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A two-dimensional theory for the analysis of surface acoustic waves in finite elastic solids

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Abstract

The analysis of surface acoustic waves in elastic solids started from a semi-infinite isotropic elastic body with solutions and techniques dated back to Lord Rayleigh. These solutions have explained the surface acoustic wave phenomena and guided its engineering applications in many fields. Research work followed have been using the same semi-infinite model with solution techniques for approximate and exact results in both analytical and numerical manners for many application problems involving finite elastic solids. On the other hand, we have noticed that various two-dimensional theories, notably plate theory by Mindlin, have been derived to study the bulk wave propagation in finite elastic solids like plates and bars with satisfactory results. The lack of such a two-dimensional theory has made the analysis of surface acoustic wave propagation in various waveguides primitive and difficult because the precise solutions cannot be obtained through any analytical effort and numerical solutions are also difficult to obtain because of the complicated nature of problems. To meet the need of a simplified analytical method for surface acoustic waves in finite elastic solids, we start the derivation from the well-known three-dimensional solutions of semi-infinite elastic solids. Using exact solutions as the basis for the two-dimensional expansion, we found that the usual procedure of dimension reduction works perfectly in this case because the unknown wavenumber is eventually removed in the expansion process, and the depth of the solid is considered through integration constants that are exponentially decaying functions of the wavenumber, which is to be specified. In addition, we also found the two-dimensional equations give the exact phase velocity without corrections of any kind in the limiting case in which the solid is semi-infinite. With these equations, two-dimensional solutions of mechanical displacements and phase velocity can be obtained analytically from four coupled differential equations that are similar to Mindlin plate equations. We demonstrated the applications of the two-dimensional theory with the analysis of surface waves in isotropic elastic rectangular strips. The velocity and displacement results offer a chance to understand surface acoustic waves in finite solids and the presence of overtone modes and transition zones.

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1. Introduction

Wave propagation in elastic solids has a long and rich history in elasticity, and branches based on the nature of the waves, materials, applications, and other factors have been established in the course of research with many different theories, methods, and solutions [1–4]. The basic theory and equations are generally derived from infinite elastic solids, and we are also overwhelmed by the relatively simple, elegant, and useful equations that can illustrate the characteristics and essential concepts of wave propagation and provide solutions for various problems and applications. To apply the fundamental theory to practical problems arising from various engineering applications, creative efforts have resulted in many approximate methods and theories which can be used to study the wave propagation in finite elastic solids with equations of reduced dimension and reasonably accurate results. One of the most popular research fields is the bulk acoustic wave propagation in finite elastic plates, or the vibration of plates, which have been studied with various plate theories like the classical plate [5], Mindlin plate [6,7], Lee plate [8,9], and others in elasticity and related solution techniques [10]. The plate theories have been widely applied to many practical engineering problems like structural design and analysis, and there are extensive research efforts in improving these theories with various considerations of the static and dynamic behavior for specific applications. Indeed, the plate theories, methods, and solutions are considered as a model approach to engineering problems, especially for wave propagation in solids, and they can be expanded to cover problems that are different from bulk acoustic waves we are familiar with. Actually, such efforts can be easily recognized, if the research work on using the same techniques to problems of bars and beams by Mindlin and others [11,12] and some one-dimensional theories [13] are taking into account.

One of the less popular but equally important problems is the surface acoustic waves in elastic solids, which has been started by Load Rayleigh [1,2,14], and found applications in earthquake and lately electronic devices widely used for frequency generation and reference. The initial study on the surface acoustic waves is done with semi-infinite isotropic solids, and many later researchers, more or less, took the same approach and the results based on three-dimensional elasticity and piezoelectricity equations are employed to study the surface acoustic waves in solids and its unique nature and wave velocity. Practical applications of surface acoustic waves are concentrated on resonators by anisotropic piezoelectric materials for frequency control purpose and new devices like sensors and transducers [3,15–18], which are designed and built with simple analysis and empirical estimation until recently, when the accurate analysis and modeling are being emphasized for fast, accurate, and efficient design and performance improvement. Expectedly, the only available analytical method is the three-dimensional elasticity equations which cannot be applied to finite solids without employing numerical methods like the finite element method [19–24]. For other applications such as acoustic waves in layered structures, which are widely used as key elements of intelligent or smart structures, extensive efforts have been made through analytical but mostly numerical methods based with finite element techniques [25–31], as detailed by Liu and Xi [19], for solutions to aid the analysis and design. Not surprisingly, even the finite element method cannot handle the surface acoustic waves propagating in finite solids with complications like the presence of periodic electrodes, which can be neglected in the vibration analysis, efficiently, because of the high frequency and relatively large sizes in comparison to the wavelength. As a result, most of research work is done with one period of the structure for simplicity and fast results in numerical computation [32,33]. Some efforts on analytical solutions are done with oversimplified equations with the acceptable assumption that the relatively large sizes can justify the neglecting of width and depth variables in the equations [17,34–36]. Accuracy aside, we clearly see the lack of a two-dimensional theory for surface acoustic waves in finite solids similar to plate theory for bulk acoustic waves in finite plates has made it very difficult to have a sophisticated method and solutions for practical applications, thus posing an opportunity to develop such a theory to make the surface acoustic wave analysis in an equally simple and accurate manner. This idea has come to our research for a few years, and we found the solution to this challenging problem from the plate theories we are familiar with, like Mindlin [6,7] and Lee [8,9].

With the objective of developing a two-dimensional theory specifically for surface acoustic waves in finite elastic solids in mind, we turn to the two-dimensional theories [6–9] for bulk acoustic waves in elastic solids for directions based on the assumption that there must exist a way to expand the displacements for the surface acoustic waves in a manner similar to the high frequency vibrations of plates. Of course, we need to find the important basis functions so the two-dimensional expansion can be made properly to handle exponential decaying displacements in depth and the dispersive nature of surface acoustic waves in finite solids. This requirement, i.e. the best approximation of propagation along the depth direction, comes from a direct observation from plate theories because the essential consideration in the development of plate theories is the accurate representation of thickness deformation. This has been the starting point of the devise of plate theory, because both Mindlin and Lee plate theories [6–9] can handle the thickness deformation well, especially the thickness-shear displacements which are important in applications. The selection of power series and trigonometric functions is for the best representation of the thickness-shear deformation and the related vibration frequencies. It is not difficult to imagine that the right expansion function can only come from a close examination of surface acoustic wave solutions, just like in the plate theories the deformation of important vibrations modes are from a direct observation of the three-dimensional exact solutions, where the thickness-shear mode and its overtone modes are well represented by either a linear or trigonometric function of the thickness coordinate. This is a basic principle for the development of two-dimensional theories, and one of the closest applications of this beside the familiar plate theories is the eigenmode expansion method for the vibrations of crystal plates by Peach [37]. In this study, we also start from this basic principle and apply it to surface acoustic waves, which differ from plate vibration problems, or bulk acoustic waves, in some basic aspects but the derivation is actually very close.

