the Van der Pol oscillator [9]. The study of nonlinear systems in the frequency

*Corresponding author.
E-mail address: z.lang@sheffield.ac.uk (Z.Q. Lang).

0022-460X/$ - see front matter
doi:10.1016/j.jsv.2006.01.041

domain is based on the concept of generalised frequency response functions (GFRFs) [10] that extend the linear FRF concept to the nonlinear case. Many studies in the frequency domain have focused on system modelling which involves the determination of the GFRFs from input–output data or the establishment of system frequency domain models from input–output spectra [11–16]. The output frequency response of nonlinear systems was recently studied by Lang and Billings [17–19] and Billings and Lang [20–22]. These studies extended the above basic linear relationship between the input and output spectra and introduced explicit relationships between input and output frequencies of nonlinear systems. Based on these relationships, Billings and Lang [23] proposed the concept of energy transfer filters and developed a general procedure for the design of energy transfer filters which can be implemented using the NARX (Nonlinear AutoRegressive with eXogenous input) model with input nonlinearities.

Unlike linear systems, the relationship between the input and output spectra of nonlinear systems is much more complicated. The relationship involves complex multidimensional integration known as association of variables and a summation with a possibly infinite number of terms [2]. This complicates the effect of the system parameters on the output frequency response. Consequently, the linear system frequency domain analysis and design approaches cannot easily be extended to the nonlinear case.

In this paper a case study is conducted based on a single-degree-of-freedom (sdof) spring damper system with a nonlinear damping characteristic. An analytical relationship between the system output frequency response and the characteristic parameters of the system damping nonlinearity is derived, for the first time, using the frequency domain theories of nonlinear systems. The results explicitly reveal how the system output frequency response depends on the damping characteristic parameters which define the system nonlinearity. Simulation studies are performed to evaluate the accurate system output frequency response for different linear and nonlinear damper parameters and different input frequencies and magnitudes, and to compare these with the analytically determined results. The results verify the effectiveness and significance of the theoretical derivations. The study is focused on a relatively simple sdof system to demonstrate the idea of the approach, but the results can be extended to general cases. The work provides an important basis for the analytical study and the design of nonlinear engineering systems in the frequency domain.

2. System description

In order to demonstrate the analysis of the effects of system nonlinearities on the output frequency response, a simple sdof system will be considered, as shown in Fig. 1. The mass, m, supported on the nonlinear damper and parallel spring, is subject to a harmonic disturbance of amplitude, F_d, and frequency, Ω. The nonlinear damping effect is described by a third-order polynomial [29] such that

$$f(.) = a_1(.) + a_2(.)^2 + a_3(.)^3, \tag{1}$$

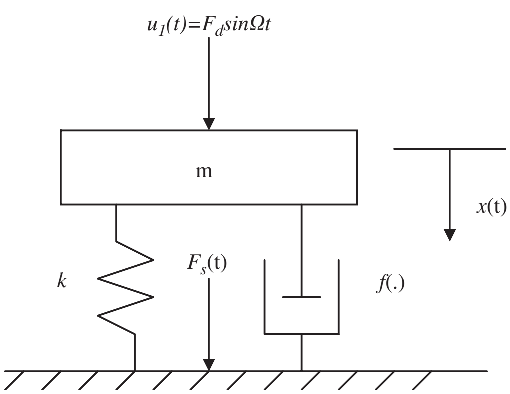

Fig. 1. The single degree of freedom spring damper system considered in the study.

where a_1, a_2, a_3 are the parameters of the damping characteristic, and a_2 and a_3 represent the system nonlinearity. The analysis of the effects of the parameters a_2 and a_3 on the system output frequency response is the focus of the present study. The spring with characteristic parameter k in parallel with the nonlinear damper $f(.)$ provides an isolation between the disturbance force, $F_d \sin \Omega t$, and the force transmitted to the support, $F_s(t)$.

The equilibrium equation for the system in Fig. 1, and the corresponding force at the support, can be expressed as

$$m\ddot{x}(t) + a_1\dot{x}(t) + a_2\dot{x}^2(t) + a_3\dot{x}^3(t) + kx(t) = F_d \sin(\Omega t), \tag{2}$$

$$F_s(t) = a_1\dot{x}(t) + a_2\dot{x}^2(t) + a_3\dot{x}^3(t) + kx(t). \tag{3}$$

For convenience of analysis, denote

$$y_1(t) = x(t), \tag{4}$$

$$y_2(t) = F_s(t) \tag{5}$$

and

$$u_1(t) = F_d \sin(\Omega t). \tag{6}$$

The system can then be described by a single input two output system:

$$m\ddot{y}_1(t) + a_1\dot{y}_1(t) + a_2\dot{y}_1^2(t) + a_3\dot{y}_1^3(t) + ky_1(t) = u_1(t), \tag{7}$$

$$y_2(t) = a_1\dot{y}_1(t) + a_2\dot{y}_1^2(t) + a_3\dot{y}_1^3(t) + ky_1(t). \tag{8}$$

What is interesting in this study is how the spectrum of the second system output $y_2(t)$ depends on the parameters a_2,a_3 of the nonlinear damping characteristic $f(.)$. Although this appears to be a relatively simple problem, surprisingly, there are no results in the literature that can address this fundamental problem. The reason for this omission is the complexity that is introduced by the nonlinearities even for this apparently simple system. The objective therefore is to establish an analytical relationship between the output spectrum and the system parameters.

3. Volterra modelling of the system in the time and frequency domain

The output $y(t)$ of a single input single output analytical system can be expressed as a Volterra functional polynomial of the input $u(t)$ [24] to give

$$y(t) = \sum_{n=1}^{N} y^{(n)}(t), \tag{9}$$

where N is the maximum order of the system nonlinearity. The nth order output of the system $y^{(n)}(t)$ is given by

$$y^{(n)}(t) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} h_n(\tau_1, \ldots, \tau_n) \prod_{i=1}^{n} u(t-\tau_i)\, \mathrm{d}\tau_i, \quad n>0 \tag{10}$$

and $h_n(\tau_1,\ldots,\tau_n)$ is a real valued function of $\tau_1,\ldots,\tau_n$ called the nth order impulse response function or Volterra kernel of the system.

The multidimensional Fourier transform of the nth order impulse response function yields the nth order transfer function or GFRF

$$H_n(\mathrm{j}\omega_1, \ldots, \mathrm{j}\omega_n) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} h_n(\tau_1, \ldots, \tau_n) \mathrm{e}^{-\mathrm{j}(\omega_1\tau_1 + \ldots + \omega_n\tau_n)}\, \mathrm{d}\tau_1 \ldots \mathrm{d}\tau_n. \tag{11}$$

Using the concept of GFRF, the relationship between the input spectrum $U(\mathrm{j}\omega)$ and the output spectrum $Y(\mathrm{j}\omega)$, that is the frequency domain input–output description of the system, can be obtained as [17]

$$Y(\mathrm{j}\omega) = \sum_{n=1}^{N} \frac{1/\sqrt{n}}{(2\pi)^{n-1}} \int_{\omega_1+\cdots+\omega_n=\omega} H_n(\mathrm{j}\omega_1,\ldots,\mathrm{j}\omega_n) \prod_{i=1}^{n} U(\mathrm{j}\omega_i)\,\mathrm{d}\sigma_{\omega n}, \tag{12}$$

where $\int_{\omega_1+\cdots+\omega_n=\omega}(.)\,\mathrm{d}\sigma_{\omega n}$ denotes the integration of (.) over the n-dimensional hyperplane $\omega_1+\ldots+\omega_n=\omega$.

When the system is subject to a multi-tone input such that

$$u(t) = \sum_{i=1}^{K} |A_i| \cos(\omega_i t + \angle A_i). \tag{13}$$

Lang and Billings [17] showed that Eq. (12) can be expressed as

$$Y(\mathrm{j}\omega) = \sum_{n=1}^{N} \frac{1}{2^n} \sum_{\omega_{k_1}+\cdots+\omega_{k_n}=\omega} H_n\big(\mathrm{j}\omega_{k_1},\ldots,\mathrm{j}\omega_{k_n}\big) A(\omega_{k_1})\ldots A(\omega_{k_n}), \tag{14}$$

where

$$k_l \in \{-K,\ldots,-1,1,\ldots,K\}, \quad l = 1,\ldots,n,$$

$$A(\omega) = \begin{cases} |A_k|\mathrm{e}^{\mathrm{j}\angle A_k} & \text{if } \omega \in \{\omega_k, k = \pm 1,\ldots,\pm K\}, \\ 0 & \text{otherwise,} \end{cases}$$

$$\omega_{-k} = -\omega_k$$

and

$$|A_{-k}|\mathrm{e}^{\mathrm{j}\angle A_{-k}} = |A_k|\mathrm{e}^{-\mathrm{j}\angle A_k}$$

The extension of the above theoretical results to the single input multiple output nonlinear system case is straightforward. The results in the time domain, which are an extension of Eqs. (9) and (10), are given in Refs. [25,26]

$$y_{j_1}(t) = \sum_{n=1}^{N} y_{j_1}^{(n)}(t), \quad j_1 = 1,2,\ldots,M \quad (M \geqslant 2), \tag{15}$$

where

$$y_{j_1}^{(n)}(t) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} h_n\big(j_1, 1^{(n)}; \tau_1,\ldots,\tau_n\big) \prod_{i=1}^{n} u_1(t-\tau_i),\ \mathrm{d}\tau_i \quad j_1 = 1,2,\ldots,M \tag{16}$$

and $h_n\big(j_1, 1^{(n)}; \tau_1,\ldots,\tau_n\big)$ is the nth order Volterra kernel of the system corresponding to the j_1th output. The results in the frequency domain, which are an extension of Eqs. (12) and (14), are

$$Y_{j_1}(\mathrm{j}\omega) = \sum_{n=1}^{N} \frac{1/\sqrt{n}}{(2\pi)^{n-1}} \int_{\omega_1+\ldots+\omega_n=\omega} H_n(\mathrm{j}_1, 1^{(n)}; \mathrm{j}\omega_1,\ldots,\mathrm{j}\omega_n) \prod_{i=1}^{n} U(\mathrm{j}\omega_i)\,\mathrm{d}\sigma_{\omega n}, \quad j_1 = 1,2,\ldots,M \tag{17}$$

and

$$Y_{j_1}(\mathrm{j}\omega) = \sum_{n=1}^{N} \frac{1}{2^n} \sum_{\omega_{k_1}+\ldots+\omega_{k_n}=\omega} H_n\big(\mathrm{j}_1, 1^{(n)}; \mathrm{j}\omega_{k_1},\ldots,\mathrm{j}\omega_{k_n}\big) A(\omega_{k_1})\ldots A(\omega_{k_n}), \quad j_1 = 1,2,\ldots,M, \tag{18}$$

where

$$H_n(\mathrm{j}_1, 1^{(n)}; \mathrm{j}\omega_1,\ldots,\mathrm{j}\omega_n) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} h_n\big(\mathrm{j}_1, 1^{(n)}; \tau_1,\ldots,\tau_n\big) \mathrm{e}^{-\mathrm{j}(\omega_1\tau_1+\ldots+\omega_n\tau_n)}\,\mathrm{d}\tau_1\ldots\,\mathrm{d}\tau_n \tag{19}$$

is the nth order GFRF of the system corresponding to the j_1th output.

