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The linearization method based on the equivalence of dissipated energies for nonlinearly damped structural systems ☆

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Abstract

Even though the nonlinear damping characteristics inherently pertain to most structural systems, many useful dynamic models are still linear. As a consequence, a method that is able to equivalently linearize (EQL) the nonlinear damping so that it can be directly applied in these existing linear models is essential. In the present paper, a novel EQL method for nonlinearly damped and single-degree-of-freedom systems is developed. The method is theoretically derived by modulating the steady-state responses of the original nonlinear system. During the linearization, the new EQL method requires the dissipated energy of the target linear system to equal that of the original nonlinear one. In effect, this criterion is equivalent to forcing both the phase angles and amplitudes of the two systems to be equal, or at least to within a small allowable error. Furthermore, the present paper proves that it is possible for the dissipative energy to be expressed in terms of the Fourier coefficients of the modulated signal. Thus, the equivalent viscous damping ratio can be computed from these Fourier coefficients. This new EQL method is numerically tested using examples of bi-linear damping models, and subjected to experimental measurements. Both the simulation results and the experimental data verify the validity of the method. They also prove that the current method gives the equivalent viscous damping with good accuracy. In addition, a quality index that signifies how well the EQL system reaches is appropriately added.

1. Introduction

In most engineering practice, the use of a linear model for the application leads to fairly simple and useful results. And, there is a comprehensive linear theory for treating problems of this type for quite some time already. Unfortunately, there is no actual problem that is exactly linear. Thus, the easiest way is to assume that the system is almost linear or the nonlinearity is negligibly small to be able to apply the linear theory. In general, nonlinearity of a system becomes more significant as vibration increases. As soon as the nonlinear effect is too large to be neglected, one is faced with the problem of predicting the responses of more complicated nonlinear systems. A simple way of attacking these nonlinear problems is to replace the nonlinear elements by linear ones, which in some cases give approximately the same responses, so that the linear theory can be reasonably applied.

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Nomenclature

a, b	Fourier coefficients of $g(t)$
$f(t)$	excitation forcing function in time domain
f_s	signal sampling frequency in Hz
$g.(t)$	modulated response in time domain
F	amplitude of the external forcing function
L, L	length in time domain
n	number of points of a sampled signal
r	frequency ratio $= \Omega/\omega_n$
Δt_s	sampling resolution $= 1/f_s$
T	period in time domain
w_d	average dissipated energies per radian (J/rad)
$y(t)$	responses of the equivalent linear system in time domain
Y	amplitude of the linear system response
$z(t)$	responses of the original nonlinear system in time domain
Z	amplitudes of the nonlinear system response
ε,ε*	allowable errors
ϕ	lag angle due to system damping
ω_n	natural frequency in rad/s
Ω	excitation frequency in rad/s
ζ	damping ratios of nonlinear systems
ζ_{EQL}	equivalent damping ratio

Mathematically, any criterion of linearization must preserve the characteristics of the original nonlinear system as much as possible [1]. There is no way to know exactly who is the first to introduce the so-called linearization method, as far as the authors knowledge goes. Nevertheless, almost all investigation reports in engineering practices concerning linearization directly or indirectly cite the paper from Krylov and Bogoliulov [2] of 1937, regardless of whether the reports are in deterministic or stochastic systems. The asymptotic method given by Krylov–Bogoliulov [2] is also known later on as the averaging method. Since then, several different versions of linearization methods have been developed. For example, the techniques for deterministic dynamics include steps like those of the perturbation methods (See for e.g. Refs. [3,4]), the harmonic balancing methods (See for e.g. Ref. [5]) and the methods of multiple scales [6], etc. These methods are analytical rather than numerical and provide an alternative to computer solutions. In addition, there are also linearization methods that combine both analytical and numerical techniques. These algorithms [7] include the so-called phase-space linearization method [8], which is based on replacing the nonlinear vector field by a set of linear ones and are valid over a sufficiently small interval of time. Thus, it is actually a method that decomposes the nonlinear governing equation into a set of linear equations, each valid and evolving over a segment of trajectory. This linearization idea was later generalized and reported in Ref. [9].

On the other hand, there are significant numbers of researches devoted to the study of linearization techniques for stochastic systems during the last two decades. The pioneering works include (See for e.g. Refs. [10,11]). Their methods for stochastic nonlinear systems are later called the statistical linearization [12], since the nonlinear function of the original system is replaced by a linear function. In addition, the work on the equivalent linearization (EQL) of Ref. [13] is a suggestion from Ref. [12] to replace the nonlinear system by an equivalent one. Extensive literature reviews concerning EQL with different criteria can be found in Roberts and Spanos [14], or Socha and Soong [15].

Based on the linearization knowledge from single-degree-of-freedom (sdof) systems, the method has been generalized to multi-dof (mdof) ones. For example, the studies [16–18] are proposed taking computational efficiency into consideration. Besides, there are also group of researchers who developed linearization techniques incorporating numerical methods. The elegant work [17] even employed the linearization idea to decompose the non-Gaussian distribution into a sum of Gaussian ones for statistical systems. By doing so, the method [17] is becomes applicable to statistical problems with larger dimensions.