Through solving three-dimensional elasticity equations specifically for surface acoustic waves by assuming displacements decay exponentially along the depth coordinate, we obtain the parameters related to exponential functions, or the decaying parameters, for all displacements. With these known parameters, we can obtain the velocity of surface acoustic waves in semi-infinite solids by applying the traction-free boundary conditions on the free surface. This is the well-known procedure for surface acoustic waves in semi-infinite solids, and this is our starting point for the two-dimensional theory derivation, which will utilize the decaying parameters to represent the exponentially decaying displacements in finite solids. It is natural to assume that in a finite solid, existing surface acoustic waves will also have displacements decaying along the depth, or thickness, coordinate exponentially, and the precise determination of parameters of these decaying functions are critical for the accurate representation of the nature of waves. It is apparent that these parameters cannot be decided analytically, because the determination of the displacement variation in a finite body is something we are trying to achieve by looking beyond the three-dimensional elasticity equations we are familiar with. On the other hand, the essence of Mindlin and Lee plate theories [6–9] are that the complicated deformation of a finite elastic body can be approximated by a series of functions that are close to the dominant modes we are interested in. Starting with this important idea, and knowing that the important expansion functions in plate theories are actually from the three-dimensional elasticity solutions of infinite elastic plates, we want to explore the possibility to use the decaying functions from a semi-infinite solid for the deformation in a finite one. In other words, we assume the displacement decaying in the depth or thickness direction of a finite solid is the same as they are in semi-infinite solids. Since there are two decaying parameters, or two exponential functions, in displacements of a semi-infinite solid, we can use the combination of the two functions in a manner similar to Mindlin plate theory, i.e. assuming the depth or thickness dependence of the displacement decaying is already known, and we need to follow the dimension reduction procedure used by Mindlin [6,7] for the derivation. As we know, the discrepancy of the true displacement variation in the depth or thickness direction should be compensated by the presence of integration constants and multiple equations, and possibly correction factors also, in plate theories. This approach and basic principle outlined here are closely related to Mindlin and Lee plate theories [6–9], and they are going to be adopted and followed in the derivation, thus establishing the foundation of the two-dimensional theory for surface acoustic waves in finite elastic solids. The solution method related to this two-dimensional theory will also be similar to those of the plate theories. This will be demonstrated in later sections as we carry out the derivation and related operations successively.

2. Surface acoustic waves in isotropic semi-infinite solids

Surface acoustic wave was discovered in a semi-infinite elastic solid with displacements in depth direction decaying exponentially along the depth coordinate, or the waves are confined in the surface of solids. The derivation of the basic equations for the wave propagation and their solutions can be found in many popular

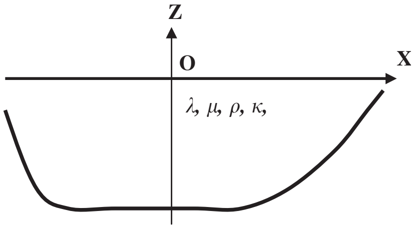

Fig. 1. A semi-infinite isotropic solid.

textbooks such as Achenbach [1] and Graff [2], and we present the results here briefly to enhance the description of the principle and method we shall use in this study.

For the surface acoustic wave solutions in a semi-infinite isotropic elastic solid, or a half-space, as shown in Fig. 1, the procedure demonstrated by Victorov [14] and Achenbach [1] assumes the three displacements are

$$u_1 = A\mathrm{e}^{k\beta z}\mathrm{e}^{\mathrm{i}k(x-ct)}, \quad u_2 = 0, \quad u_3 = B\mathrm{e}^{k\beta z}\mathrm{e}^{\mathrm{i}k(x-ct)}, \tag{1}$$

where $u_j(j = 1, 2, 3), A, B, \beta, k, c$ and t are displacements, amplitudes of displacements, decaying index, wavenumber, phase velocity, and time, respectively. It is important to note that we have the wavenumber k in the exponential term of z coordinate explicitly, highlighting the need to have the decaying parameter as an independent variable in the equations. This appearance is different from the textbooks [1,2] because we want to signify the importance of the parameters and the related physical insights.

With displacement solutions given in Eq. (1), we have the stress components of isotropic materials, in Lamé material constants and abbreviated notations, as

$$\begin{aligned} T_1 &= k[(\lambda + 2\mu)\mathrm{i}A + \lambda\beta B]\mathrm{e}^{k\beta z}\mathrm{e}^{\mathrm{i}k(x-ct)}, \\ T_3 &= k[\lambda\mathrm{i}A + (\lambda + 2\mu)\beta B]\mathrm{e}^{k\beta z}\mathrm{e}^{\mathrm{i}k(x-ct)}, \\ T_5 &= k\mu(\beta A + \mathrm{i}B)\mathrm{e}^{k\beta z}\mathrm{e}^{\mathrm{i}k(x-ct)}. \end{aligned} \tag{2}$$

The stress equations of motion, in this case with the disappearance of the transverse displacement, are

$$T_{1,1} + T_{5,3} = \rho\ddot{u}_1, \quad T_{5,1} + T_{3,3} = \rho\ddot{u}_3, \tag{3}$$

where ρ is the density of material.

Substituting Eq. (2) into Eq. (3), we can rewrite the equations to

$$\begin{aligned} &k^2\left[\beta^2 c_T^2 + \left(c^2 - c_L^2\right)\right]A + \mathrm{i}k^2\beta\left(c_L^2 - c_T^2\right)B = 0, \\ &\mathrm{i}k^2\beta\left(c_L^2 - c_T^2\right)A + k^2\left[\beta^2 c_L^2 + \left(c^2 - c_T^2\right)\right]B = 0, \end{aligned} \tag{4}$$

with

$$c_L^2 = \frac{\lambda + 2\mu}{\rho}, \quad c_T^2 = \frac{\mu}{\rho}, \tag{5}$$

defined as the longitudinal and transverse wave velocities of the material, respectively.

By solving Eq. (4) for the decaying index β, we have two neat and elegant solutions

$$\beta_1^2 = 1 - \frac{c^2}{c_L^2}, \quad \beta_2^2 = 1 - \frac{c^2}{c_T^2}, \tag{6}$$

and the corresponding amplitude ratios

$$\left(\frac{B}{A}\right)_1 = -\mathrm{i}\beta_1, \left(\frac{B}{A}\right)_2 = \frac{1}{\mathrm{i}\beta_2}. \tag{7}$$

As we already know, they represent the longitudinal and transverse modes of surface acoustic waves propagating in solids, respectively. The phase velocity c is to be determined by boundary conditions of the free surface.

Then the solutions of displacement and stress based on Eqs. (6) and (7) are:

$$\begin{aligned}
u_1 &= \left(A_1 e^{k\beta_1 z} + A_2 e^{k\beta_2 z}\right) e^{ik(x-ct)},\\
u_3 &= \left(-i\beta_1 A_1 e^{k\beta_1 z} + \frac{1}{i\beta_2} A_2 e^{k\beta_2 z}\right) e^{ik(x-ct)},\\
T_1 &= ik\left\{\left[(\lambda+2\mu) - \lambda\beta_1^2\right] A_1 e^{\beta_1 z} + 2\mu A_2 e^{k\beta_2 z}\right\} e^{ik(x-ct)},\\
T_3 &= ik\left\{\left[\lambda - (\lambda+2\mu)\beta_1^2\right] A_1 e^{\beta_1 z} - 2\mu A_2 e^{k\beta_2 z}\right\} e^{ik(x-ct)},\\
T_5 &= k\mu\left[2\beta_1 A_1 e^{k\beta_1 z} + \frac{1}{\beta_2}\left(\beta_2^2+1\right) A_2 e^{k\beta_2 z}\right] e^{ik(x-ct)},
\end{aligned} \tag{8}$$

where the amplitudes A_1 and A_2 are to be determined with boundary conditions.

Now applying the traction-free boundary conditions on the free surface

$$T_3 = T_5 = 0 \quad \text{for } z = 0, \tag{9}$$

with the stresses in Eq. (8), we have

$$\left[\lambda - (\lambda + 2\eta)\beta_1^2\right] A_1 - 2\mu A_2 = 0, \quad 2\beta_1 A_1 + \frac{1}{\beta_2}\left(\beta_2^2 + 1\right) A_2 = 0. \tag{10}$$

This basically gives the familiar surface acoustic wave phase velocity equation

$$\left(2 - \frac{c^2}{c_T^2}\right)^2 = 4\sqrt{\left(1 - \frac{c^2}{c_L^2}\right)\left(1 - \frac{c^2}{c_T^2}\right)}, \tag{11}$$

as the well-known Rayleigh surface waves [1,2,14].