It is obvious that the Volterra time domain model of the system (2) and (3) is given by Eq. (15) with $M=2$, and the output frequency response of the system when subject to the input in Eq. (6) is given by Eq. (18) with $M=2$,

$$k_l \in \{-1,+1\}, \quad l=1,\ldots,n, \tag{20}$$

$$A(\omega)=\begin{cases} |A_k|e^{j\angle A_k} & \text{if } \ \omega\in\{\omega_k, k=\pm 1\}, \ \text{where } |A_{\pm 1}|=F_d, \ \omega_{\pm 1}=\pm\Omega, \ \text{and } \angle A_{\pm 1}=\mp\pi/2, \\ 0 & \text{otherwise.}\end{cases} \tag{21}$$

This is the starting point for the derivation of an analytical expression for the effects of nonlinearity on the output frequency response of the system (2) and (3).

4. The effects of system nonlinearity on the output frequency response

The focus of this section is to investigate the effects of the nonlinear damping characteristic of the system (2) and (3) on the output frequency response when the system is subject to a multi-tone or a harmonic input under the condition that the system can be described by the frequency domain Volterra model (18) where $M=2$. This study involves two steps. First the GFRF matrices of the system

$$\left\lfloor H_n(1,1^{(n)};\mathrm{j}\omega_1,\ldots,\mathrm{j}\omega_n), \ \ H_n(2,1^{(n)};\mathrm{j}\omega_1,\ldots,\mathrm{j}\omega_n)\right\rfloor, \quad n=1,2,3,\ldots$$

are derived using the probing method [27]. Then an analytical relationship between the system output frequency response $Y_2(\mathrm{j}\omega)$ and the parameters of the nonlinear damping characteristic is determined.

4.1. The probing method

Given a parametric model of a nonlinear system, the GFRFs of the system can be derived analytically using the probing method. In the case of single input single output nonlinear systems, the basic idea of the probing method can be introduced as below.

It was shown by Rugh [2] that for nonlinear systems which are described by the Volterra model (9) and (10) and excited by a combination of exponentials

$$u(t)=\sum_{i=1}^{R}\mathrm{e}^{\mathrm{j}\omega_i t}, \quad 1\leqslant R\leqslant N, \tag{22}$$

the output response can be written as

$$\begin{aligned} y(t) &= \sum_{n=1}^{N}\sum_{i_1=1}^{R}\cdots\sum_{i_n=1}^{R}H_n\left(\mathrm{j}\omega_{i_1},\ldots,\mathrm{j}\omega_{i_n}\right)\mathrm{e}^{\mathrm{j}(\omega_{i_1}+\ldots+\omega_{i_n})t} \\ &= \sum_{n=1}^{N}\sum_{m(n)}G_{m_1(n)\ldots m_R(n)}(\mathrm{j}\omega_1,\ldots,\mathrm{j}\omega_R)\mathrm{e}^{\mathrm{j}[m_1(n)\omega_1+\ldots+m_R(n)\omega_R]t}, \end{aligned} \tag{23}$$

where $\sum_{m(n)}$ indicates a R-fold sum over all integer indices $m_1(n),\ldots,m_R(n)$ such that

$$0\leqslant m_i(n)\leqslant n, \quad m_1(n)+\ldots+m_R(n)=n, \text{ and}$$

$$G_{m_1(n)\ldots m_R(n)}(\mathrm{j}\omega_1,\ldots,\mathrm{j}\omega_R)=\frac{n!}{m_1(n)!\cdots m_R(n)!}H_n(\underbrace{\mathrm{j}\omega_1,\ldots,\mathrm{j}\omega_1}_{m_1(n)},\ldots,\underbrace{\mathrm{j}\omega_R,\ldots,\mathrm{j}\omega_R}_{m_R(n)}). \tag{24}$$

Notice that in Eq. (24) when $n=R$, $m_i(n)=1, \ i=1,\ldots,R$, therefore

$$G_{m_1(R)\ldots m_R(R)}(\mathrm{j}\omega_1,\ldots,\mathrm{j}\omega_R)=R!H_R(\mathrm{j}\omega_1,\ldots,\mathrm{j}\omega_R). \tag{25}$$

Considering Eq. (25), Eq. (23) can be written as

$$
\begin{aligned}
y(t) = & \sum_{n=1, n \neq R}^{N} \sum_{m(n)} G_{m_1(n) \ldots m_R(n)}(\mathrm{j}\omega_1, \ldots, \mathrm{j}\omega_R) \mathrm{e}^{\mathrm{j}[m_1(n)\omega_1 + \ldots + m_R(n)\omega_R]t} \\
& + R! H_R(\mathrm{j}\omega_1, \ldots, \mathrm{j}\omega_R) \mathrm{e}^{\mathrm{j}(\omega_1 + \ldots + \omega_R)t}.
\end{aligned} \tag{26}
$$

For nonlinear systems which have a parametric model with parameter vector θ:

$$
y(t) = f_0(t, \theta, y(t), u(t)) \tag{27}
$$

and which can also be described by the Volterra model (9) and (10), substituting Eqs. (22) and (26) into Eq. (27) for $y(t)$ and $u(t)$, and extracting the coefficient of $\mathrm{e}^{\mathrm{j}(\omega_1+\ldots+\omega_R)t}$ from the resulting expression produces an equation from which the GFRF $H_R(\mathrm{j}\omega_1, \ldots, \mathrm{j}\omega_R)$ can be obtained.

For single input multiple output systems, which are described by the Volterra model (15) and (16), and excited by input (22), it can be shown, based on the same idea as used for the single input single output system case above, that the output response is given by

$$
\begin{aligned}
y_{j_1}(t) = & \sum_{n=1, n \neq R}^{N} \sum_{m(n)} G_{m_1(n) \ldots m_R(n)}\left(j_1, 1^{(n)}; \mathrm{j}\omega_1, \ldots, \mathrm{j}\omega_R\right) \mathrm{e}^{\mathrm{j}[m_1(n)\omega_1 + \ldots + m_R(n)\omega_R]t} \\
& + R! H_R\left(j_1, 1^{(R)}; \mathrm{j}\omega_1, \ldots, \mathrm{j}\omega_R\right) \mathrm{e}^{\mathrm{j}(\omega_1 + \ldots + \omega_R)t} \quad j_1 = 1, 2, \ldots, M.
\end{aligned} \tag{28}
$$

where

$$
G_{m_1(n) \ldots m_R(n)}\left(j_1, 1^{(n)}; \mathrm{j}\omega_1, \ldots, \mathrm{j}\omega_R\right) = \frac{n!}{m_1(n)! \ldots m_R(n)!} H_n(j_1, 1^{(n)}; \underbrace{\mathrm{j}\omega_1, \ldots, \mathrm{j}\omega_1}_{m_1(n)}, \ldots, \underbrace{\mathrm{j}\omega_R, \ldots, \mathrm{j}\omega_R}_{m_R(n)}). \tag{29}
$$

If the system is of a single input and two outputs and can be described by the parametric model

$$
\begin{cases}
y_1(t) = f_1(t, \theta, y_1(t), y_2(t), u_1(t)), \\
y_2(t) = f_2(t, \theta, y_1(t), y_2(t), u_1(t)),
\end{cases} \tag{30}
$$

then substituting $u_1(t) = \sum_{i=1}^{R} \mathrm{e}^{\mathrm{j}\omega_i t}$, and $y_1(t)$ and $y_2(t)$ given by Eq. (28) into Eq. (30), and extracting the coefficient of $\mathrm{e}^{\mathrm{j}(\omega_1+\ldots+\omega_R)t}$ from the resulting expressions produces two coupled equations for which the GFRF matrix

$$
\left\lfloor H_R(1, 1^{(R)}; \mathrm{j}\omega_1, \ldots, \mathrm{j}\omega_R), \quad H_R(2, 1^{(R)}; \mathrm{j}\omega_1, \ldots, \mathrm{j}\omega_R) \right\rfloor
$$

can be obtained.

4.2. Derivation of the system GFRF matrices

Following the probing method for single input two output systems introduced above, the GFRF matrices of the system (2) and (3) can be determined. The detailed derivation for the GFRF matrices up to third order is given below to demonstrate how the probing method is applied to achieve this objective.