Amongst all these aforementioned linearization methods, the system nonlinearity, which the most methods are faced with, are focused either in localized or small areas, or in stiffness. Only few reports specifically mention the applicability of their methods on nonlinear damping. Apparently, damping or friction induces many nonlinear dynamic phenomena [1,15,19,20]. However, to the authors' knowledge, there exist only a few reports that discuss the linearization of this kind. The main reason may stem from the fact that the true damping of system is very difficult to obtain accurately. Unlike its counterparts, mass and stiffness properties can be directly measured or analytically computed by numerical models such as the finite element methods through computer-aided tools. In general, damping characteristics, whether they are internal (material, micro-structural effects, friction, etc.) or external (boundary, fluid contact, fluid/structure interaction, etc.) can be observed only by experimental measurements. As a result, only those linearization methods that are able to physically incorporate measured signals can be used in engineering practices.

Motivated by this, the present study attempts to give a novel EQL method specifically for nonlinearly damped structural systems. Nevertheless, a method [21] is provided to evaluate the system damping when it is viscous. However, there is no way to determine if the damping of a system is linear or nonlinear before actual measurements are made. In addition, the system is still restricted to be deterministic and sdof in the present report. For this reason, the present method is yet to be verified for random or mdof systems.

2. Derivation of the method

In general, a unit mass, sdof system with nonlinear damping can be represented together with the initial conditions by

$$\ddot{z} + f_d(z, \dot{z}) + \omega_n^2 z = f(t), \tag{1}$$

where $f_d(z, \dot{z})$ denotes the nonlinear damping function, ω_n is the undamped circular natural frequency, and $f(t)$ the externally applied forcing function, respectively. If the system is excited by a harmonic forcing function, without loss of the generality, one can express the function as

$$f(t) = F \sin(\Omega t). \tag{2}$$

Thus, the solution of system (1) can be written as

$$z(t) = z_0(t) + z_p(t), \tag{3}$$

in which $z_0(t)$ is the free or zero-input response characterized by the system itself and the initial conditions, and $z_p(t)$ is the forced or zero-state response. It is also clear that the latter depends mainly on the characteristics of excitation. Furthermore, $z(t) \approx z_p(t)$ if the system is stable and the steady-state responses only are of interest. Since the external forcing function is periodic, as shown in Eq. (2), the total work done by/to system (1) for the response per cycle can be represented by integrating both sides of Eq. (1) to obtain

$$\oint (\ddot{z} + f_d(z, \dot{z}) + \omega_n^2 z)\,\mathrm{d}z = \oint f(t)\,\mathrm{d}z, \tag{4}$$

where the circle-integral denotes the integration for one complete cycle. Note that the expression on the RHS of Eq. (4) actually represents the energy input to the system by the mechanism of the forcing function $f(t)$. Because of the energy conservation in each cycle, the total work done by them is zero. Thus, the averaged dissipated energy per radian can be evaluated from the LHS of Eq. (4) to give

$$w_{d,z} = \frac{1}{2\pi} \oint f_d(z, \dot{z})\,\mathrm{d}z = \frac{1}{2\pi} \int_t^{t+T_1} f_d(z, \dot{z})\,\dot{z}\mathrm{d}t, \tag{5}$$

in which T_1 is the period of the $f(t)$. Notice from Eqs. (4) and (5) that $w_{d,z}$ is also equal to the averaged energy input to the system. Or, the energy loss due to the damper is equal to the one supplied by the excitation under the steady state [22]. Since the external excitation is assumed to be sinusoidal, it is then reasonable to write the nonlinear responses in two parts: the time invariant and periodically alternating parts. In other words, one writes the steady state of $z(t)$ as

$$z(t) = Z_0 + z_a(t) = Z_0 + Z_a \sin(\Omega t - \phi), \tag{6}$$

where ϕ is the phase angle which is mainly due to the nonlinear damping forces. Substituting Eq. (6) back to Eq. (4), one computes the input energy per radian from the RHS of Eq. (4) and obtains

$$w_{d,z} = \frac{1}{2\pi} \int_t^{t+T_1} f(t)\dot{z}(t)\,\mathrm{d}t = \frac{FZ_a}{2} \sin\phi. \tag{7}$$

That is, one can compute the dissipated energy of the nonlinearly system from its input energy, Eq. (7), instead of directly computing from Eq. (5).

One applies the input modulation concept [21,23], or modulates the steady state of $z(t)$ by its input. In other words, the product modulation of $f(t)$ and $z(t)$ has the form

$$\begin{aligned} g_z(t) &= f(t)\,z(t) \\ &= \frac{F\,Z_a}{2}\cos\phi + Z_0 F\sin(\Omega t) - \frac{F\,Z_a}{2}\cos(2\Omega t - \phi). \end{aligned} \tag{8}$$

Notice that the first term on the RHS of Eq. (8) is time-invariant, and others sinusoidal with frequencies of 1 Ω and 2 Ω, respectively. And, in fact, Eq. (8) is the weak form of Eq. (7). Referring to the Fourier series, the coefficients of $g_z(t)$ in Eq. (8) can be denoted a_0, b_1 and α_2, respectively. And the last coefficient α_2 can be expressed as

$$\alpha^2 = \sqrt{a_2^2 - b_2^2}, \tag{9}$$

in which a_0, a_2, b_1 and b_2 are known as the Fourier coefficients associated with the cosine and sine functions. Substituting these Fourier coefficient notations back to Eq. (7), the averaged input energy $w_{d,z}$ can now be represented by

$$w_{d,z} = \sqrt{\alpha_2^2 - a_0^2}. \tag{10}$$

By comparing Eq. (8) with Eq. (10), one can conclude that the second term (b_1) on RHS of $g_z(t)$ does not dissipate energy. It may be regarded as a sort of elastic strain energy resulting from the nonlinear damping force.