It should be kept in mind that these solutions are for the exponentially decaying surface waves only. If the negative roots are taken into consideration also, there will be terms exponentially growing by depth, leading to solutions for infinite plates of finite thickness. It should be emphasized that there are only two displacement modes with thickness dependence in surface acoustic waves in an isotropic semi-infinite solid.

The solutions, notably the decaying parameters given in Eq. (6) and the phase velocity in Eq. (11), are the important results we are going to use in the development of our two-dimensional theory in different ways. The decaying parameters, which define the exponentially decaying displacements along the depth coordinate, are going to be used for the displacement representation in an expansion manner as in plate theories by Mindlin [6,7], Lee [8,9], and Peach [37]. We take this approach because comparing with other possible expansion schemes we think the solutions from three-dimensional solutions of semi-infinite solids are the best and closest for the approximation of displacements in the thickness direction of a finite solid. In other words, the decaying parameters and the corresponding exponential functions will be chosen as basis functions of the expansion in next section. The exact velocity in Eq. (11) will be used as a reference to validate the resulting two-dimensional equations.

3. Two-dimensional expansion of displacements

With the principle for the derivation of the two-dimensional theory outlined in detail and the three-dimensional solutions of surface acoustic waves in a semi-infinite elastic solid presented in the previous section, we can easily go to next step, which is to expand the displacements in a finite elastic solid with known decaying functions from the same material. Since we have decaying functions as two exponential functions associated with decaying indices β_1 and β_2 in Eq. (6), the displacements now are

$$u_j(x,y,z,t) = \sum_{n=1}^{2} u_j^{(n)}(x,y,t)\, e^{k\beta_n z}, \quad j = 1,2,3, \tag{12}$$

where $u_j^{(n)}$ are the higher-order displacement components which are functions of plane coordinates and time only. This is to say that we expand the displacements in the z-direction with solutions in a half-space in a manner similar to two-dimensional plate theories with characteristic expansion functions for deformation in the thickness direction based on solutions of infinite plates. Since our exponential functions are from semi-infinite solids, this approximation is valid for cases the solid is semi-infinite or the thickness is large. In case the solid is anisotropic, there will be more expansion modes [38]. For complications such as anisotropic and piezoelectric materials and thin metal layers, similar two-dimensional equations have been established and utilized [39,40].

Actually from Eq. (12), the expansion functions

$$\varphi_1 = \mathrm{e}^{\sqrt{1-(c^2/c_L^2)}kz}, \quad \varphi_2 = \mathrm{e}^{\sqrt{1-(c^2/c_T^2)}kz}, \tag{13}$$

with known surface wave velocity c to be decided, represent two strongly coupled components of surface acoustic wave variation functions present in the half-space. It is conceivable that the solutions in a finite body can also be best represented by the combination of these two solutions, consistent with the observation that there are two modes in Rayleigh waves. For finite solids, it is a fact that there are many wave modes due to interactions on the surfaces, but it is also reasonable to assume that there will be two dominant modes in the surface acoustic waves, and they are represented by Eq. (13). In addition, the exponential functions are valid for large thickness. If the thickness is finite, there will be exponentially growing modes to be included also [41].

With known displacements in Eq. (12), the standard procedure for the derivation of two-dimensional plate theories by Mindlin [6,7], Lee [8,9], and others [10] includes the variational equations of motion of elasticity

$$\int_{-h}^{0} \mathrm{d}z \int_A \left(T_{ij,i} - \rho \ddot{u}_j\right)\delta u_j \,\mathrm{d}A = 0, \quad i,j = 1,2,3, \tag{14}$$

where h is the depth of the solid and A is the surface plane, as illustrated in Fig. 2. By substituting Eq. (12) into Eq. (14) and carry out the integration, we have the two-dimensional equations of motion of the elastic solid as

$$\sum_{n=1}^{2} \int_A \left(T_{ij,i}^{(n)} - k\beta_n T_{3j}^{(n)} + F_j^{(n)} - \rho \sum_{m=1}^{2} A_{mn} \ddot{u}_j^{(m)} \right) \delta u_j^{(n)} \,\mathrm{d}A = 0, \quad i,j = 1,2,3. \tag{15}$$

The two-dimensional equations and quantities appeared in Eq. (15) are

$$\begin{aligned} &T_{ij,i}^{(n)} - k\beta_n T_{3j}^{(n)} + F_j^{(n)} = \rho \sum_{m=1}^{2} A_{mn} \ddot{u}_j^{(m)}, \quad i,j = 1,2,3, \\ &T_{ij}^{(n)} = \int_{-h}^{0} T_{ij} \mathrm{e}^{k\beta_n z} \,\mathrm{d}z, \quad F_j^{(n)} = T_{3j}(0) - \mathrm{e}^{-k\beta_n h} T_{3j}(-h), \\ &A_{mn} = \frac{1}{k(\beta_m + \beta_n)} \left[1 - \frac{1}{\mathrm{e}^{(\beta_m+\beta_n)kh}} \right], \quad m,n = 1,2. \end{aligned} \tag{16}$$

Not surprisingly, we find the equations and the quantities in Eq. (16) are very close to similar systems from both Mindlin [6,7] and Lee [8,9] plate theories except some specific terms related to the basis functions, which are exponential in this study, such as the higher-order stress $T_{ij}^{(n)}$, face-traction $F_j^{(n)}$, and integration constants A_{mn}.

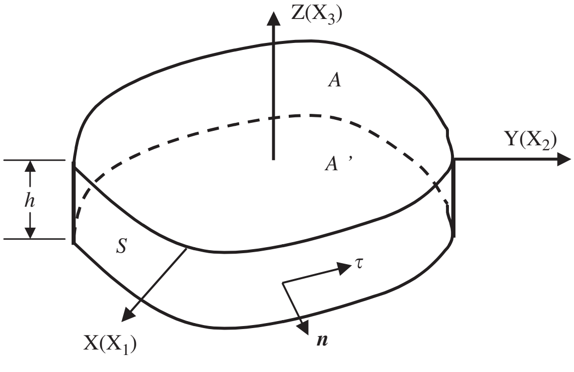

Fig. 2. An elastic plate configuration.

This is expected because the derivation process is based on the same variational equations of motion, which is the foundation for plate theories also. The simplicity of basis functions actually makes the two-dimensional equations and quantities having simpler appearances for easy interpretation of the physics insight and further derivation. It is important to note that the integration constants are dependent on wavenumber k, a phenomena which is new in comparison to plate theories of Mindlin and Lee [6–9]. Of course, this is directly linked to the appearance of wavenumber in basis functions, and it is vital to explore the way to make the integration constants explicit to facilitate applications of the equations.

Now for given depth h, from Eq. (16) we have

$$A_{mn} = \frac{1}{k(\beta_m + \beta_n)}\left[1 - \frac{1}{e^{(\beta_m+\beta_n)kh}}\right] = \frac{1}{k(\beta_m + \beta_n)}\left[1 - \frac{1}{e^{2\pi(\beta_m+\beta_n)H}}\right], \quad m,n = 1,2. \tag{17}$$

Note we use the relationship

$$H = \frac{h}{\zeta}, \quad k\zeta = 2\pi, \tag{18}$$

here to simplify Eq. (17). In any case, if the wavelength ζ or wavenumber k is given, we can evaluate the integration constants with the relationship in Eq. (17). As H is large enough or even approaches to infinity, A_{mn} will be constants independent of the depth ratio as

$$A_{mn} = \frac{1}{k(\beta_m + \beta_n)}, \quad m,n = 1,2, \tag{19}$$

with a simple relationship of wavenumber k and decaying parameters $\beta_n (n = 1,2)$. It is also important to note that the wavenumber will no longer appear as an unknown when the larger depth of solid is specified to neglect the exponential terms. Also the explicit appearance of wavenumber k in the denominator is significant because this is the key step in the elimination of the wavenumber in the two-dimensional equations. We also need to emphasize again that the thickness of the substrate will be an important parameter in the equations. Since we know that surface waves diminish essentially beyond three wavelengths, we can safely predict that such two-dimensional equations will be a good approximation for problems in which the substrate thickness is larger than three wavelengths.