To determine the first-order GFRF matrix

$$
\left\lfloor H_1(1, 1^{(1)}; \mathrm{j}\omega_1), \quad H_1(2, 1^{(1)}; \mathrm{j}\omega_1) \right\rfloor,
$$

the probing input

$$
u_1(t) = \mathrm{e}^{\mathrm{j}\omega_1 t} \tag{31}
$$

is used and, by taking $M = 2$ and $R = 1$, Eq. (28) can be written as

$$
\begin{cases}
y_1(t) = H_1(1, 1^{(1)}; \mathrm{j}\omega_1) \mathrm{e}^{\mathrm{j}\omega_1 t} + \ldots, \\
y_2(t) = H_1(2, 1^{(1)}; \mathrm{j}\omega_1) \mathrm{e}^{\mathrm{j}\omega_1 t} + \ldots.
\end{cases} \tag{32}
$$

Substituting Eqs. (31) and (32) into Eqs. (7) and (8) for $u_1(t)$, $y_1(t)$, and $y_2(t)$, and extracting the coefficient of $e^{j(\omega_1)t}$ from the resulting expressions yields two equations for $[H_1(1, 1^{(1)}; j\omega_1),\ H_1(2, 1^{(1)}; j\omega_1)]$ which can be expressed in a matrix form such that

$$\begin{bmatrix} m(j\omega_1)^2 & 1 \\ -k - a_1(j\omega_1) & 1 \end{bmatrix} \begin{bmatrix} H_1(1, 1^{(1)}; j\omega_1) \\ H_1(2, 1^{(1)}; j\omega_1) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}. \tag{33}$$

Consequently the first-order GFRF matrix is determined as

$$\begin{bmatrix} H_1(1, 1^{(1)}; j\omega_1) \\ H_1(2, 1^{(1)}; j\omega_1) \end{bmatrix} = \begin{bmatrix} 1/\left(m(j\omega_1)^2 + a_1 j\omega_1 + k\right) \\ (a_1 j\omega_1 + k)/\left(m(j\omega_1)^2 + a_1 j\omega_1 + k\right) \end{bmatrix}. \tag{34}$$

To determine the second-order GFRF matrix

$$\left[H_2(1, 1^{(2)}; j\omega_1, j\omega_2),\ \ H_2(2, 1^{(2)}; j\omega_1, j\omega_2)\right],$$

the probing input

$$u_1(t) = e^{j\omega_1 t} + e^{j\omega_2 t} \tag{35}$$

is used and, by taking $M = 2$ and $R = 2$, Eq. (28) can be written as

$$\begin{cases} y_1(t) = H_1(1, 1^{(1)}; j\omega_1)e^{j\omega_1 t} + H_1(1, 1^{(1)}; j\omega_2)e^{j\omega_2 t} + 2H_2(1, 1^{(2)}; j\omega_1, j\omega_2)e^{j(\omega_1+\omega_2)t} + \ldots, \\ y_2(t) = H_1(2, 1^{(1)}; j\omega_1)e^{j\omega_1 t} + H_1(2, 1^{(1)}; j\omega_2)e^{j\omega_2 t} + 2H_2(2, 1^{(2)}; j\omega_1, j\omega_2)e^{j(\omega_1+\omega_2)t} + \ldots. \end{cases} \tag{36}$$

Substituting Eqs. (35) and (36) into Eqs. (7) and (8) for $u_1(t)$, $y_1(t)$, and $y_2(t)$, and extracting the coefficient of $e^{j(\omega_1+\omega_2)t}$ from the resulting expressions yields two coupled equations for $[H_2(1, 1^{(2)}; j\omega_1, j\omega_2),\ \ H_2(2, 1^{(2)}; j\omega_1, j\omega_2)]$

$$\begin{cases} mH_2(1, 1^{(2)}; j\omega_1, j\omega_2)(j\omega_1 + j\omega_2)^2 + H_2(2, 1^{(2)}; j\omega_1, j\omega_2) = 0, \\ H_2(2, 1^{(2)}; j\omega_1, j\omega_2) - kH_2(1, 1^{(2)}; j\omega_1, j\omega_2) - a_1 H_2(1, 1^{(2)}; j\omega_1, j\omega_2)(j\omega_1 + j\omega_2) \\ \qquad -a_2 H_1(1, 1^{(1)}; j\omega_1)H_1(1, 1^{(1)} j\omega_2)(j\omega_1)(j\omega_2) = 0. \end{cases} \tag{37}$$

So the second-order GFRF matrix is obtained as

$$\begin{cases} H_2(1, 1^{(2)}; j\omega_1, j\omega_2) = -\dfrac{a_2 H_1(1, 1^{(1)}; j\omega_1)H_1(1, 1^{(1)}; j\omega_2)(j\omega_1)(j\omega_2)}{m(j\omega_1 + j\omega_2)^2 + a(j\omega_1 + j\omega_2) + k}, \\ H_2(2, 1^{(2)}; j\omega_1, j\omega_2) = \dfrac{ma_2 H_1(1, 1^{(1)}; j\omega_1)H_1(1, 1^{(1)}; j\omega_2)(j\omega_1)(j\omega_2)(j\omega_1 + j\omega_2)^2}{m(j\omega_1 + j\omega_2)^2 + a(j\omega_1 + j\omega_2) + k}. \end{cases} \tag{38}$$

Define

$$F_0(j\omega_1, j\omega_2) = \frac{H_1(1, 1^{(1)}; j\omega_1)H_1(1, 1^{(1)}; j\omega_2)(j\omega_1)(j\omega_2)}{m(j\omega_1 + j\omega_2)^2 + a(j\omega_1 + j\omega_2) + k}, \tag{39}$$

the second-order GFRF matrix can be described in a more concise form as

$$\begin{cases} H_2(1, 1^{(2)}; j\omega_1, j\omega_2) = -a_2 F_0(j\omega_1, j\omega_2), \\ H_2(2, 1^{(2)}; j\omega_1, j\omega_2) = ma_2(j\omega_1 + j\omega_2)^2 F_0(j\omega_1, j\omega_2). \end{cases} \tag{40}$$

To determine the third-order GFRF matrix

$$[H_3(1, 1^{(3)}; j\omega_1, j\omega_2, j\omega_3),\ \ H_3(2, 1^{(3)}; j\omega_1, j\omega_2, j\omega_3)],$$

the probing input

$$u_1(t) = e^{j\omega_1 t} + e^{j\omega_2 t} + e^{j\omega_3 t} \tag{41}$$

is used and, by taking $M=2$ and $R=3$, Eq. (28) can be written as

$$
\begin{cases}
y_1(t)=\begin{bmatrix}
H_1(1,1^{(1)};\mathrm{j}\omega_1)\mathrm{e}^{\mathrm{j}\omega_1 t}+H_1(1,1^{(1)};\mathrm{j}\omega_2)e^{\mathrm{j}\omega_2 t}+H_1(1,1^{(1)};\mathrm{j}\omega_3)\mathrm{e}^{\mathrm{j}\omega_3 t}+2H_2(1,1^{(2)};\mathrm{j}\omega_1,\mathrm{j}\omega_2)\mathrm{e}^{\mathrm{j}(\omega_1+\omega_2)t}\\
\quad+2H_2(1,1^{(2)};\mathrm{j}\omega_2,\mathrm{j}\omega_3)\mathrm{e}^{\mathrm{j}(\omega_2+\omega_3)t}+2H_2(1,1^{(2)};\mathrm{j}\omega_1,\mathrm{j}\omega_3)e^{\mathrm{j}(\omega_1+\omega_3)t}\\
\quad+6H_3(1,1^{(3)};\mathrm{j}\omega_1,\mathrm{j}\omega_2,\mathrm{j}\omega_3)\mathrm{e}^{\mathrm{j}(\omega_1+\omega_2+\omega_3)t}+\ldots\ \ldots
\end{bmatrix},\\
y_2(t)=\begin{bmatrix}
H_1(2,1^{(1)};\mathrm{j}\omega_1)\mathrm{e}^{\mathrm{j}\omega_1 t}+H_1(2,1^{(1)};\mathrm{j}\omega_2)\mathrm{e}^{\mathrm{j}\omega_2 t}+H_1(2,1^{(1)};\mathrm{j}\omega_3)\mathrm{e}^{\mathrm{j}\omega_3 t}+2H_2(2,1^{(2)};\mathrm{j}\omega_1,\mathrm{j}\omega_2)\mathrm{e}^{\mathrm{j}(\omega_1+\omega_2)t}\\
\quad+2H_2(2,1^{(2)};\mathrm{j}\omega_2,\mathrm{j}\omega_3)\mathrm{e}^{\mathrm{j}(\omega_2+\omega_3)t}+2H_2(2,1^{(2)};\mathrm{j}\omega_1,\mathrm{j}\omega_3)e^{\mathrm{j}(\omega_1+\omega_3)t}\\
\quad+6H_3(2,1^{(3)};\mathrm{j}\omega_1,\mathrm{j}\omega_2,\mathrm{j}\omega_3)\mathrm{e}^{\mathrm{j}(\omega_1+\omega_2+\omega_3)t}+\ldots\ \ldots
\end{bmatrix}.
\end{cases}
\tag{42}
$$