On the other hand, one assumes that there exists a linear system that is considered equivalent to system (1) and has the governing equation

$$\ddot{y} + 2\zeta_{\mathrm{EQL}}\omega_n\dot{y} + \omega_n^2 y = f(t), \tag{11}$$

subjected to the same initial conditions as the nonlinear one. ζ_{EQL} in Eq. (10) is considered as the equivalent viscous damping ratio to be determined. Note that the forcing function and the undamped natural frequency of systems in (1) and (10) are all kept the same. Similar to the nonlinear system (1), the response of the linear one takes the form

$$y(t) = y_0(t) + y_p(t). \tag{12}$$

In addition, the steady state of $y(t)$ is considered to have the form

$$y(t) \approx y_p(t) = Y\sin(\Omega t - \phi_L), \tag{13}$$

where ϕ_L represents the phase angle due to the existence of the equivalent viscous damping. Applying the same modulation as that in the nonlinear case, the modulated responses of the equivalent linear system is

$$\begin{aligned} g_y(t) &= f(t)\,y(t) \\ &= \frac{FY}{2}\cos\phi_{\mathrm{L}} - \frac{FY}{2}\cos(2\Omega t - \phi_{\mathrm{L}}). \end{aligned} \tag{14}$$

Again, the first term on RHS of Eq. (14) is time-invariant. Analogous to Eq. (8), Eq. (14) can be expressed in terms of the Fourier coefficient notations as

$$g_y(t) = (a_0') - \alpha_2' \, cos(2\Omega t - \phi_L), \tag{15}$$

where the prime is added to distinguish those coefficients of the linear from the nonlinear. Correspondingly, the averaged dissipated energy per cycle of the EQL system can be written in terms of these coefficients as

$$w_{d,y} = \sqrt{(\alpha_2')^2 - (a_0')^2}. \tag{16}$$

In order to establish the equivalent relation between the nonlinear and linear systems, which are governed by Eqs. (1) and (11), respectively, one considers that the differences in their amplitudes at some t are within a small tolerance ε, or

$$\left|z(t)-y(t)\right|\leqslant\varepsilon. \tag{17}$$

However, Eq. (17) may be generalized and equivalently written as

$$\left|\frac{1}{2\pi}\oint f(t)[z(t)-y(t)]\,\mathrm{d}t\right|\leqslant\varepsilon*, \tag{18}$$

if the computation is carried out in terms of complete number of cycles from the steady states. Note, parameter ε* in Eq. (18) also represents a small tolerance similar to ε since $f(t)$ is a bounded periodic function. Furthermore, one may simply compute the differences between the square of these two dissipative energies. Using Eqs. (10) and (16) to Eq. (18), one has

$$\left|(w_{d,z})^2-(w_{d,y})^2\right|=\left|\left[a_2^2-a_0^2\right]-\left[(a_2')^2-(a_0')^2\right]\right|\leqslant\varepsilon*. \tag{19}$$

To satisfy Eq. (19), a sufficient condition can be easily found, which is

$$a_2'\to a_2 \text{ together with } a_0'\to a_0. \tag{20,21}$$

Both conditions must be valid at the same time. However, if one looks closely, the former condition indirectly suggests $Y\to Z_a$. That is to require the output amplitudes of the two systems to be equal. In the meantime, the latter condition in Eq. (21) implies that

$$\cos\phi_L\to\frac{a_0}{a_2}. \tag{22}$$

Or, the condition is to force the two-phase angles, ϕ and ϕ_L, to be equal. Notice that if $a_0=0$ from Eq. (22), $\phi_L=\pi/2$ or 90°. Readers are referred to Ref. [23] for the detailed discussion in this case. The tangent of the phase angle for the linear system is well known and can be expressed as (See for e.g. Ref. [20])

$$\tan\phi_L=\frac{2r\zeta_L}{1-r^2}. \tag{23}$$

Thus, one is able to solve the equivalent viscous damping ratio (ζ_L) from Eqs. (23) and (22), obtains

$$\zeta_{\mathrm{EQL}}(r)=\frac{(1-r^2)}{2r}\left(\frac{a_2}{a_0}\sqrt{1-(a_0/a_2)^2}\right)\quad\text{when } a_0\neq 0, \tag{24}$$

where $r=\Omega/\omega_n$. Using Eq. (10) to Eq. (24), the upper equation is in fact equivalent to

$$\zeta_{\mathrm{EQL}}=\frac{(1-r^2)}{2r}\left(\frac{w_{d,z}}{a_0}\right). \tag{25}$$

Thus, if one directly measures the dissipative energy from the nonlinear system, only one coefficient a_0 is enough to compute the equivalent viscous damping ratio. In addition, only the positive value of Eq. (25) can be taken in accordance with the definition of the damping ratio.