With the two-dimensional equations and quantities derived and defined, now we return to Eq. (12) for strain components in two-dimensional variables, like we have seen in the plate theories. By following the definition of strain tensor in elasticity, we have

$$S_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right) = \sum_{n=1}^{2} e^{k\beta_n z} S_{ij}^{(n)},$$
$$S_{ij}^{(n)} = \frac{1}{2}\left[\left(u_{i,j}^{(n)} + u_{j,i}^{(n)}\right) + k\beta_n\left(\delta_{3j}u_i^{(n)} + \delta_{3i}u_j^{(n)}\right)\right], \quad i,j = 1,2,3, \quad n = 1,2. \tag{20}$$

By substituting Eq. (20) into Eq. (16), we have the higher-order stress components in two-dimensional variables and abbreviated notation for general anisotropic materials as

$$T_p^{(n)} = \int_{-h}^{0} c_{pq} S_q e^{k\beta_n z}\, dz = \sum_{m=1}^{2} A_{mn} c_{pq} S_q^{(m)}, \quad p,q = 1,2,3,4,5,6, \quad n,m = 1,2, \tag{21}$$

where c_{pq} are elastic constants of a general anisotropic solid.

Through expansion, this will be

$$T_p^{(n)} = \sum_{m=1}^{2} A_{mn}\left[c_{p1}u_{1,1}^{(m)} + c_{p2}u_{2,2}^{(m)} + k\beta_m c_{p3}u_3^{(m)} + c_{p4}\left(u_{3,2}^{(m)} + k\beta_m u_2^{(m)}\right) + c_{p5}\left(u_{3,1}^{(m)} + k\beta_m u_1^{(m)}\right) + c_{p6}\left(u_{2,1}^{(m)} + u_{1,2}^{(m)}\right)\right]. \tag{22}$$

It is important to note that the integration constants do not disappear for a particular order of equations, thus leading to the strong coupling of terms. Fortunately, the expansion is done only for two terms, so the coupling do not introduce the unlimited number of terms which require a truncation procedure as we have seen in plate theories [6–9]. On the other hand, the cancellation of some terms due to the disappearance of integration constants as an advantage of decoupling in plate theories cannot be utilized here either. What makes this different from plate theories is that there are only two basis functions, and the variables are strongly coupled due to the properties of expansion functions, implying there is no need to employ a truncation procedure in the system.

4. Two-dimensional equations

We now have the framework of the two-dimensional theory for surface acoustic waves in finite elastic solids, resembling the popular plate theories by Mindlin, Lee, and others [6–10]. These equations, with two sets of strongly coupled two-dimensional displacement variables, are complete for the surface acoustic waves propagating in finite elastic solids with large thickness. It should be pointed out that we have not made any assumption on material properties, thus implying its applicability to anisotropic materials we can think of, as we have demonstrated in Eqs. (21) and (22). To make these equations easy to understand and demonstrate the fundamentals of the two-dimensional theory, we need to recapitulate them in a systematic manner. Besides, these equations need to be examined for their accuracy in comparison with the three-dimensional theory of elasticity.

For these reasons, we shall derive the complete equations with isotropic materials for simplicity and make comparison with three-dimensional solutions. A formal expansion to anisotropic material is implied, and a particular derivation has been devoted to certain cuts of quartz crystal for their applications in surface acoustic wave resonator modeling by Wang and Hashimoto [38].

4.1. Strain–displacement relations

The strain-displacement relations in the two-dimensional formulation of elasticity equations with exponential expansion functions are given in Eq. (20), and it is important to note that the decaying parameters $\beta_n (n = 1, 2)$ appearing in the equations with the wavenumber together. This differs from the plate theories significantly, because in those equations the expansion is done with functions of constant parameters. However, as we shall demonstrate in derivations followed, these variables actually do not complicate the equations, because we have explained the automatic cancellation of the wavenumber in two-dimensional variables and equations for large thickness of the plate earlier.

Using the definitions of strains in Eq. (20), we can write out all of the components, again in abbreviated notation, as

$$\begin{aligned}
&S_1^{(1)} = u_{1,1}^{(1)}, \quad S_2^{(1)} = u_{2,2}^{(1)}, \quad S_3^{(1)} = k\beta_1 u_3^{(1)},\\
&S_4^{(1)} = u_{3,2}^{(1)} + k\beta_1 u_2^{(1)}, \quad S_5^{(1)} = u_{3,1}^{(1)} + k\beta_1 u_1^{(1)}, \quad S_6^{(1)} = u_{1,2}^{(1)} + u_{2,1}^{(1)},\\
&S_1^{(2)} = u_{1,1}^{(2)}, \quad S_2^{(2)} = u_{2,2}^{(2)}, \quad S_3^{(2)} = k\beta_2 u_3^{(2)},\\
&S_4^{(2)} = u_{3,2}^{(2)} + k\beta_2 u_2^{(2)}, \quad S_5^{(2)} = u_{3,1}^{(2)} + k\beta_2 u_1^{(2)}, \quad S_6^{(2)} = u_{1,2}^{(2)} + u_{2,1}^{(2)}.
\end{aligned} \tag{23}$$

It is clear that for strains in Eq. (23) that only displacements of the same order appear, which is another important feature concerning the coupling of terms in comparison with popular plate theories. This will certainly simplify the cross-term coupling of the variables in equations significantly. Of course, as we stated before, only two sets of variables are closely coupled, which does not have serious complicating effect on the equations in comparison with Mindlin and Lee plate theories [6–9]. This is also true regarding to other equations, like the stresses, in this study.

4.2. Stress–displacement equations

We have presented the two-dimensional stresses in strain components in Eqs. (21) and (22) for anisotropic materials, and the following formulation will be done for isotropic materials. The actual equations for isotropic materials are simple and straightforward with the stress equations in Eqs. (21) and (22) and the two-dimensional strain components in Eq. (23). With Lamé elastic constants, we present the following two-dimensional stress components in displacements as