Substituting Eqs. (41) and (42) into Eqs. (7) and (8) for $u_1(t)$, $y_1(t)$, and $y_2(t)$, and extracting the coefficient of $\mathrm{e}^{\mathrm{j}(\omega_1+\omega_2+\omega_3)t}$ from the resulting expressions yield two coupled equations for $[H_3(1,1^{(3)};\ \mathrm{j}\omega_1,\ \mathrm{j}\omega_2,\ \mathrm{j}\omega_3)$, $H_3(2,1^{(3)};\mathrm{j}\omega_1,\mathrm{j}\omega_2,\mathrm{j}\omega_3)]$

$$
\begin{cases}
H_3(2,1^{(3)};\mathrm{j}\omega_1,\mathrm{j}\omega_2,\mathrm{j}\omega_3)-m[\mathrm{j}(\omega_1+\omega_2+\omega_3)]^2H_3(1,1^{(3)};\mathrm{j}\omega_1,\mathrm{j}\omega_2,\mathrm{j}\omega_3)=0,\\[2ex]
6H_3(2,1^{(3)};\mathrm{j}\omega_1,\mathrm{j}\omega_2,\mathrm{j}\omega_3)-6kH_3(1,1^{(3)};\mathrm{j}\omega_1,\mathrm{j}\omega_2,\mathrm{j}\omega_3)\\
\quad-6a_1H_3(1,1^{(3)};\mathrm{j}\omega_1,\mathrm{j}\omega_2,\mathrm{j}\omega_3)[\mathrm{j}(\omega_1+\omega_2+\omega_3)]\\
\quad-4a_2H_1(1,1^{(1)};\mathrm{j}\omega_1)H_2(1,1^{(2)};\mathrm{j}\omega_2,\mathrm{j}\omega_3)(\mathrm{j}\omega_1)(\mathrm{j}\omega_2+\mathrm{j}\omega_3)\\
\quad-4a_2H_1(1,1^{(1)};\mathrm{j}\omega_2)H_2(1,1^{(2)};\mathrm{j}\omega_1,\mathrm{j}\omega_3)(\mathrm{j}\omega_2)(\mathrm{j}\omega_1+\mathrm{j}\omega_3)\\
\quad-4a_2H_1(1,1^{(1)};\mathrm{j}\omega_3)H_2(1,1^{(2)};\mathrm{j}\omega_1,\mathrm{j}\omega_2)(\mathrm{j}\omega_3)(\mathrm{j}\omega_1+\mathrm{j}\omega_2)\\
\quad-6a_3H_1(1,1^{(1)};\mathrm{j}\omega_1)H_2(1,1^{(1)};\mathrm{j}\omega_2)H_1(1,1^{(1)};\mathrm{j}\omega_3)(\mathrm{j}\omega_1)(\mathrm{j}\omega_2)(\mathrm{j}\omega_3)=0.
\end{cases}
\tag{43}
$$

Thus the third-order GFRF matrix is obtained as

$$
\begin{cases}
H_3(1,1^{(3)};\mathrm{j}\omega_1,\mathrm{j}\omega_2,\mathrm{j}\omega_3)=\dfrac{1}{\left\{m(\mathrm{j}\omega_1+\mathrm{j}\omega_2+\mathrm{j}\omega_3)^2+a(\mathrm{j}\omega_1+\mathrm{j}\omega_2+\mathrm{j}\omega_3)+k\right\}}\\
\qquad\times\left\{\begin{array}{l}
-\frac{2a_2}{3}\begin{bmatrix}
H_1(1,1^{(1)};\mathrm{j}\omega_1)H_2(1,1^{(2)};\mathrm{j}\omega_2,\mathrm{j}\omega_3)(\mathrm{j}\omega_1)(\mathrm{j}\omega_2+\mathrm{j}\omega_3)\\
+H_1(1,1^{(1)};\mathrm{j}\omega_2)H_2(1,1^{(2)};\mathrm{j}\omega_1,\mathrm{j}\omega_3)(\mathrm{j}\omega_2)(\mathrm{j}\omega_1+\mathrm{j}\omega_3)\\
+H_1(1,1^{(1)};\mathrm{j}\omega_3)H_2(1,1^{(2)};\mathrm{j}\omega_1,\mathrm{j}\omega_2)(\mathrm{j}\omega_3)(\mathrm{j}\omega_1+\mathrm{j}\omega_2)
\end{bmatrix}\\
-a_3H_1(1,1^{(1)};\mathrm{j}\omega_1)H_1(1,1^{(1)};\mathrm{j}\omega_2)H_1(1,1^{(1)};\mathrm{j}\omega_3)(\mathrm{j}\omega_1)(\mathrm{j}\omega_2)(\mathrm{j}\omega_3)
\end{array}\right\},\\
H_3(2,1^{(3)};\mathrm{j}\omega_1,\mathrm{j}\omega_2,\mathrm{j}\omega_3)=-m[\mathrm{j}(\omega_1+\omega_2+\omega_3)]^2H_3(1,1^{(3)};\mathrm{j}\omega_1,\mathrm{j}\omega_2,\mathrm{j}\omega_3).
\end{cases}
\tag{44}
$$

Denote

$$
\begin{aligned}
&\frac{1}{3}\begin{bmatrix}
H_1(1,1^{(1)};\mathrm{j}\omega_1)H_2(1,1^{(2)};\mathrm{j}\omega_2,\mathrm{j}\omega_3)(\mathrm{j}\omega_1)(\mathrm{j}\omega_2+\mathrm{j}\omega_3)\\
+H_1(1,1^{(1)};\mathrm{j}\omega_2)H_2(1,1^{(2)};\mathrm{j}\omega_1,\mathrm{j}\omega_3)(\mathrm{j}\omega_2)(j\omega_1+\mathrm{j}\omega_3)\\
+H_1(1,1^{(1)};\mathrm{j}\omega_3)H_2(1,1^{(2)};\mathrm{j}\omega_1,\mathrm{j}\omega_2)(\mathrm{j}\omega_3)(\mathrm{j}\omega_1+\mathrm{j}\omega_2)
\end{bmatrix}=\\
&-\frac{a_2}{3}\begin{bmatrix}
H_1(1,1^{(1)};\mathrm{j}\omega_1)F_0(\mathrm{j}\omega_2,\mathrm{j}\omega_3)(\mathrm{j}\omega_1)(\mathrm{j}\omega_2+\mathrm{j}\omega_3)\\
+H_1(1,1^{(1)};\mathrm{j}\omega_2)F_0(\mathrm{j}\omega_1,\mathrm{j}\omega_3)(\mathrm{j}\omega_2)(\mathrm{j}\omega_1+\mathrm{j}\omega_3)\\
+H_1(1,1^{(1)};\mathrm{j}\omega_3)F_0(\mathrm{j}\omega_1,\mathrm{j}\omega_2)(\mathrm{j}\omega_3)(\mathrm{j}\omega_1+\mathrm{j}\omega_2)
\end{bmatrix}=-a_2F_1(\mathrm{j}\omega_1,\mathrm{j}\omega_2,\mathrm{j}\omega_3),
\end{aligned}
\tag{45}
$$

where

$$F_1(j\omega_1, j\omega_2, j\omega_3) = \frac{1}{3}\begin{bmatrix} H_1(1,1^{(1)};j\omega_1)F_0(j\omega_2,j\omega_3)(j\omega_1)(j\omega_2+j\omega_3) \\ +H_1(1,1^{(1)};j\omega_2)F_0(j\omega_1,j\omega_3)(j\omega_2)(j\omega_1+j\omega_3) \\ +H_1(1,1^{(1)};j\omega_3)F_0(j\omega_1,j\omega_2)(j\omega_3)(j\omega_1+j\omega_2) \end{bmatrix}$$

and define

$$F_2(j\omega_1, j\omega_2, j\omega_3) = H_1(1,1^{(1)};j\omega_1)H_1(1,1^{(1)};j\omega_2)H_1(1,1^{(1)};j\omega_3)(j\omega_1)(j\omega_2)(j\omega_3). \tag{46}$$

Substituting these results into Eq. (44) yields

$$\begin{cases} H_3(1,1^{(3)};j\omega_1,j\omega_2,j\omega_3) & = \dfrac{1}{\beta(j\omega_1+j\omega_2+j\omega_3)} \\ & \quad \left\{a_2^2F_1(j\omega_1,j\omega_2,j\omega_3) - a_3F_2(j\omega_1,j\omega_2,j\omega_3)\right\}, \\ H_3(2,1^{(3)};j\omega_1,j\omega_2,j\omega_3) & = \dfrac{-m[j(\omega_1+\omega_2+\omega_3)]^2}{\beta(j\omega_1+j\omega_2+j\omega_3)} \\ & \quad \left\{a_2^2F_1(j\omega_1,j\omega_2,j\omega_3) - a_3F_2(j\omega_1,j\omega_2,j\omega_3)\right\}, \end{cases} \tag{47}$$

where $\beta(j\omega_1+j\omega_2+j\omega_3) = \left\{m(j\omega_1+j\omega_2+j\omega_3)^2 + a_1(j\omega_1+j\omega_2+j\omega_3)+k\right\}$.

Eqs. (34), (40) and (47) give the system GFRF matrices up to third order. Notice that $F_0(.,.)$, $F_1(.,.,.)$, $F_2(.,.,.)$, and $\beta(.)$ only depend on m, a_1, k, the parameters which describe the system linear characteristics. Therefore, given the system linear characteristics, Eqs. (40) and (47) explicitly reveal how the second and third order GFRF matrices depend on the parameters a_2 and a_3 of the system nonlinear damping characteristic.

Following the same principle, the GFRF matrices of the system (2) and (3) up to any higher order can be determined and represented in terms of the nonlinear damping parameters a_2 and a_3. However considerable symbolic computations are involved, and the results may consist of equations of several pages. Under the condition of $a_3 = 0$, the fifth-order GFRF $H_5(2,1^{(5)}; j\omega_1,\ldots,j\omega_5)$ of the system (2) and (3) has been determined for this study. The specific expression of this higher order GFRF is omitted here due to space limitations, but the result will be used in the next section to obtain a more accurate description for the system output frequency response.