3. Evaluation of the Fourier coefficients

3.1. Coefficient a_0

The relation in Eq. (24) or Eq. (25) shows that the equivalent linear damping ratio of the EQL system can be expressed in terms of the Fourier coefficients or the dissipated energy of the original nonlinear system. Now, the main problem is how to correctly evaluate these coefficients so that the relation can be properly applied. Notice that the first Fourier coefficient, a_0, is time-invariant, which is dc of the modulated signal. The easiest way to obtain its value is to design a low-pass filter (LPF) and to get rid of those periodic terms from the signal $g_z(t)$. By doing that, one obtains

$$a_0=\left\|g_z(t)\right\|_{\mathrm{LPF}}=\frac{FZ_a}{2}\cos\phi, \tag{26}$$

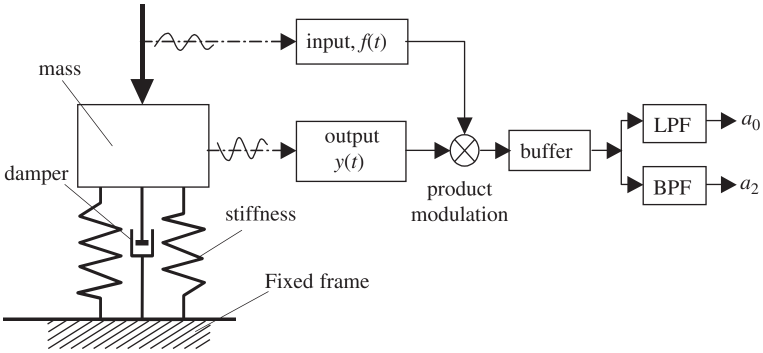

Fig. 1. Schematic diagram for obtaining the Fourier coefficients.

where $\|\cdots\|$ denotes the filtered value from the RHS of Eq. (8). Fig. 1 graphically shows the concept of using filters to obtain the Fourier coefficients.

Alternately, one may compute the following integration from RHS of Eq. (8), or

$$\begin{aligned}\frac{1}{L}\int_{L_0}^{L_0+L}(\text{RHS})\,\mathrm{d}t &= \frac{1}{L}\left[\frac{FZ_a}{2}\int_{L_0}^{L_0+L}\cos\phi\,\mathrm{d}t\right.\\ &\quad\left.+\,Z_0F\int_{L_0}^{L_0+L}\sin(\Omega t)\,\mathrm{d}t-\frac{FZ_a}{2}\int_{L_0}^{L_0+L}\cos(2\Omega t-\phi)\,\mathrm{d}t\right]\\ &= \frac{FZ_a}{2}\cos\phi.\end{aligned} \tag{27}$$

In the above equation, all terms in RHS are zero except the first time-invariant one if L is taken in terms of complete cycles of Ωt. If the signals are discrete, one can practically compute the value of the LHS of Eq. (8) from the modulated signal $g_z(t)$ similar to that of its RHS. In other words, the result takes the form

$$a_0 = \tfrac{1}{L}\int_{L_0}^{L_0+L} g_z(t)\,\mathrm{d}t \cong \frac{1}{n}\sum_{j=1}^{n} g_z(j\Delta t_s), \tag{28}$$

where $\Delta t_s = (1/f_s)$ and f_s is the sampling frequency, n the number of sampling points in L. In fact, a_0 in Eq. (28) is the averaged value or the dc offset of the modulated signal $g_z(t)$ in $[L_0,\ L_0+L]$. During the averaging, one may just take L to be large enough, instead of determining if the sampling is in complete cycles. By doing so, the error is bounded and proved in Ref. [21]. And, the longer the L takes, the smaller is error a_0.

3.2. Coefficient α_2

The evaluation of the coefficient α_2 is not as easy as its counterpart a_0. However, one still has very useful tools available. Consider the Fourier transform of Eq. (8) from the modulated signal, one gets

$$\begin{aligned}\frac{F_T[g_z(t)]}{2\pi} &= a_0\delta(\omega) + i(b_1/2)[\delta(\omega+\Omega)-\delta(\omega-\Omega)]\\ &\quad+(\alpha_2/2)[\delta(\omega+2\Omega)\mathrm{e}^{-\mathrm{i}\phi}+\delta(\omega-2\Omega)\mathrm{e}^{\mathrm{i}\phi}],\end{aligned} \tag{29}$$

where $F_T[.]$ denotes the Fourier transform from t into ω, $\delta(.)$ the delta (or impulse) function. Therefore, the first method of obtaining α_2 is from the Fourier transform of $g_z(t)$. One is thus able to identify the value at 2Ω after it is transformed into the frequency domain. That value is actually one-half of α_2, based on Eq. (29).

In addition to the above-mentioned method, by examining Eq. (29), it is also possible to design a band pass filter (BPF) similar to Eq. (26), i.e., to obtain α_2 by a BPF

$$\alpha_2 = \left\|g_z(t)\right\|_{\mathrm{BPF}}, \tag{30}$$

with its central frequency at 2Ω. Refer to Fig. 1 for detail.

3.3. The quality index

It is also worthwhile to emphasize that the aforementioned linearization procedure is actually based on the equivalence of the dissipated energies. In other words, the accuracy of the equivalent viscous damping ratio in Eq. (25) depends completely on how accurately $W_{d,y}$ can represent $w_{d,z}$. This also means that the ratio of $w_{d,z}$ to $w_{d,y}$, or

$$S_{\mathrm{EQL}} = \frac{w_{d,z}}{w_{d,y}}, \tag{31}$$

can be used as the quality index to signify how well the nonlinearly damped system is represented. The ratio index has a value of unity if the two systems are precisely identical. If $S_{\mathrm{EQL}} > 1.0$ implies the present EQL method under-estimates the original dissipated energy, or $w_{d,y} < w_{d,z}$.