$$
\begin{aligned}
T_1^{(1)} &= A_{11}\left[(\lambda+2\mu)u_{1,1}^{(1)}+\lambda u_{2,2}^{(1)}+\lambda k\beta_1 u_3^{(1)}\right]+A_{21}\left[(\lambda+2\mu)u_{1,1}^{(2)}+\lambda u_{2,2}^{(2)}+\lambda k\beta_2 u_3^{(2)}\right],\\
T_2^{(1)} &= A_{11}\left[\lambda u_{1,1}^{(1)}+(\lambda+2\mu)u_{2,2}^{(1)}+\lambda k\beta_1 u_3^{(1)}\right]+A_{21}\left[\lambda u_{1,1}^{(2)}+(\lambda+2\mu)u_{2,2}^{(2)}+\lambda k\beta_2 u_3^{(2)}\right],\\
T_3^{(1)} &= A_{11}\left[\lambda u_{1,1}^{(1)}+\lambda u_{2,2}^{(1)}+(\lambda+2\mu)k\beta_1 u_3^{(1)}\right]+A_{21}\left[\lambda u_{1,1}^{(2)}+\lambda u_{2,2}^{(2)}+(\lambda+2\mu)k\beta_2 u_3^{(2)}\right],\\
T_4^{(1)} &= A_{11}\mu\left(u_{3,2}^{(1)}+k\beta_1 u_2^{(1)}\right)+A_{21}\mu\left(u_{3,2}^{(2)}+k\beta_2 u_2^{(2)}\right),\\
T_5^{(1)} &= A_{11}\mu\left(u_{3,1}^{(1)}+k\beta_1 u_1^{(1)}\right)+A_{21}\mu\left(u_{3,1}^{(2)}+k\beta_2 u_1^{(2)}\right),\\
T_6^{(1)} &= A_{11}\mu\left(u_{1,2}^{(1)}+u_{2,1}^{(1)}\right)+A_{21}\mu\left(u_{1,2}^{(2)}+u_{2,1}^{(2)}\right),\\
T_1^{(2)} &= A_{12}\left[(\lambda+2\mu)u_{1,1}^{(1)}+\lambda u_{2,2}^{(1)}+\lambda k\beta_1 u_3^{(1)}\right]+A_{22}\left[(\lambda+2\mu)u_{1,1}^{(2)}+\lambda u_{2,2}^{(2)}+\lambda k\beta_2 u_3^{(2)}\right],\\
T_2^{(2)} &= A_{12}\left[\lambda u_{1,1}^{(1)}+(\lambda+2\mu)u_{2,2}^{(1)}+\lambda k\beta_1 u_3^{(1)}\right]+A_{22}\left[\lambda u_{1,1}^{(2)}+(\lambda+2\mu)u_{2,2}^{(2)}+\lambda k\beta_2 u_3^{(2)}\right],\\
T_3^{(2)} &= A_{12}\left[\lambda u_{1,1}^{(1)}+\lambda u_{2,2}^{(1)}+(\lambda+2\mu)k\beta_1 u_3^{(1)}\right]+A_{22}\left[\lambda u_{1,1}^{(2)}+\lambda u_{2,2}^{(2)}+(\lambda+2\mu)k\beta_2 u_3^{(2)}\right],\\
T_4^{(2)} &= A_{12}\mu\left(u_{3,2}^{(1)}+k\beta_1 u_2^{(1)}\right)+A_{22}\mu\left(u_{3,2}^{(2)}+k\beta_2 u_2^{(2)}\right),\\
T_5^{(2)} &= A_{12}\mu\left(u_{3,1}^{(1)}+k\beta_1 u_1^{(1)}\right)+A_{22}\mu\left(u_{3,1}^{(2)}+k\beta_2 u_1^{(2)}\right),\\
T_6^{(2)} &= A_{12}\mu\left(u_{1,2}^{(1)}+u_{2,1}^{(1)}\right)+A_{22}\mu\left(u_{1,2}^{(2)}+u_{2,1}^{(2)}\right).
\end{aligned}
\tag{24}
$$

Again, as in the strain equations in Eq. (23), strong coupling of the two sets of two-dimensional displacement components are exhibited. With Eq. (24), now it is clear that the stress equations in two-dimensional displacement components will also be strongly coupled in a consistent manner.

4.3. Stress equations of motion

The successive sets of two-dimensional stress equations of motion in Eq. (16) can be expanded into six equations as

$$
\begin{aligned}
T_{1,1}^{(1)}+T_{6,2}^{(1)}-k\beta_1 T_5^{(1)}+F_1^{(1)} &= A_{11}\rho\ddot{u}_1^{(1)}+A_{21}\rho\ddot{u}_1^{(2)},\\
T_{6,1}^{(1)}+T_{2,2}^{(1)}-k\beta_1 T_4^{(1)}+F_2^{(1)} &= A_{11}\rho\ddot{u}_2^{(1)}+A_{21}\rho\ddot{u}_2^{(2)},\\
T_{5,1}^{(1)}+T_{4,2}^{(1)}-k\beta_1 T_3^{(1)}+F_3^{(1)} &= A_{11}\rho\ddot{u}_3^{(1)}+A_{21}\rho\ddot{u}_3^{(2)},\\
T_{1,1}^{(2)}+T_{6,2}^{(2)}-k\beta_2 T_5^{(2)}+F_1^{(2)} &= A_{12}\rho\ddot{u}_1^{(1)}+A_{22}\rho\ddot{u}_1^{(2)},\\
T_{6,1}^{(2)}+T_{2,2}^{(2)}-k\beta_2 T_4^{(2)}+F_2^{(2)} &= A_{12}\rho\ddot{u}_2^{(1)}+A_{22}\rho\ddot{u}_2^{(2)},\\
T_{5,1}^{(2)}+T_{4,2}^{(2)}-k\beta_2 T_3^{(2)}+F_3^{(2)} &= A_{12}\rho\ddot{u}_3^{(1)}+A_{22}\rho\ddot{u}_3^{(2)}.
\end{aligned}
\tag{25}
$$

These equations are very close to those of plate theories except the strong coupling of two sets of variables in both two-dimensional stress and inertia terms. It should be emphasized that in the cases we are interested such as the determination of surface acoustic wave velocity in finite elastic solids, the face-traction $F_j^{(n)}(j=1,2,3;n=1,2)$ are not considered for the free upper and lower faces. However, if the excitation forces on the surface through metal electrodes in resonators are to be considered, the face-traction terms are directly related to the alternative current and they are the input source for the forced vibrations. We leave these discussions to the piezoelectric materials and electrodes to be studied in the future.

4.4. Displacement equations of motion

With stress components in displacements in Eq. (24) and equations of motion in Eq. (25), a simple substitution will give the following stress equations of motion in displacements