4.3. The effects of system nonlinearity on the output frequency response

The expressions for the system GFRF matrices in terms of the nonlinear damping characteristic parameters a_2 and a_3 can now be used to derive an expression for the output spectrum $Y_2(j\omega)$. Substituting Eqs. (40) and (47), and the expressions for higher order GFRFs, as required, into Eq. (18) for $H_2(2,1^{(2)};j\omega_1,j\omega_2)$, $H_3(2,1^{(3)};j\omega_1,j\omega_2,j\omega_3)$, and $H_n(2,1^{(n)};j\omega_1,\ldots,j\omega_n)$, $n>3$, yields

$$\begin{aligned} Y_2(j\omega) &= \sum_{n=1}^{N}\frac{1}{2^n}\sum_{\omega_{k_1}+\ldots+\omega_{k_n}=\omega} H_n\left(2,1^{(n)};j\omega_{k_1},\ldots,j\omega_{k_n}\right)A(\omega_{k_1})\ldots A(\omega_{k_n}) \\ &= \frac{1}{2}H_1(2,1^{(1)};j\omega)A(\omega) + \frac{1}{2^2}\sum_{\omega_{k_1}+\omega_{k_2}=\omega} H_2\left(2,1^{(2)};j\omega_{k_1},j\omega_{k_2}\right)A(\omega_{k_1})A(\omega_{k_2}) \\ &\quad + \frac{1}{2^3}\sum_{\omega_{k_1}+\omega_{k_2}+\omega_{k_3}=\omega} H_3\left(2,1^{(3)};j\omega_{k_1},j\omega_{k_2},j\omega_{k_3}\right)A(\omega_{k_1})A(\omega_{k_2})A(\omega_{k_3}) + \ldots \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2}H_1(2,1^{(1)};\mathrm{j}\omega)A(\omega) + \frac{ma_2(\mathrm{j}\omega)^2}{2^2}\sum_{\omega_{k_1}+\omega_{k_2}=\omega} A(\omega_{k_1})A(\omega_{k_2})F_0(\mathrm{j}\omega_{k_1},\mathrm{j}\omega_{k_2}) \\
&\quad - \frac{m(\mathrm{j}\omega)^2 a_2^2}{2^3\beta(\mathrm{j}\omega)}\sum_{\omega_{k_1}+\omega_{k_2}+\omega_{k_3}=\omega} F_1(\mathrm{j}\omega_{k_1},\mathrm{j}\omega_{k_2},\mathrm{j}\omega_{k_3})A(\omega_{k_1})A(\omega_{k_2})A(\omega_{k_3}) \\
&\quad + \frac{m(\mathrm{j}\omega)^2 a_3}{2^3\beta(\mathrm{j}\omega)}\sum_{\omega_{k_1}+\omega_{k_2}+\omega_{k_3}=\omega} F_2(\mathrm{j}\omega_{k_1},\mathrm{j}\omega_{k_2},\mathrm{j}\omega_{k_3})A(\omega_{k_1})A(\omega_{k_2})A(\omega_{k_3}) + \dots \\
&= p_1(\mathrm{j}\omega) + p_2(\mathrm{j}\omega)a_2 - p_3(\mathrm{j}\omega)a_2^2 + p_4(\mathrm{j}\omega)a_3 + \dots \qquad (48)
\end{aligned}
$$

where

$$
\begin{aligned}
p_1(\mathrm{j}\omega) &= \frac{1}{2}H_1(2,1^{(1)};\mathrm{j}\omega)A(\omega), \\
p_2(\mathrm{j}\omega) &= \frac{m(\mathrm{j}\omega)^2}{2^2}\sum_{\omega_{k_1}+\omega_{k_2}=\omega} A(\omega_{k_1})A(\omega_{k_2})F_0(\mathrm{j}\omega_{k_1},\mathrm{j}\omega_{k_2}), \\
p_3(\mathrm{j}\omega) &= \frac{m(\mathrm{j}\omega)^2}{2^3\beta(\mathrm{j}\omega)}\sum_{\omega_{k_1}+\omega_{k_2}+\omega_{k_3}=\omega} F_1(\mathrm{j}\omega_{k_1},\mathrm{j}\omega_{k_2},\mathrm{j}\omega_{k_3})A(\omega_{k_1})A(\omega_{k_2})A(\omega_{k_3}), \\
p_4(\mathrm{j}\omega) &= \frac{m(\mathrm{j}\omega)^2}{2^3\beta(\mathrm{j}\omega)}\sum_{\omega_{k_1}+\omega_{k_2}+\omega_{k_3}=\omega} F_2(\mathrm{j}\omega_{k_1},\mathrm{j}\omega_{k_2},\mathrm{j}\omega_{k_3})A(\omega_{k_1})A(\omega_{k_2})A(\omega_{k_3}).
\end{aligned}
$$

Note that $p_i(\mathrm{j}\omega)$, $i = 1,2,3,4$, depend on the applied multi-tone input and the parameters which describe the linear characteristics of the system but are independent of a_2 and a_3.

Equation (48) is a very important result which describes the relationship between the system frequency response and the characteristic parameters of the system nonlinearity. As far as we are aware, little effort if any has previously been made to arrive at such an explicit description for this relationship. The result extends the fundamental analytical relationship between the linear characteristic parameters and the output frequency response to the nonlinear case for the system (2) and (3) when the system is subject to a multi-tone input, and can be further extended to other general situations.

If $a_2 = a_3 = 0$ in Eq. (48), then the system reduces to a simple linear sdof spring and damper system, and

$$
Y_2(\mathrm{j}\omega) = p_1(\mathrm{j}\omega) = \frac{1}{2}H_1(2,1^{(1)};\mathrm{j}\omega)A(\omega) = \frac{(a_1\mathrm{j}\omega_1 + k)}{2\big(m(\mathrm{j}\omega_1)^2 + a_1\mathrm{j}\omega_1 + k\big)}A(\omega). \qquad (49)
$$

Eq. (49) reveals the explicit analytical relationship between the system output frequency response and the linear characteristic parameters m, a_2, and k for the applied multi-tone input. This relationship is well-known and is used in analysis and design of linear sdof spring and damper systems.

In the case of nonlinear systems for example when $a_2 \neq 0$ and/or $a_3 \neq 0$, it has generally been believed that the relationship between the system output frequency response and the system characteristic parameters will always be very complicated. Researchers and engineers therefore tended to rely on numerical analysis, rather than analytical studies, to investigate the effects of system parameters on the output frequency response. Eq. (48), however, shows that the analysis of the nonlinear sdof system described by Eqs. (2) and (3) can be achieved in two steps. At the first step, the analysis of the effects of the system linear characteristic parameters on the output frequency response is conducted based on Eq. (49). This can be achieved in the case where the system is excited by an input with a low amplitude such that the system operates only over the linear regime. Secondly, the analysis of the effects of the system nonlinear characteristic parameters on the output frequency response is performed based on a truncated representation of Eq. (48) with fixed linear characteristic parameters. This covers the operating scenarios where the system works under a regime where the truncated representation is valid or approximately valid. The first step is straightforward and is the same as the widely applied linear system analysis approach. The second step allows the nonlinear analysis to be completed based on the explicit analytical expression given in Eq. (48). These results allow the frequency domain design of nonlinear systems to be performed in a totally new and systematic manner. This will be studied in more detail in a later publication. In the following analysis, however, the emphasis will be on studying how well a truncated representation of Eq. (48) can be used to represent the effects of the nonlinear characteristic parameters a_2 and a_3 on the system output spectrum.

For a given multi-tone input and the linear characteristic parameters m, a_1, k, $p_i(\mathrm{j}\omega)$ $i = 1, 2, 3, 4$ in Eq. (48) are known functions of frequency ω. Eq. (48) indicates that at each frequency component the system output spectrum is a polynomial function of the nonlinear damping characteristic parameters a_2 and a_3.

When the system is subject to the harmonic input (6), and the output frequency of interest in the analysis is the same as the input frequency Ω, $p_i(\mathrm{j}\omega)$ $i = 1, 2, 3, 4$ can be written as

$$p_1(\mathrm{j}\Omega) = \frac{1}{2} H_1(2, 1^{(1)}; \mathrm{j}\Omega) A(\Omega), \tag{50}$$

$$p_2(\mathrm{j}\Omega) = 0, \tag{51}$$

$$p_3(\mathrm{j}\Omega) = \frac{\Omega^6 m \left|H_1^{1:1}(\mathrm{j}\Omega)\right|^2 H_1^{1:1}(\mathrm{j}\Omega)}{2\beta(\mathrm{j}\Omega)\beta(2\mathrm{j}\Omega)} \left|A(\Omega)\right|^2 A(\Omega), \tag{52}$$

$$p_4(\mathrm{j}\Omega) = -\frac{3\mathrm{j}\Omega^5 m \left|H_1(1, 1^{(1)}; \mathrm{j}\Omega)\right|^2 H_1(1, 1^{(1)}; \mathrm{j}\Omega)}{2^3 \beta(\mathrm{j}\Omega)} \left|A(\Omega)\right|^2 A(\Omega). \tag{53}$$

Consider two relatively simple cases to illustrate the basic ideas. In the first case, a_3 will be assumed to be zero and a truncated representation of Eq. (48) which includes system nonlinearities up to fifth order is used to describe the frequency response of the system (2) and (3) to the harmonic input (6). In the second case, the effects of both a_2 and a_3 on the system output frequency response to the harmonic input are studied assuming that a truncated representation of Eq. (48) with system nonlinearity up third order can be used to describe the system response. The results show how the damping characteristic parameters a_2 and a_3 analytically determine the system output frequency response, and to what extent an analytical expression can be used to conduct system analysis and design.

For the first case, by considering the effects of system nonlinearities up to fifth order, Eq. (48) can be written as

$$\begin{aligned} Y_2(\mathrm{j}\Omega) &= \sum_{n=1}^{5} \frac{1}{2^n} \sum_{\omega_{k_1}+\ldots+\omega_{k_n}=\Omega} A(\omega_{k_1}) \ldots A(\omega_{k_n}) H_n(2, 1^{(n)}; \mathrm{j}\omega_{k_1}, \ldots, \mathrm{j}\omega_{k_n}) \\ &= p_1(\mathrm{j}\Omega) + p_2(\mathrm{j}\Omega) a_2 - p_3(\mathrm{j}\Omega) a_2^2 \\ &\quad + \frac{1}{2^5} \sum_{\omega_{k_1}+\ldots+\omega_{k_5}=\Omega} A(\omega_{k_1}) \ldots A(\omega_{k_5}) H_5(2, 1^{(5)}; \mathrm{j}\omega_{k_1}, \cdots, \mathrm{j}\omega_{k_5}). \end{aligned} \tag{54}$$