4. Numerical examples

In order to verify the validity of the present EQL method, numerical simulations are conducted. The nonlinearly damped system is considered to possess a bi-linear damping, or $\zeta = \zeta_1$if $|\dot{z}| \leq Z_c$ and $\zeta = \zeta_2$ if $|\dot{z}| > Z_c$, as shown in Fig. 2. The nonlinearities of simulations can be controlled by specifying the combination of the two different damping ratios. Moreover, in the case of $\zeta_1 = \zeta_2$, the system becomes linear with a viscous damping ratio ζ_1. During the simulations, the following parameters are employed:

- $\omega_n = 10\pi$ (or 5 Hz),
- $\Omega = 0.95\omega_n$ (or 4.75 Hz),
- $F = 10$,
- $\dot{Z}_c = 1.0$

with consistent units, and the sampling frequency is set to 200 Hz.

Once the EQL method is checked with the case of $\zeta_1 = \zeta_2$, several bi-linear simulations are conducted. However, in order to delve further into the results predicted by the current EQL method, a typical time response is presented in Fig. 3(a). This figure is selected because $\zeta_2/\zeta_1 = 3$ and the quality index, S_{EQL}, is approximately 1.1. In other words, the original nonlinearly damped system dissipates 10% higher energy than the EQL one. As can be clearly seen from Fig. 3(a), the time-domain response indicates that the EQL system response (shown by solid line) is somewhat smaller than the original nonlinear one (dotted line). Actually, this is already indicated by the quality index. $S_{\mathrm{EQL}} > 1$ means that the new EQL method underestimates the

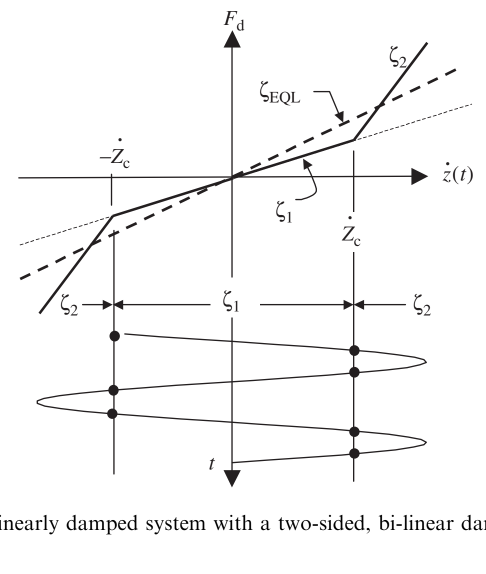

Fig. 2. A nonlinearly damped system with a two-sided, bi-linear damping function.

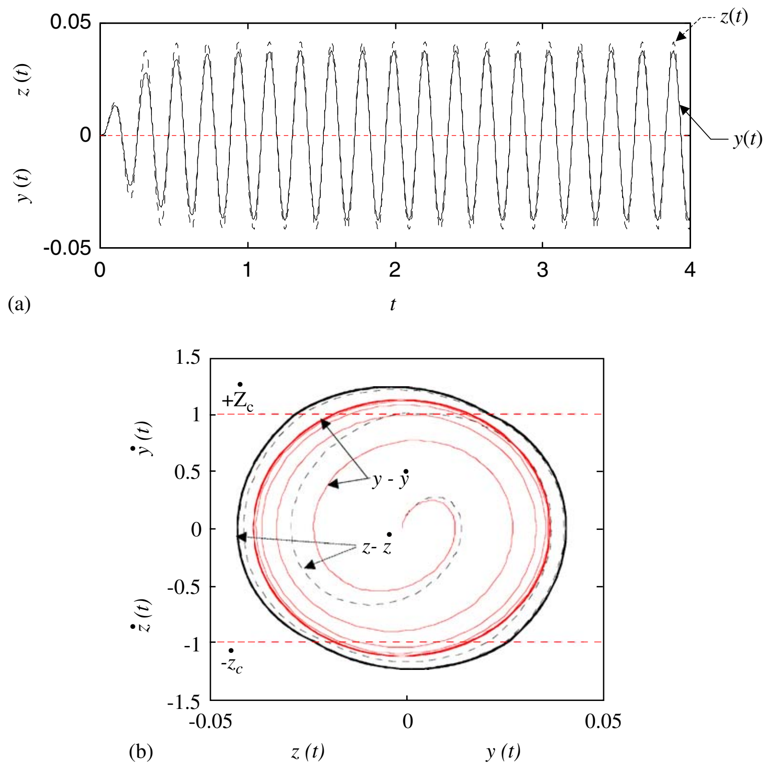

Fig. 3. Typical $z(t)$ and $y(t)$ for $\zeta_1 = 0.05$, $\zeta_2 = 0.15$, $\zeta_{\mathrm{EQL}} = 0.1326$, and $S_{\mathrm{EQL}} = 1.1$: (a) time history; (b): phase portrait.

average dissipated energy. The phase portraits of Fig. 3(a) are shown in Fig. 3(b), wherein the portrait of the EQL system is completely enclosed within the original one. As a consequence, the amplitude of $y(t)$ is smaller than that of $z(t)$. However, the phase angles of the both systems coincide precisely with each other even from the beginning of the transient response, shown in Fig. 3(b). In other words, this method has correctly forced the EQL system to have the same lag phase angle as the original one. However, the amplitude of the EQL system is somewhat sacrificed for the sake of$w_{d,y} < w_{d,z}$.