$$\begin{aligned}
&A_{11}(\lambda+2\mu)u^{(1)}_{1,11}+A_{11}\mu u^{(1)}_{1,22}-A_{11}\mu k^2\beta_1^2u^{(1)}_1+A_{11}(\lambda+\mu)u^{(1)}_{2,21}+A_{11}(\lambda-\mu)k\beta_1u^{(1)}_{3,1}\\
&\quad+A_{21}(\lambda+2\mu)u^{(2)}_{1,11}+A_{21}\mu u^{(2)}_{1,22}-A_{21}\mu k^2\beta_1\beta_2u^{(2)}_1+A_{21}(\lambda+\mu)u^{(2)}_{2,21}+A_{21}\left(\lambda\beta_2-\mu\beta_1\right)ku^{(2)}_{3,1}+F^{(1)}_1\\
&=A_{11}\rho\ddot{u}^{(1)}_1+A_{21}\rho\ddot{u}^{(2)}_1,\\
&A_{11}\mu u^{(1)}_{2,11}+A_{11}(\lambda+\mu)u^{(1)}_{1,12}+A_{11}(\lambda+2\mu)u^{(1)}_{2,22}+A_{11}(\lambda-\mu)k\beta_1u^{(1)}_{3,2}-A_{11}\mu k^2\beta_1^2u^{(1)}_2\\
&\quad+A_{21}\mu u^{(2)}_{2,11}+A_{21}(\lambda+\mu)u^{(2)}_{1,12}+A_{21}(\lambda+2\mu)u^{(2)}_{2,22}+A_{21}\left(\lambda\beta_2-\mu\beta_1\right)ku^{(2)}_{3,2}-A_{21}\mu k^2\beta_1\beta_2u^{(2)}_2+F^{(1)}_2\\
&=A_{11}\rho\ddot{u}^{(1)}_2+A_{21}\rho\ddot{u}^{(2)}_2,\\
&A_{11}\mu u^{(1)}_{3,11}+A_{11}\mu u^{(1)}_{3,22}-A_{11}(\lambda-\mu)k\beta_1u^{(1)}_{1,1}-A_{11}(\lambda-\mu)k\beta_1u^{(1)}_{2,2}-A_{11}(\lambda+2\mu)k^2\beta_1^2u^{(1)}_3\\
&\quad+A_{21}\mu u^{(2)}_{3,11}+A_{21}\mu u^{(2)}_{3,22}-A_{21}\left(\lambda\beta_1-\mu\beta_2\right)ku^{(2)}_{1,1}-A_{21}\left(\lambda\beta_1-\mu\beta_2\right)ku^{(2)}_{2,2}-A_{21}(\lambda+2\mu)k^2\beta_1\beta_2u^{(2)}_3+F^{(1)}_3\\
&=A_{11}\rho\ddot{u}^{(1)}_3+A_{21}\rho\ddot{u}^{(2)}_3,\\
&A_{12}(\lambda+2\mu)u^{(1)}_{1,11}+A_{12}\mu u^{(1)}_{1,22}-A_{12}\mu k^2\beta_1\beta_2u^{(1)}_1+A_{12}(\lambda+\mu)u^{(1)}_{2,21}+A_{12}\left(\lambda\beta_1-\mu\beta_2\right)ku^{(1)}_{3,1}\\
&\quad+A_{22}(\lambda+2\mu)u^{(2)}_{1,11}+A_{22}\mu u^{(2)}_{1,22}-A_{22}\mu k^2\beta_2^2u^{(2)}_1+A_{22}(\lambda+\mu)u^{(2)}_{2,21}+A_{22}(\lambda-\mu)k\beta_2u^{(2)}_{3,1}+F^{(2)}_1\\
&=A_{12}\rho\ddot{u}^{(1)}_1+A_{22}\rho\ddot{u}^{(2)}_1,\\
&A_{12}\mu u^{(1)}_{2,11}+A_{12}(\lambda+\mu)u^{(1)}_{1,12}+A_{12}(\lambda+2\mu)u^{(1)}_{2,22}+A_{12}\left(\lambda\beta_1-\mu\beta_2\right)ku^{(1)}_{3,2}-A_{12}\mu k^2\beta_1\beta_2u^{(1)}_2\\
&\quad+A_{22}\mu u^{(2)}_{2,11}+A_{22}(\lambda+\mu)u^{(2)}_{1,12}+A_{22}(\lambda+2\mu)u^{(2)}_{2,22}+A_{22}(\lambda-\mu)k\beta_2u^{(2)}_{3,2}-A_{22}\mu k^2\beta_2^2u^{(2)}_2+F^{(2)}_2\\
&=A_{12}\rho\ddot{u}^{(1)}_2+A_{22}\rho\ddot{u}^{(2)}_2,\\
&A_{12}\mu u^{(1)}_{3,11}+A_{12}\mu u^{(1)}_{3,22}-A_{12}\left(\lambda\beta_2-\mu\beta_1\right)ku^{(1)}_{1,1}-A_{12}\left(\lambda\beta_2-\mu\beta_1\right)ku^{(1)}_{2,2}-A_{12}(\lambda+2\mu)k^2\beta_1\beta_2u^{(1)}_3\\
&\quad+A_{22}\mu u^{(2)}_{3,11}+A_{22}\mu u^{(2)}_{3,22}-A_{22}(\lambda-\mu)k\beta_2u^{(2)}_{1,1}-A_{22}(\lambda-\mu)k\beta_2u^{(2)}_{2,2}-A_{22}(\lambda+2\mu)k^2\beta_2^2u^{(2)}_3+F^{(2)}_3\\
&=A_{12}\rho\ddot{u}^{(1)}_3+A_{22}\rho\ddot{u}^{(2)}_3.
\end{aligned}\tag{26}$$

These equations are the complete set of equations of the coupled two-dimensional displacement components. All the equations in this section will be complete in comparison with the equations of elasticity except all the variables are two-dimensional, just as in popular plate theories. For validation, as we can easily consider, we need to check some simple results for the accuracy in some particular modes related to the surface acoustic wave propagation.

4.5. Boundary conditions

With the establishment of two-dimensional equations, we now need to have the related boundary conditions for these equations and the two-dimensional variables. Similar to the plate theories, the derivation of the boundary conditions is also straightforward with the basic equations from the three-dimensional theory of elasticity. To derive the boundary conditions related to the two-dimensional equations, we return to the geometric configuration of a solid in Fig. 2. We start with the familiar boundary conditions on the cylindrical surface

$$\int_{-h}^{0}\int_C n_iT_{ij}\delta u_j\,\mathrm{d}z\,\mathrm{d}s=\int_{-h}^{0}\int_C n_iT_{ij}\sum_{n=1}^{2}\delta u_j^{(n)}\mathrm{e}^{k\beta_n z}\,\mathrm{d}z\,\mathrm{d}s=\sum_{n=1}^{2}\int_C n_iT_{ij}^{(n)}\delta u_j^{(n)}\,\mathrm{d}s=0,\quad n=1,2;\quad i,j=1,2,3,\tag{27}$$

where C is the border curve of the faces, with thickness coordinate representing the cylindrical surface shown in Fig. 2.

Now it is clear from Eq. (27) that we need to specify either the traction terms $T_{ij}^{(n)}(n=1,2)$ or the displacement terms $u_j^{(n)}$ on the cylindrical surface. The boundary conditions of the upper- and lower-face have already been incorporated into the equations of motion through Eq. (16). Again, the boundary conditions are also similar with Mindlin and Lee plate theories [6–9].

5. Straight-crested wave solutions

The two-dimensional equations in the previous section have established the foundation of the two-dimensional theory for the analysis of surface acoustic waves in finite elastic solids. It is clear that this theory offers an opportunity for the analysis that cannot be matched by other available approximations and methods, thus providing a chance for more accurate results in a much efficient manner, as demonstrated in the finite element analysis of bulk acoustic wave resonators with the Mindlin plate theory [42,43]. This, of course, is the sole objective of this study.

Customarily, we start with the straight-crested solutions from the two-dimensional theory for analytical solutions for cases where the effect of the width can be neglected. In addition, solutions from the straight-crested waves can be used to make certain validations and corrections in limiting cases, as we have been familiar with two-dimensional plate theories. This is certainly an important concern in our study, but so far we have not touched the accuracy issue, although the importance cannot be over stated, yet.

We now examine the coupled straight-crested waves $u_1^{(1)}, u_1^{(2)}, u_3^{(1)}$ and $u_3^{(2)}$, traveling along the x direction. As in the semi-infinite case in Section 2, we only consider the in-plane displacements. For simplicity, and the solutions are assumed as

$$\begin{aligned} u_1^{(1)} &= A_1 \sin k\bar{k}x \exp(ikct), \quad u_3^{(1)} = A_2 \cos k\bar{k}x \exp(ikct), \\ u_1^{(2)} &= A_3 \sin k\bar{k}x \exp(ikct), \quad u_3^{(2)} = A_4 \cos k\bar{k}x \exp(ikct), \end{aligned} \tag{28}$$

where A_i $(i=1,2,3,4), k, \bar{k}, c,$ and t, are amplitudes, wavenumber, associated wavenumber, velocity, and time, respectively. The displacements $u_2^{(n)}(n=1,2)$ are not coupled with others, so we do not consider them here. Note the usual circular frequency in the time factor has been replaced with kc here.

It should be emphasized that the associated wavenumbers $\bar{k}$ only appear in the trigonometric function arguments of x, not the time factor, thus only affecting the spatial variations of these displacements. This is anticipated because the solutions in the depth or the thickness direction are almost known already with the exponential functions. The allowed variations in the displacements are reflected in the amplitudes, which are constants, and the wavenumbers, which can be complex, to accommodate the complicated mode shapes. These extra variables are the results of the two-dimensional theory and the extra equations related to the appearance of the higher-order terms. On the other hand, the solutions of this form will not change the velocity of wave propagation in the solids directly because the essential relationship between wavenumber and frequency is preserved as before.