The last term in Eq. (54) represents the effect on the output response of the fifth order system nonlinearity and can be further expressed as

$$\begin{aligned} &\frac{1}{2^5} \sum_{\omega_{k_1}+\ldots+\omega_{k_5}=\Omega} A(\omega_{k_1}) \ldots A(\omega_{k_5}) H_5(2, 1^{(5)}; \mathrm{j}\omega_{k_1}, \ldots, \mathrm{j}\omega_{k_5}) \\ &= \frac{1}{2^5} \left|A(\Omega)\right|^4 A(\Omega) \sum H_5(2, 1^{(5)}; .) \end{aligned} \tag{55}$$

where

$$\sum H_5(2, 1^{(5)}; .) = \begin{bmatrix} H_5(2, 1^{(5)}; -\mathrm{j}\Omega, \mathrm{j}\Omega, \mathrm{j}\Omega, \mathrm{j}\Omega, -\mathrm{j}\Omega) + H_5(2, 1^{(5)}; -\mathrm{j}\Omega, \mathrm{j}\Omega, \mathrm{j}\Omega, -\mathrm{j}\Omega, \mathrm{j}\Omega) \\ +H_5(2, 1^{(5)}; -\mathrm{j}\Omega, \mathrm{j}\Omega, -\mathrm{j}\Omega, \mathrm{j}\Omega, \mathrm{j}\Omega) + H_5(2, 1^{(5)}; -\mathrm{j}\Omega, -\mathrm{j}\Omega, \mathrm{j}\Omega, \mathrm{j}\Omega, \mathrm{j}\Omega) \\ +H_5(2, 1^{(5)}; \mathrm{j}\Omega, \mathrm{j}\Omega, \mathrm{j}\Omega, -\mathrm{j}\Omega, -\mathrm{j}\Omega) + H_5(2, 1^{(5)}; \mathrm{j}\Omega, \mathrm{j}\Omega, -\mathrm{j}\Omega, \mathrm{j}\Omega, -\mathrm{j}\Omega) \\ +H_5(2, 1^{(5)}; \mathrm{j}\Omega, \mathrm{j}\Omega, -\mathrm{j}\Omega, -\mathrm{j}\Omega, \mathrm{j}\Omega) + H_5(2, 1^{(5)}; \mathrm{j}\Omega, -\mathrm{j}\Omega, \mathrm{j}\Omega, \mathrm{j}\Omega, -\mathrm{j}\Omega) \\ +H_5(2, 1^{(5)}; \mathrm{j}\Omega, -\mathrm{j}\Omega, \mathrm{j}\Omega, -\mathrm{j}\Omega, \mathrm{j}\Omega) + H_5(2, 1^{(5)}; \mathrm{j}\Omega, -\mathrm{j}\Omega, -\mathrm{j}\Omega, \mathrm{j}\Omega, \mathrm{j}\Omega) \end{bmatrix}. \tag{56}$$

An analytical expression for $H_5(2,1^{(5)};\mathrm{j}\omega_1,\ldots,\mathrm{j}\omega_5)$ in terms of the damping characteristic parameter a_2 has been derived using the probing method. This expression is quite long and will not be given here to save space. However, from this complicated analytical expression, the summation of $\sum H_5(2,1^{(5)};.)$ can be expressed in a very compact form as

$$\sum H_5(2,1^{(5)};.) = \frac{a_2^4 m\Omega^{10}}{\beta^4(\Omega)\beta^2(-\Omega)\beta(2\Omega)}\left[\begin{array}{l}\dfrac{32}{\beta(\Omega)\beta(2\Omega)} - \dfrac{48}{\beta(3\Omega)\beta(-2\Omega)} \\ +\dfrac{48}{\beta(3\Omega)\beta(2\Omega)} + \dfrac{16}{\beta(-\Omega)\beta(-2\Omega)}\end{array}\right] \tag{57}$$

Therefore, in this case, the analytical expression for the system output frequency response is given as

$$Y_2(\mathrm{j}\Omega) = p_1(\mathrm{j}\Omega) - p_3(\mathrm{j}\Omega)a_2^2 + p_5(\mathrm{j}\Omega)a_2^4, \tag{58}$$

where $p_1(\mathrm{j}\Omega)$ and $p_3(\mathrm{j}\Omega)$ are as defined in Eqs. (50) and (52), and

$$p_5(\mathrm{j}\Omega) = \frac{m\Omega^{10}|A(\Omega)|^4 A(\Omega)}{2^5\beta^4(\Omega)\beta^2(-\Omega)\beta(2\Omega)}\left[\begin{array}{l}\dfrac{32}{\beta(\Omega)\beta(2\Omega)} - \dfrac{48}{\beta(3\Omega)\beta(-2\Omega)} \\ +\dfrac{48}{\beta(2\Omega)\beta(3\Omega)} + \dfrac{16}{\beta(-\Omega)\beta(-2\Omega)}\end{array}\right]. \tag{59}$$

For the second case, by ignoring all terms contributed by nonlinear effects higher than third order, Eq. (48) can be written as

$$Y_2(\mathrm{j}\Omega) = p_1(\mathrm{j}\Omega) - p_3(\mathrm{j}\Omega)a_2^2 + p_4(\mathrm{j}\Omega)a_3, \tag{60}$$

where $p_1(\mathrm{j}\Omega)$, $p_3(\mathrm{j}\Omega)$ and $p_4(\mathrm{j}\Omega)$ are as defined in Eqs. (50), (52), and (53).

Simulation studies will be conducted in the next section for the system (2) and (3) to evaluate the output frequency response to the harmonic input (6) for different values of a_2 and a_3. The results will then be compared with the output spectrum $Y_2(\mathrm{j}\Omega)$ determined using Eq. (58) or (60). The objective is to verify the effectiveness of the theoretically derived analytical relationships and to show the potential of using these relationships in system analysis and design.

5. Simulation studies

Consider the system (2) and (3) subject to the harmonic input (6). Take the system linear characteristic parameters m and k to be $m = 240$ kg and $k = 16\,000$ N/m with two choices for the linear damping parameter of $a_1 = 2960$ s N/m and $a_1 = 1960$ s N/m, respectively. For a range of nonlinear damping parameters of a_2 and a_3, the system was simulated to generate the output frequency response $Y_2(\mathrm{j}\Omega)$. This was then compared with the analytical result from Eq. (58) or (60) for the following five cases:

(i) $\Omega = 8.1$ rad/s, $F_d = 80$ N, $a_3 = 0$, $a_2 \neq 0$;
(ii) $\Omega = 8.1$ rad/s, $F_d = 100$ N, $a_3 = 0$, $a_2 \neq 0$;
(iii) $\Omega = 10$ rad/s, $F_d = 100$ N, $a_3 = 0$, $a_2 \neq 0$;
(iv) $\Omega = 10$ rad/s, $F_d = 120$ N, $a_3 = 0$, $a_2 \neq 0$;
(v) $\Omega = 8.1$ rad/s, $F_d = 100$ N, $a_3 \neq 0$, $a_2 \neq 0$;

Figs. 2 and 4 show the results for cases (i) and (ii) for $a_1 = 2960$ s N/m and $a_1 = 1960$ s N/m, respectively. Figs. 3 and 5 show the results for cases (iii) and (iv) for the same two different choices of a_1.

Fig. 6(a) and (b) show the analytically determined output spectrum for case (v) for $a_1 = 1960$ s N/m and $a_1 = 2960$ s N/m, respectively. Fig. 7(a) and (b) show a comparison of the analytically determined output spectra with the simulation results for case (v) for the same two choices of a_1, and for $a_2 = -20000\,\mathrm{s}^2\,\mathrm{N/m}^2$ and $a_2 = 0$, respectively.

Notice that $2|Y_2(\mathrm{j}\Omega)|$ not $|Y_2(\mathrm{j}\Omega)|$ is used to show the output spectrum. This is because $2|Y_2(\mathrm{j}\Omega)|$ represents the physical magnitude of the system output $y_2(t)$ at frequency Ω.

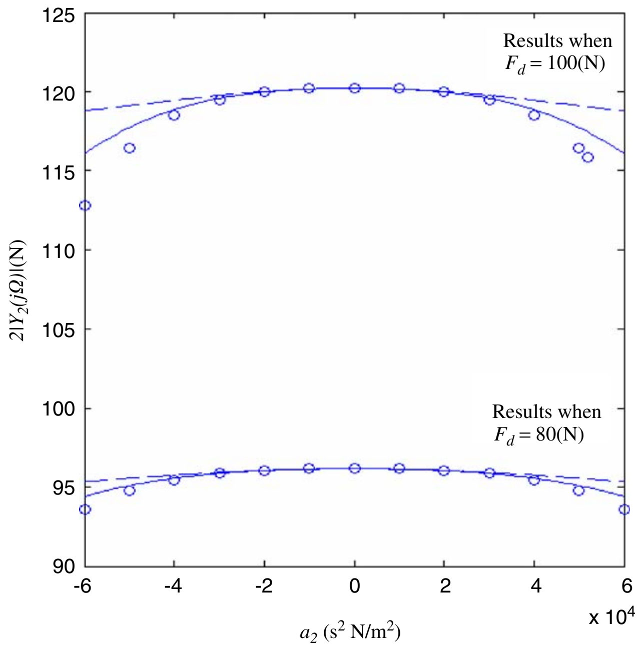

Fig. 2. The effect of the nonlinear damping characteristic parameter a_2 on the system output frequency response when $\Omega = 8.1\,\mathrm{rad/s}$, $a_1 = 2960\,\mathrm{s\,N/m}$, and $a_3 = 0$. Solid lines: analytically determined results using nonlinear terms up to fifth order; dashed lines: analytically determined results using nonlinear terms up to third order; circles: simulation results.

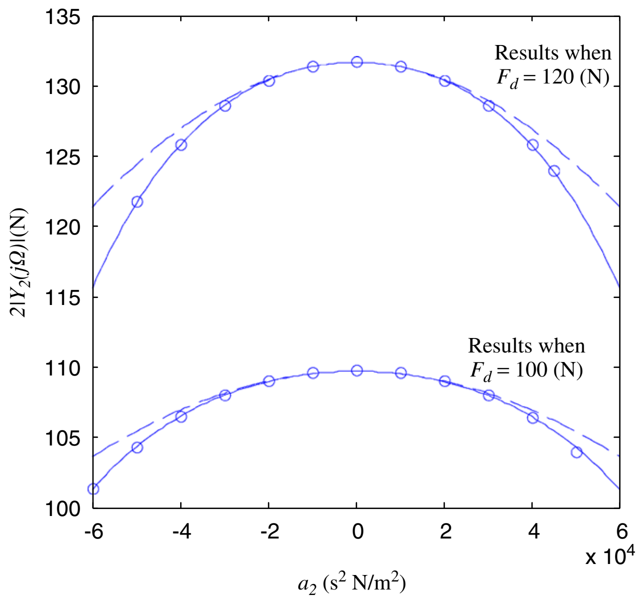

Fig. 3. The effect of the nonlinear damping characteristic parameter a_2 on the system output frequency response when $\Omega = 10\,\mathrm{rad/s}$, $a_1 = 2960\,\mathrm{s\,N/m}$, and $a_3 = 0$. Solid lines: analytically determined results using nonlinear terms up to fifth order; dashed lines: analytically determined results using nonlinear terms up to third order; circles: simulation results.