The fast Fourier transform (FFT) is performed to the modulated steady state response, $g_z(t)$, of the time responses shown in Fig. 3 to further study the modulated signal. The transformed result is shown in Fig. 4. It can be seen from the figure that there are two high peaks located at ω equals to 0 and 2Ω. The former is the one that is denoted by a_0, while the latter one-half of α_2 when referred to Eq. (29). Not surprisingly, this result is exactly the same as that given in Eq. (8). At the same time, it also indicates that the appropriate values of LPF and BPF from $g_z(t)$ can be used to get a_0 and α_2. However, one cannot see any peak at Ω since Z_0 in Eq. (8) is zero or too small to be seen. Besides, $z(t)$ in Fig. 3(a) does not appear dc offset either.

If the nonlinear system is slightly altered by relaxing the nonlinear damping to one-sided bilinear, i.e., the system damping is changed to $\zeta = \zeta_1$ when $\dot{z} \leqslant Z_c$ and $\zeta = \zeta_2$ when $\dot{z} > Z_c$. The FFT result of the modulated signals is shown in Fig. 5. Comparing Fig. 5 with Fig. 4, the change of damping is accompanied with an increasing in a_0. In the meantime, the peak located at Ω can now be clearly seen. The height of this peak is one-half of b_1. One can notice that the coefficient b_1 does not play any role in the EQL procedures. However, it can be used as an index to signify how strong the nonlinearity can bring the system away from zero mean. The larger b_1 indicates that the system will behalf a stronger nonlinear characteristics and have a larger dc offset, as in Eq. (8). Fig. 6 substantiates this argument in which the dc offset Z_0 can be clearly noticed. Obviously, there is no dc offset response for the EQL system response $y(t)$ since the external excitation has the zero mean and the system is linear. However, a non-zero response may exist for a nonlinearly damped system even though the excitation has a zero mean. This dc offset becomes zero if $\zeta_1 = \zeta_2$, which is linear.

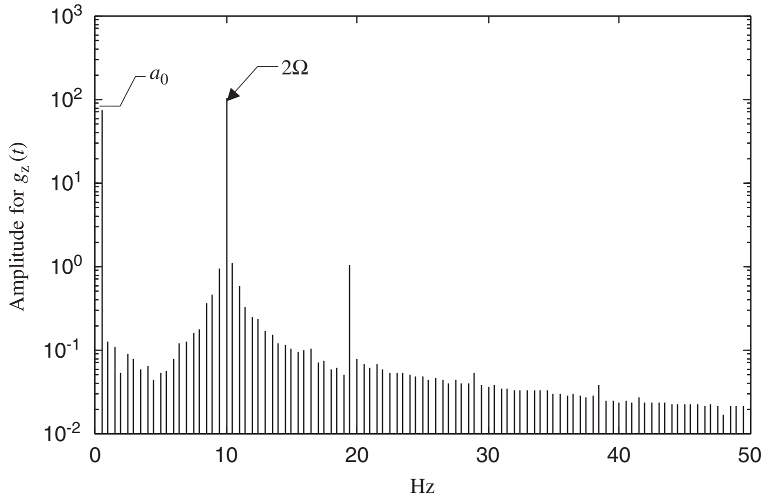

Fig. 4. Typical FFT of $g_z(t)$ for systems with two-sided bilinear damping.

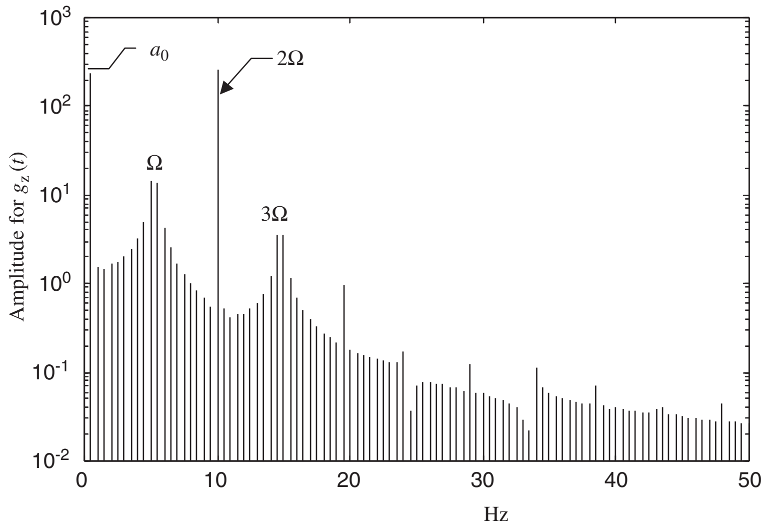

Fig. 5. Typical FFT of $g_z(t)$ for systems with one-sided bilinear damping.