Now we substitute Eq. (28) into Eq. (27) and drop the face-traction terms for free vibrations with traction-free faces, the four coupled equations of motion can be simplified to

$$\begin{aligned} & A_{11}\left[(\lambda+2\mu)\bar{k}^2+\mu\beta_1^2-c^2\rho\right]A_1+A_{11}\beta_1(\lambda-\mu)\bar{k}A_2+A_{21}\left[(\lambda+2\mu)\bar{k}^2+\mu\beta_1\beta_2-c^2\rho\right] \\ & \quad \times A_3+A_{21}\left(\lambda\beta_2-\mu\beta_1\right)\bar{k}A_4=0, \\ & A_{11}\beta_1(\lambda-\mu)\bar{k}A_1+A_{11}\left[\mu\bar{k}^2+(\lambda+2\mu)\beta_1^2-c^2\rho\right]A_2+A_{21}\left(\lambda\beta_1-\mu\beta_2\right)\bar{k}A_3 \\ & \quad +A_{21}\left[\mu\bar{k}^2+(\lambda+2\mu)\beta_1\beta_2-c^2\rho\right]A_4=0, \\ & A_{12}\left[(\lambda+2\mu)\bar{k}^2+\mu\beta_1\beta_2-c^2\rho\right]A_1+A_{12}\left(\lambda\beta_1-\mu\beta_2\right)\bar{k}A_2+A_{22}\left[(\lambda+2\mu)\bar{k}^2+\mu\beta_2^2-c^2\rho\right] \\ & \quad \times A_3+A_{22}\beta_2(\lambda-\mu)\bar{k}A_4=0, \\ & A_{12}\left(\lambda\beta_2-\mu\beta_1\right)\bar{k}A_1+A_{12}\left[\mu\bar{k}^2+(\lambda+2\mu)\beta_1\beta_2-c^2\rho\right]A_2+A_{22}\beta_2(\lambda-\mu)\bar{k}A_3 \\ & \quad +A_{22}\left[\mu\bar{k}^2+(\lambda+2\mu)\beta_2^2-c^2\rho\right]A_4=0. \end{aligned} \tag{29}$$

In matrix notation, Eq. (29) can be rewritten as

$$\begin{bmatrix} A_{11}\left[(\lambda+2\mu)\bar{k}^2+\mu\beta_1^2-c^2\rho\right] & A_{11}\beta_1(\lambda-\mu)\bar{k} & A_{21}\left[(\lambda+2\mu)\bar{k}^2+\mu\beta_1\beta_2-c^2\rho\right] & A_{21}(\lambda\beta_2-\mu\beta_1)\bar{k} \\ A_{11}\beta_1(\lambda-\mu)\bar{k} & A_{11}\left[\mu\bar{k}^2+(\lambda+2\mu)\beta_1^2-c^2\rho\right] & A_{21}(\lambda\beta_1-\mu\beta_2)\bar{k} & A_{21}\left[\mu\bar{k}^2+(\lambda+2\mu)\beta_1\beta_2-c^2\rho\right] \\ A_{12}\left[(\lambda+2\mu)\bar{k}^2+\mu\beta_1\beta_2-c^2\rho\right] & A_{12}(\lambda\beta_1-\mu\beta_2)\bar{k} & A_{22}\left[(\lambda+2\mu)\bar{k}^2+\mu\beta_2^2-c^2\rho\right] & A_{22}\beta_2(\lambda-\mu)\bar{k} \\ A_{12}(\lambda\beta_2-\mu\beta_1)\bar{k} & A_{21}\left[\mu\bar{k}^2+(\lambda+2\mu)\beta_1\beta_2-c^2\rho\right] & A_{22}\beta_2(\lambda-\mu)\bar{k} & A_{22}\left[\mu\bar{k}^2+(\lambda+2\mu)\beta_2^2-c^2\rho\right] \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{bmatrix} = 0. \tag{30}$$

Now we consider a special case with Eq. (30). If the thickness of the solid h goes to infinity, it represents the surface acoustic waves in the semi-infinite elastic solid we have studied before. If the theory represented by Eq. (30) is accurate, the solution of wave velocity c should be the precise result from the three-dimensional solution in Eq. (11). To verify this, we set the associated wavenumber $\bar{k}$ to 1, and using the integration constants given in Eq. (19) with Eq. (30). Since boundary conditions on the free surface are already incorporated into the equations in Eq. (30), we now have a case that the two-dimensional theory should confirm the three-dimensional results, otherwise we have to correct the equations systematically. Not surprisingly, through solving Eq. (30) for the wave velocity c, we find it is indeed the precise wave velocity of the semi-infinite elastic solids, proving that the two-dimensional theory can handle the limiting case with accuracy. As a result, additional efforts in making necessary corrections are no longer needed. This is certainly important to this study, because the equations without correction factors are more elegant and simple. For surface acoustic waves, there are no specific vibration modes in comparison with plates, so the accuracy can only be checked with this limiting case alone.

With Eqs. (28) and (30), we can solve for the associated wavenumbers for given material with known decaying parameters, thus having the stresses as defined in Eq. (24). Since we only have displacement components of two vibration modes, with time factor omitted, the associated wavenumbers from Eq. (30) change Eq. (28) to

$$\begin{aligned} u_1^{(1)} &= \sum_{i=1}^{4} A_{4i}\alpha_{1i}\sin(k\bar{k}_i x), \quad u_3^{(1)} = \sum_{i=1}^{4} A_{4i}\alpha_{2i}\cos(k\bar{k}_i x), \\ u_1^{(2)} &= \sum_{i=1}^{4} A_{4i}\alpha_{3i}\sin(k\bar{k}_i x), \quad u_3^{(2)} = \sum_{i=1}^{4} A_{4i}\cos(k\bar{k}_i x), \end{aligned} \tag{31}$$

where the amplitude ratios are defined as

$$\alpha_{ij} = \frac{A_{ij}(\bar{k}_i)}{A_{4j}(\bar{k}_i)}, \quad i = 1, 2, 3, \quad j = 1, 2, 3, 4. \tag{32}$$

Consequently, the stress components defined in Eq. (24) will be

$$\begin{aligned} T_1^{(1)} &= \sum_{i=1}^{4} A_{4i}k\left[(\lambda+2\mu)(A_{11}\alpha_{1i}+A_{21}\alpha_{3i})\bar{k}_i+\lambda\left(A_{11}\beta_1\alpha_{2i}+A_{21}\beta_2\right)\right]\cos k\bar{k}_i x, \\ T_5^{(1)} &= \sum_{i=1}^{4} A_{4i}\mu k\left[-(A_{11}\alpha_{2i}+A_{21})\bar{k}_i+\left(A_{11}\beta_1\alpha_{1i}+A_{21}\beta_2\alpha_{3i}\right)\right]\sin k\bar{k}_i x, \\ T_1^{(2)} &= \sum_{i=1}^{4} A_{4i}k\left[(\lambda+2\mu)(A_{21}\alpha_{1i}+A_{22}\alpha_{3i})\bar{k}_i+\lambda\left(A_{21}\beta_1\alpha_{2i}+A_{22}\beta_2\right)\right]\cos k\bar{k}_i x, \\ T_5^{(2)} &= \sum_{i=1}^{4} A_{4i}\mu k\left[-(A_{21}\alpha_{2i}+A_{22})\bar{k}_i+\left(A_{21}\beta_1\alpha_{1i}+A_{22}\beta_2\alpha_{3i}\right)\right]\sin k\bar{k}_i x, \end{aligned} \tag{33}$$

and the corresponding traction-free boundary conditions at the ends of a strip based on Eq. (27) are

$$T_1^{(1)} = T_5^{(1)} = T_1^{(2)} = T_5^{(2)} = 0 \quad \text{at} \quad x = \pm a, \tag{34}$$

where α is the length of the strip. Again, we adopt a normalization scheme for the length coordinate as

$$X = \frac{x}{\zeta}, \quad kx = 2\pi X, \quad k\bar{k}_i x = 2\pi \bar{k}_i X. \tag{35}$$

With boundary conditions in Eq. (34), we further have the displacement solutions in Eq. (31) as

$$\begin{aligned} u_1^{(1)} &= \sum_{i=1}^{4} A_{44}\gamma_i\alpha_{1i} \sin\left(2\pi\bar{k}_i X\right), \quad u_3^{(1)} = \sum_{i=1}^{4} A_{44}\gamma_i\alpha_{2i} \cos\left(2\pi\bar{k}_i X\right), \\ u_1^{(2)} &= \sum_{i=1}^{4} A_{44}\gamma_i\alpha_{3i} \sin\left(2\pi\bar{k}_i X\right), \quad u_3^{(2)} = \sum_{i=1}^{4} A_{44}\gamma_i \cos\left(2\pi\bar{k}_i X\right), \\ \gamma_r &= \frac{A_{4r}}{A_{44}}, \quad r = 1,2,3,4, \end{aligned} \tag{36}$$

where the amplitude ratios γ_r $(r = 1, 2, 3, 4)$ are obtained from the boundary conditions in Eq. (34) with stress equations in Eq. (33).