In Figs. 2–5, the solid lines show the magnitude of the output spectrum $2Y_2(\mathrm{j}\Omega)$ determined using the analytical description (58) over a range of values of a_2 when the system nonlinear effects up to fifth order (all terms in Eq. (58)) are taken into account. The dashed lines show the results determined from Eq. (58) when only the system nonlinearity up to third order (just first two terms in Eq. (58)) is considered. The circles show

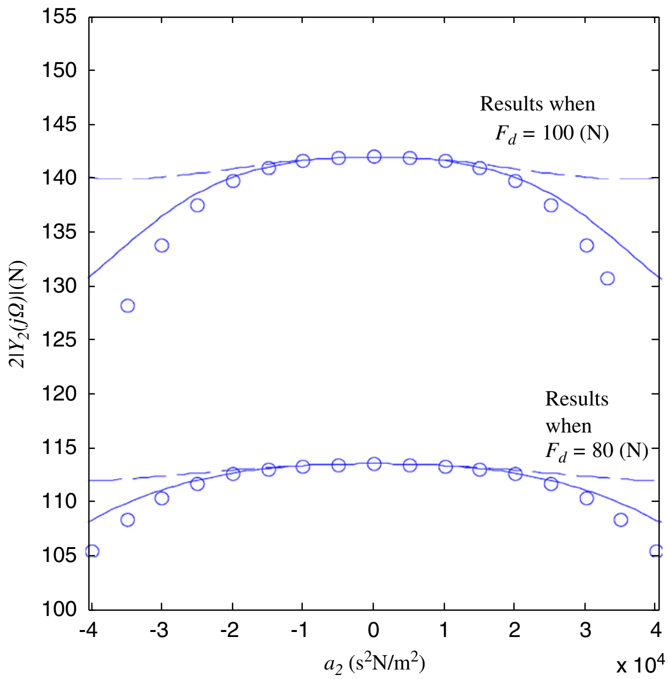

Fig. 4. The effect of the nonlinear damping characteristic parameter a_2 on the system output frequency response when $\Omega = 8.1\,\mathrm{rad/s}$, $a_1 = 1960\,\mathrm{s\,N/m}$, and $a_3 = 0$. Solid lines: analytically determined results using nonlinear terms up to fifth order; dashed lines: analytically determined results using nonlinear terms up to third order; circles: simulation results.

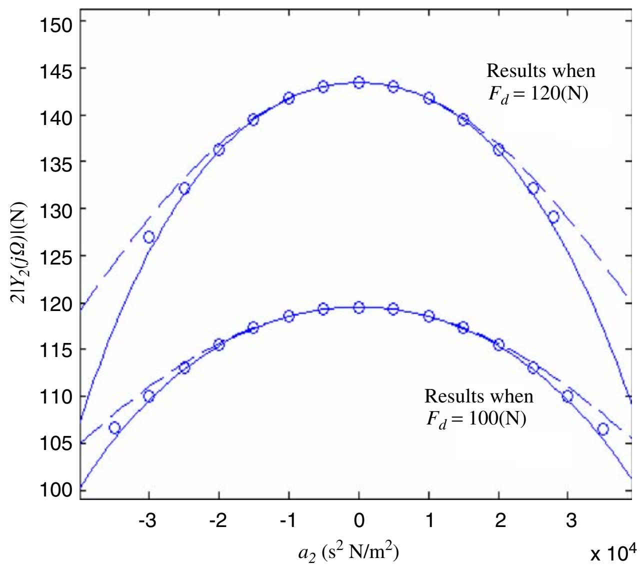

Fig. 5. The effect of the nonlinear damping characteristic parameter a_2 on the system output frequency response when $\Omega = 10\,\mathrm{rad/s}$, $a_1 = 1960\,\mathrm{s\,N/m}$, and $a_3 = 0$. Solid lines: analytically determined results using nonlinear terms up to fifth order; dashed lines: analytically determined results using nonlinear terms up to third order; circles: simulation results.

the results of the spectrum over a set of discrete values of a_2, representing the numerical simulation results of the output spectrum, and obtained by performing a FFT operation on the system time domain output $y_2(t)$. In each of the four diagrams, the output spectra for two different input magnitudes are presented to show the effect of the magnitude of the harmonic input on the analytical description for the output spectrum. These results reflect how the magnitude of the system output spectrum changes with the nonlinear damping characteristic parameter a_2, and how the analytically determined system output frequency responses match the simulation results over a considerable range of values of a_2 under the conditions of different linear damping parameters a_1, different harmonic input frequencies Ω, and different input magnitudes F_d.

Inspection of Figs. 2–5 indicates that the system output frequency response analytically determined from Eq. (58) using nonlinear terms up to third can represent the real system output spectrum well over a certain range of values of a_2, and including the fifth order nonlinear terms into the analytical expression can considerably improve the accuracy of the analytically determined results. It can also be observed from these figures that the improvement due to taking higher order nonlinear terms into account is more significant when the system is subject to an input with a greater input magnitude.

Comparing Figs. 2 and 4, or Figs. 3 and 5 shows the effect of the linear damping parameter a_1 on the accuracy of the analytical expression (58) for the output spectrum. When the linear damping parameter a_1 is increased from $a_1 = 1960$ to $2960\,\mathrm{s\,N/m}$, the same harmonic force input produces a smaller output force response at the input frequency, a less considerable variation of output force magnitudes can be observed over the same range of variation of a_2, and the range of a_2, over which the analytical expression (58) for the output spectrum can well describe the true output spectrum, becomes greater.

Comparing Figs. 2 and 3, or Figs. 4 and 5 shows the effect of the frequency Ω of the applied harmonic input on the accuracy of the analytical expression (58) for the output spectrum. When Ω changes from $\Omega = 8.1$ up to 10 rads/s, it can be observed that the analytical expression (58) becomes more accurate. This is because $\Omega = 10$ rads/s is farther from the resonant frequency of 8.16 rad/s of the system than $\Omega = 8.1$ rads/s. The nonlinear effects of the system are less significant when subject to a harmonic input farther away from the resonant frequency. In contrary, the nonlinear effects of the system are more significant when the input is closer to the resonant frequency. Consequently system nonlinearities higher than fifth order may be needed to more accurately represent the case for $\Omega = 8.1$ rads/s.

Figs. 6 and 7 show the results regarding the derived analytical expression (60) for the system output spectrum. Phenomena similar to the ones that can be observed from Figs. 2–5 can be observed. System nonlinear effects become more significant when a_1 is reduced from $a_1 = 2960$ s to $1960\,\mathrm{s\,N/m}$, and a_2 is changed from $a_2 = 0$ to $-20\,000\,\mathrm{s^2\,N/m^2}$. For this case, Fig. 7(a) indicates that higher order system nonlinearities should be considered in the analytical expression (60) to achieve a more accurate description. However, nonlinear effects become less significant when either a_1 is increased from $a_1 = 1960$ to $2960\,\mathrm{s\,N/m}$, or a_2 is changed from $a_2 = -20\,000\,\mathrm{s^2\,N/m^2}$ to $a_2 = 0$. Consequently, for the three cases of $a_1 = 2960\,\mathrm{s\,N/m}$, $a_2 = 20\,000\,\mathrm{s^2\,N/m^2}$; $a_1 = 1960\,\mathrm{s\,N/m}$, $a_1 = 0$; and $a_1 = 2960\,\mathrm{s\,N/m}$, $a_1 = 0$; Fig. 7 indicates that the analytical expression (60), which only includes nonlinearities up to third order, is sufficient to accurately represent the system output frequency response.

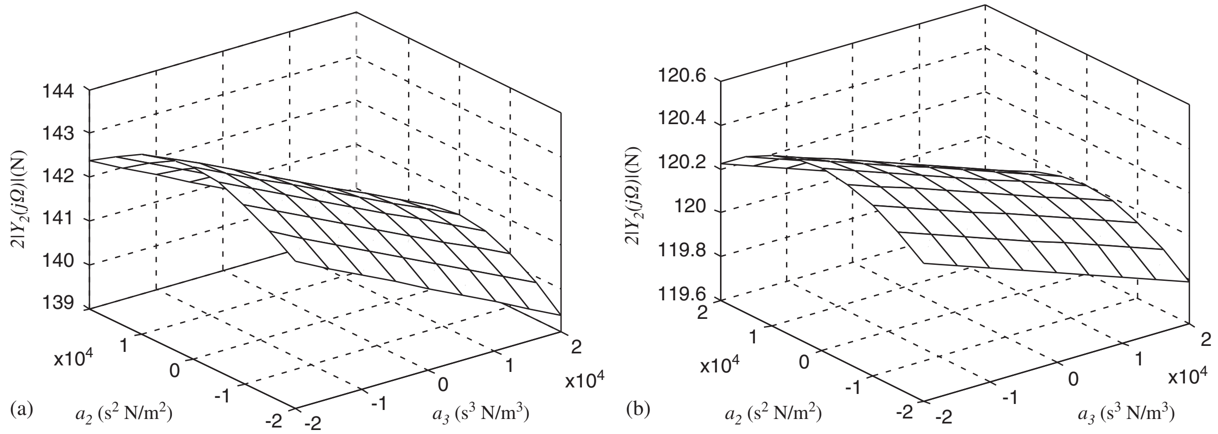

Fig. 6. The analytically determined relationship between the nonlinear damping characteristic parameters a_2, a_3 and the system output frequency response when $\Omega = 8.1$ rad/s and $F_d = 100$ N: (a) for the case of $a_1 = 1960\,\mathrm{s\,N/m}$: (b) for the case of $a_1 = 2960\,\mathrm{s\,N/m}$.