5. Experiments

In addition to the numerical verifications given in the last section, several experimental measurements are carried out. The main objective of the experiments is to verify the validity of Eq. (8). The experimental setup for the measurements is shown in Fig. 7. The test specimen is an austenite stainless-steel cantilever with dimensions $1.0^T \times 21.5^W \times 194^L$ mm (ca. 35.6 g) directly mounted to a shaker. At the end of the beam is a plastic straw with the length 96 mm and the outer diameter 8.6 mm (ca. 2 g) firmly glued to the beam. Meanwhile, in order to simulate the damping forces acting on the beam, the straw is inserted into a cup of viscid liquid during the experiments. The inserted depth is set in such a way that the straw is at the depth d when the beam is at its static equilibrium position. For example, in the case of $d = 0$ mm, the tip of the straw is just about to contact the liquid surface before the beam starts oscillating. On the other hand, the tip completely immerses in the liquid during the oscillations if $d = 20$ mm. Since the damping force is proportional

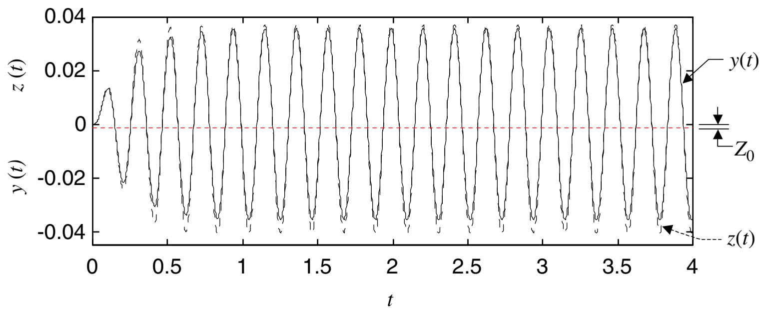

Fig. 6. Typical $z(t)$ and $y(t)$ of an one-sided bilinear system for $\zeta_1 = 0.1$, $\zeta_2 = 0.2$, $\zeta_{EQL} = 0.1402$, and $S_{EQL} = 1.09$.

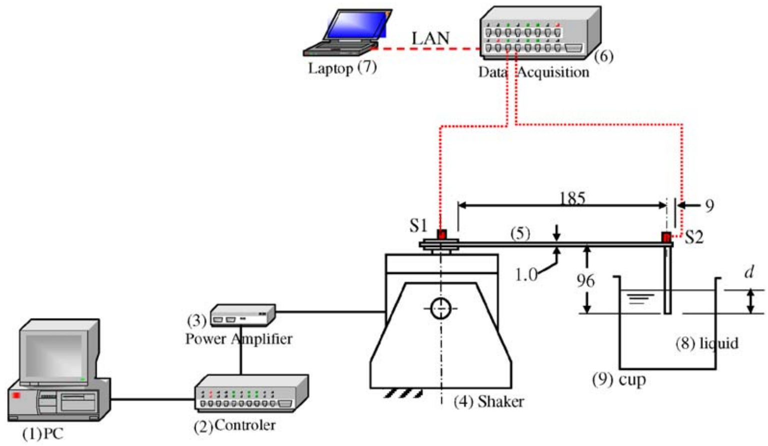

Fig. 7. The experimental set-up (1, PC; 2, signal controller, Dactron; 3, Power Amplifier, B&K 2706; 4, Exciter, B&K 4809; 5, Cantilever, 6, acquisition system, iMC μ-Musys; 7, PC and Software, FAMOS; 8, Liquid; and 9, Glass cup).

to the depth inside the liquid, the straw thus acts as the nonlinear damper in the experimental system. Refer to Fig. 7 for the details of the set-up.

During the experiments, the sampling frequency is set to 200 Hz, and sinusoidal excitations with the constant frequency (Ω) are applied from the controller to the shaker and detected by accelerometer 1 (S1), which is directly mounted on the base of the cantilever. The amplitudes of these excitations are all kept the same while its frequency may be changed if so desired. The system responses are then detected and acquired by accelerometer 2 (S2), which is located at the tip of the beam. Both signals are then acquired and sent to the laptop computer for the later analyses.

Figs. 8(a) to (c) show the typical results in which the modulated signals are transformed into the frequency domains. The peaks appear at both 1Ω and 2Ω can be clearly seen in these plots. Note that the peak heights at 2Ω are corrected since the original value can be realized as one-half of the coefficient α_2, as shown in Eq. (10). However, care must be taken when measuring b_1 from the plot, since there is also another way of affecting the peak at 1Ω. Experimentally, the digital filter that is applied during the data acquisition may also create an extra peak at the frequencies like 1Ω. This phenomenon is also known as the signal leakage [24]. Nevertheless, the experimental results further verify the validity of Eq. (10), which is the core of the present EQL method.

The three plots in Fig. 8 are intentionally shown in the same scales for comparison. The dissipative energies corresponding to the three cases are tabulated in Table 1. From the first two cases, namely (a) and (b) where $d = 0$ mm, one can find that the increase of energy loss may be due mainly to the increase of α_2, after comparing the first two plots where a_0 remains almost unchanged. Meanwhile the increase of α_2 is relatively

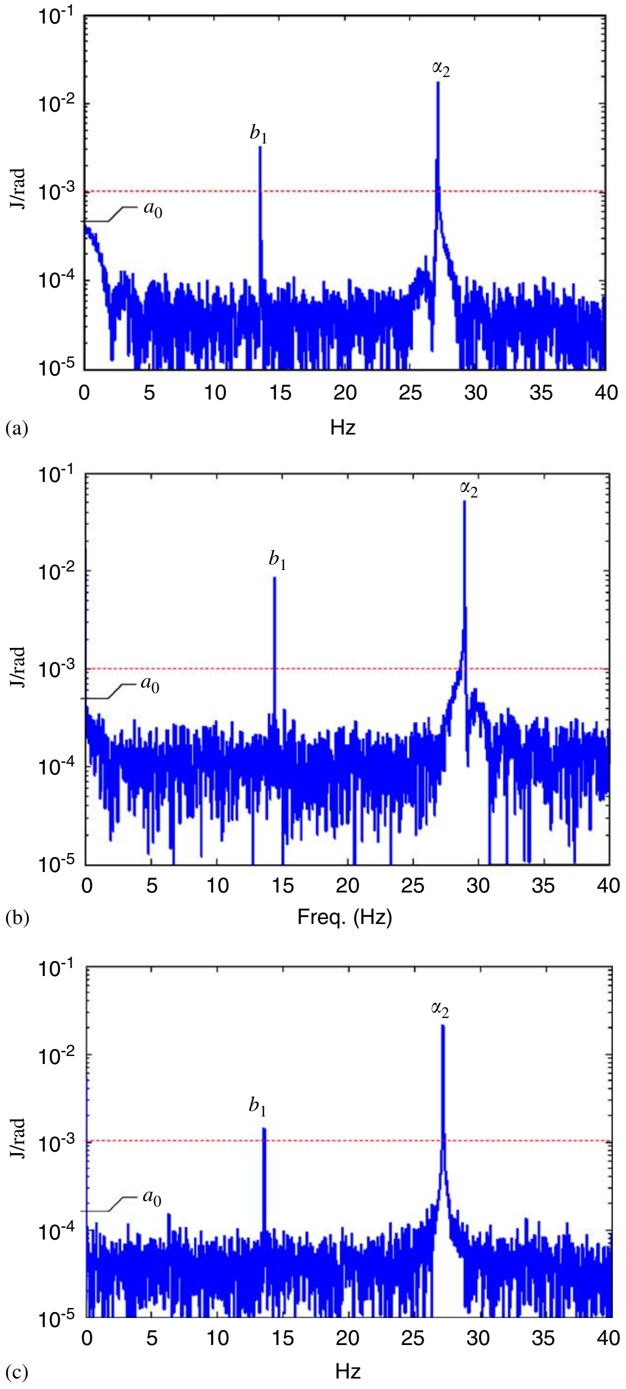

Fig. 8. A typical modulated response in the frequency domain: (a) $d = 0\,\text{mm}$, $\Omega = 13.6\,\text{Hz}$; (b) $d = 0\,\text{mm}$, $\Omega = 14.5\,\text{Hz}$; and (c) $d = 20\,\text{mm}$, $\Omega = 13.6\,\text{Hz}$.

apparent. On the other hand, one allows the straw damper to provide a larger damping force by immersing it deeper into the liquid. The result is shown in Fig. 8(c) for the case of $d = 20\,\text{mm}$. More dissipative energy is observed in this case from Table 1. However, it is also accompanied by a smaller coefficient b_1, as shown in Fig. 8(a) and (c). As mentioned earlier, this means that the value of this coefficient can be regarded as an index signifying how strong the nonlinear damping force is. Accordingly, one can conclude that the case shown in

Table 1
The measured dissipative power

d (mm)	Ω (Hz)	w_d 10^{-4} j/rad	power mW	ζ_{EQL} %	Fig.
0	13.6	0.0120 (0.0019)	0.1023 (0.0159)	–	8(a)
	14.5	0.2480 (0.0004)	2.3	10.99 (0.3)	8(b)
20	13.6	0.1166 (0.0002)	1.0	8.54 (0.13)	8(c)

(…), std. deviation.

Fig. 8(a) has a stronger nonlinearity than the one in 8(c), i.e., the coefficient b_1 of Fig. 8(a) is larger than that of 8(c). More importantly, the tabulated data assure that measuring the dissipative energy from the Fourier coefficients of the modulated signal is easy and practical.

6. Conclusions

A novel equivalent linearization method for nonlinearly damped sdof systems is developed and presented in this paper. The method is derived by modulating the steady state responses of the original nonlinear system. During the linearization, the present EQL method requires the dissipated energy of the nonlinear system to equal to that of the target linear one. As a consequence, this criterion is equivalent to forcing both the phase angles and amplitudes of the two systems to be equal, or at least within a small allowable error. The equivalent viscous damping ratio can be then computed from the modulated signal.

In addition, the present report also theoretically explains the validity of the method, i.e., the EQL method is de facto closely related to the Fourier coefficients of the modulated responses. In other words, the dissipated energy of the nonlinear system can be expressed in terms of these Fourier coefficients. As a result, the core problem of this linearization method is to correctly obtain these coefficients from the modulated signal. A few approaches are discussed in the present paper, including averaging the signals, applying the filters and the FFT methods. The last one is clearly demonstrated and shown by examples. The results of the examples indicate that the new linearization method gives the equivalent viscous damping with good accuracy. Moreover, carefully designed experiments are carried out after the numerical simulations. The experimental data also validate that the finding of the Fourier coefficients from the modulated signal is feasible. Thus, the dissipative energy of the nonlinear system damper can be computed from these Fourier coefficients of the modulated signal.

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