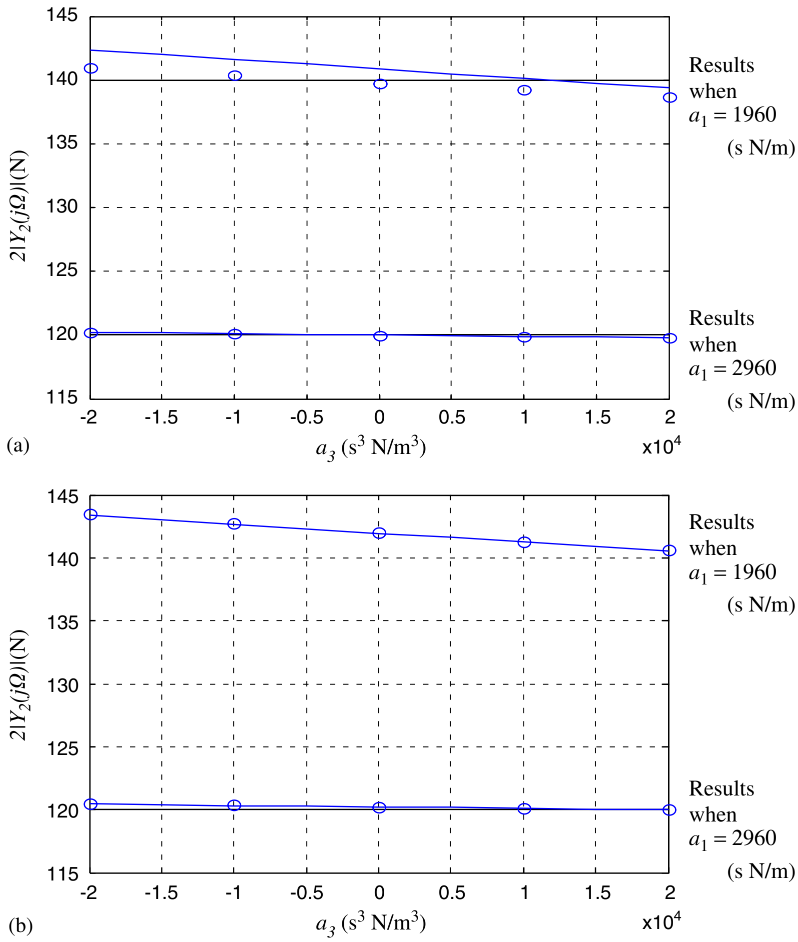

Fig. 7. Comparison of analytically determined output frequency responses and simulation results when $\Omega = 8.1\,\text{rad/s}$ and $F_d = 100\,\text{N}$: (a) for the case of $a_2 = -20\,000$; (b) for the case of $a_2 = 0$.

Overall, the results in Figs. 2–7 verify the theoretical analysis in the previous sections and demonstrate the effectiveness of the analytical descriptions (58) and (60) for the output frequency response. The analytical expressions describe the system output spectra quite well over a considerable range of values of damping parameters a_2 and a_3, for different linear damping parameters, and for different harmonic inputs. Similar results have also been obtained for different mass m's and stiffness k's, but are omitted here due to page limitations. The improvements achieved by considering the effects of fifth order nonlinear terms, as illustrated in Figs. 2–5, show the potential of producing accurate results if higher order terms are taken into account.

The analytical expressions can be used to design the damping characteristic parameters. Given a desired output frequency response, which can be realised within the range of the parameters where the analytical expressions are valid, the values of the parameters which cause the system output to reach the desired response can be determined from the expressions via an optimisation procedure.

All the derivations in the present study assume that the system under study can be described by the Volterra series model (15) and (16) under the defined operating conditions and over the range of parameter variations discussed. Volterra series can be used to represent the class of fading memory nonlinear systems [28]. This excludes systems which exhibit subharmonics and chaos, but includes the wide class of engineering systems that generate harmonics and intermodulation effects.

6. Conclusions

An analytical expression showing the effects of system nonlinearities on the output frequency response of a sdof spring damper system has been investigated. The relationship between the output frequency response and the nonlinear damping characteristic parameters has been derived for the system. The derivations are based on the frequency domain analysis of nonlinear systems. Results from simulation studies have been used to verify the theoretical analysis and to demonstrate the effectiveness of the derived relationship.

The basic ideas of this work can be extended to general situations to arrive at a comprehensive analytical description for the relationship between nonlinear system output frequency responses and model parameters provided that the system behaviours can be described by a Volterra series model under the considered operating conditions. As demonstrated in the present study, this analytical relationship can be very powerful and can be used to considerably facilitate the analysis and design of a wide range of nonlinear engineering systems and structures.

Acknowledgements

ZQL, SAB, and GRT gratefully acknowledge the support of the Engineering and Physical Science Research Council, UK, for this work.

References

[1] M. Schetzen, *The Volterra and Wiener Theories of Nonlinear Systems*, Wiley Interscience Publications, New York, 1980.
[2] W.J. Rugh, *Nonlinear System Theory—the Volterra/Wiener Approach*, Johns Hopkins University Press, Baltimore, MD, 1981.
[3] S. Boyd, L.O. Chua, Fading memory and the problem of approximating nonlinear operators with Volterra series, *IEEE Transactions on Circuits and Systems* 30 (1985) 1150–1161.
[4] S.A. Billings, S. Chen, Extended model set, global data and threshold model identification of severely nonlinear systems, *International Journal of Control* 50 (1989) 1897–1923.
[5] S. Chen, S.A. Billings, Representation of nonlinear systems: the NARMAX model, *International Journal of Control* 49 (1989) 1013–1032.
[6] D. Nauck, F. Klawonn, R. Kruse, *Foundations of Neuro-Fuzzy Systems*, Wiley, New York, 1997.
[7] G. Duffing, *Erzwungene schwingungen bei weranderlicher eigenfrequenz*, Vieweg, Braunschweig, 1918.
[8] T. Matsumoto, M. Komuro, H. Kokubu, R. Tokunaga, *Bifurcations: Sights, Sounds, and Mathematics*, Springer, Berlin, 1993.
[9] M.W. Hirsch, S. Smale, *Differential Equations, Dynamical Systems, and Linear Algebra*, Academic Press, Berlin, 1974.
[10] D.A. George, Continuous nonlinear systems, *MIT Research Laboratory Electron., Technical Report* 335 (1959).
[11] L.J. Tick, The estimation of transfer functions of quadratic systems, *Technom* 3 (1961) 503–507.
[12] K.I. Kim, E.J. Powers, A digital method of modelling quadratically nonlinear systems with a general random input, *IEEE Transactions on Acoustics, Speech, Signal Processing* ASSP-36 (1988) 1758–1769.
[13] J.C. Peyton-Jones, S.A. Billings, A recursive algorithm for computing the frequency response of a class of nonlinear difference equation models, *International Journal of Control* 50 (1989) 1925–1940.
[14] S.A. Billings, J.C. Peyton-Jones, Mapping nonlinear integro-differential equation into the frequency domain, *International Journal of Control* 54 (1990) 863–879.
[15] S.W. Nam, E.J. Powers, Application of higher order spectral analysis to cubicaly nonlinear system identification, *IEEE Transactions on Signal Processing* 42 (1994) 1745–1746.
[16] D.E. Adams, Frequency domain ARX model and multi-harmonic FRF estimators for nonlinear dynamic systems, *Journal of Sound and Vibration* 250 (2002) 935–950.
[17] Z.Q. Lang, S.A. Billings, Output frequency characteristics of nonlinear systems, *International Journal of Control* 64 (1996) 1049–1067.
[18] Z.Q. Lang, S.A. Billings, Output frequencies of nonlinear systems, *International Journal of Control* 67 (1997) 713–730.
[19] Z.Q. Lang, S.A. Billings, Evaluation of output frequency response of nonlinear systems under multiple inputs, *IEEE Transactions on Circuits and Systems II: Analog and Digital Signal Processing* 47 (2000) 28–38.
[20] S.A. Billings, Z.Q. Lang, A bound for the magnitude characteristics of nonlinear output frequency response functions, part 1: analysis and computation, *International Journal of Control* 65 (1996) 309–328.
[21] S.A. Billings, Z.Q. Lang, A bound for the magnitude characteristics of nonlinear output frequency response functions, part 2: practical computation of the bound for systems described by the NARX model, *International Journal of Control* 65 (1996) 365–384.
[22] S.A. Billings, Z.Q. Lang, Truncation of nonlinear system expansions in the frequency domain, *International Journal of Control* 68 (1997) 1019–1042.
[23] S.A. Billings, Z.Q. Lang, Nonlinear systems in the frequency domain: energy transfer filters, *International Journal of Control* 75 (2002) 1066–1081.

[24] V. Volterra, *Theory of Functions*, Blackie, London, 1930.
[25] A.K. Swain, S.A. Billings, Generalised frequency response function matrix for MIMO nonlinear systems, *International Journal of Control* 74 (2001) 829–844.
[26] K. Worden, G. Manson, G.R. Tomlinson, A harmonic probing algorithm for the multi-input Volterra series, *Journal of Sound and Vibration* 201 (1997) 67–84.
[27] E. Bedrosian, S.O. Rice, The output properties of Volterra systems (nonlinear systems with memory) driven by harmonic and Gaussian input, *Proceedings of the IEEE* 59 (1971) 1688–1707.
[28] S. Boyd, L.O. Chua, Fading memory and the problem of approximating nonlinear operators with Volterra series, *IEEE Transactions on Circuit and Systems* 32 (1985) 1150–1161.
[29] S. Cafferty, J. Giacomin, K. Worden, A nonlinear model for automotive shock absorbers, in: IUTAM Symposium on Identification of Mechanical Systems, Wuppertal, Germany, August 23–27, 1993.