The above equations are adequate for the straight-crested wave analysis in a finite isotropic elastic strip. The displacements and the phase velocity are the most important solutions for finite solids we are interested.

6. A numerical example

For a semi-infinite isotropic elastic solid with Poisson's ratio $v = 0.20$, the phase velocity of surface acoustic waves is

$$c = 0.91099c_T. \tag{37}$$

For two-dimensional analysis of surface acoustic waves in finite solids, the two decaying parameters defined in Eq. (6) are used, but it should be reminded that the velocity is to be determined.

Now we can use the two-dimensional theory for the surface acoustic wave analysis in a finite solid. For this particular material, we find the associated wavenumbers have a structure that has two real numbers include 1, representing the dominant mode, and two imaginary numbers, representing two radiating terms along the length direction. For a strip with given length ratio range and $H = 10$, we have the phase velocity $v = c/c_T$, where the transverse wave velocity c_T is defined in Eq. (5), versus the length to wavelength ratios calculated in Fig. 3.

Similar to a finite plate, there are many velocity solutions representing different vibration modes. The velocities or modes can be found in the transcendental equation from Eqs. (30) and (34) with typical root-finding techniques like bisection method.

It is clear from Fig. 3 that there are many branches of surface acoustic waves in finite elastic solids, and the phase velocity of each branch generally decreases with the increase of the length to wavelength ratio. The multiple branches, or modes, of surface acoustic waves are the direct results of interactions of all surfaces, just like bulk acoustic waves in finite plates. Each mode has slightly different properties in the displacement amplitudes and their strain energy, in addition to the obvious different phase velocity we can see directly. This is, of course, a known fact to us in plate theories in identifying the vibration modes and evaluating their dominance at a particular frequency. This is important in practical applications because based on the semi-infinite model we are used to, there will be no overtone modes shown in Fig. 3, and design parameters such as the length to wavelength ratio might fall in the transition zone shown between 100.4 and 100.5, which could cause velocity change in a unpredictable way with small perturbation of frequency or length, and these surface wave features cannot be utilized at all. Such a mode conversion phenomena also exits in bulk acoustic waves, and it is generally recognized that to obtain stable frequency, the design parameters should be selected based on the frequency spectra, or the velocity spectra in our case, to avoid mode conversion zones, which appear periodically in Fig. 3. The validation of Fig. 3 requires either the detailed calculation of vibration frequencies in the strip or through experiments, but none of the above can be easily accomplished because there are too many wave branches for us to identify. This challenge will have to be dealt with coordinated computational and experimental approaches we are working on.

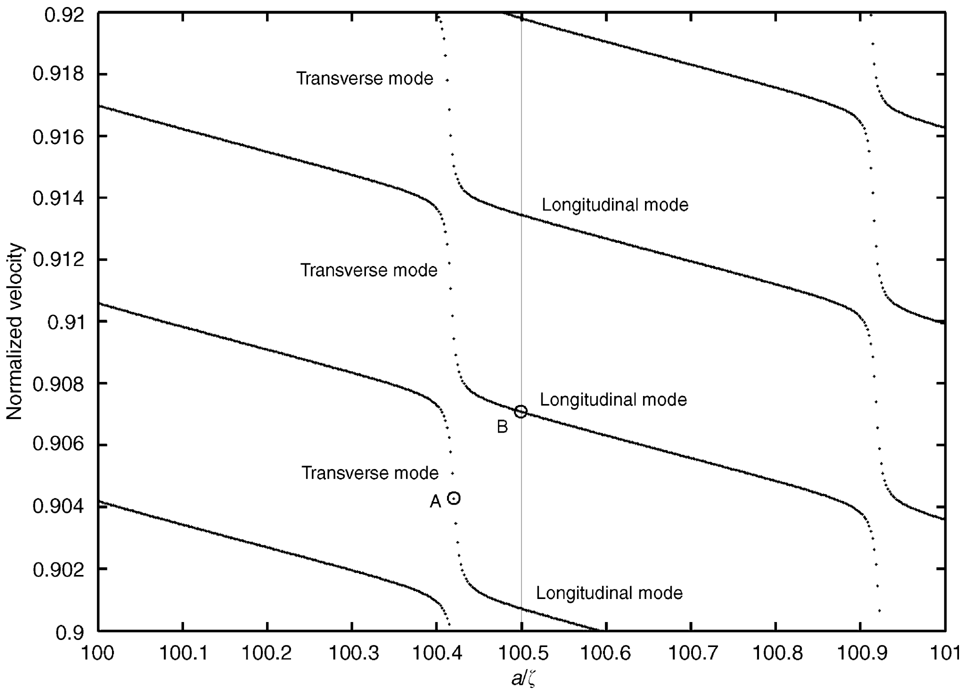

Fig. 3. Surface acoustic wave normalized phase velocity versus length to wavelength ratio for a strip with normalized thickness $H = 10$ and Poisson's ratio $\nu = 0.2$.

To further examine the vibration modes in Fig. 3, we plot displacements u_1 and u_3 on the upper surface of the strip of two representative modes designated as A and B in and out of the transition zone in Figs. 4–7. Basically it is clear that one particular mode will gradually lower its phase velocity until it reaches to a point that the mode properties approach to another one, and the conversion is completed by going through a transition zone. Figs. 4 and 5 are displacements on surface of the strip inside the transition zone at point A in Fig. 3, and for comparison we have displacements of another surface acoustic wave point B in Figs. 6 and 7. By comparison, it is clear that mode A is a predominately transverse mode with relatively larger displacement u_3. In contrast, point B is a predominantly longitudinal mode with relatively larger displacement u_1. The fundamental differences of the modes are clearly shown, and this is important in knowing the inside of complicated wave modes and frequency spectra, or velocity spectra, in Fig. 3. Similar to bulk acoustic waves, the displacements and the velocity spectra will provide practical guidance in applications like device design, including utilizing the overtone modes, which are not discussed and analyzed before due to limitations of the semi-infinite model.

7. Conclusions

Using the three-dimensional solutions of surface acoustic waves in semi-infinite elastic solids, displacements in finite elastic solids are expressed with known decaying functions in the depth or the thickness direction in a manner similar to the popular higher-order plate theories of Mindlin, Lee, and others. Applying variational equations of elasticity, the equations with two-dimensional variables are integrated over plane domain, thus resulting a two-dimensional theory for surface acoustic waves in a finite strip with equations and solution procedures resembling the plate theories also. Through validating the two-dimensional equations at a limiting case where the thickness of the strip is set to infinity, we find the phase velocity solution is the same with the equations of three-dimensional elasticity. It shows that there is no need to make corrections as we have used to in plate theories, demonstrating the advantage in adopting such an expansion scheme and creating the corresponding two-dimensional theory.

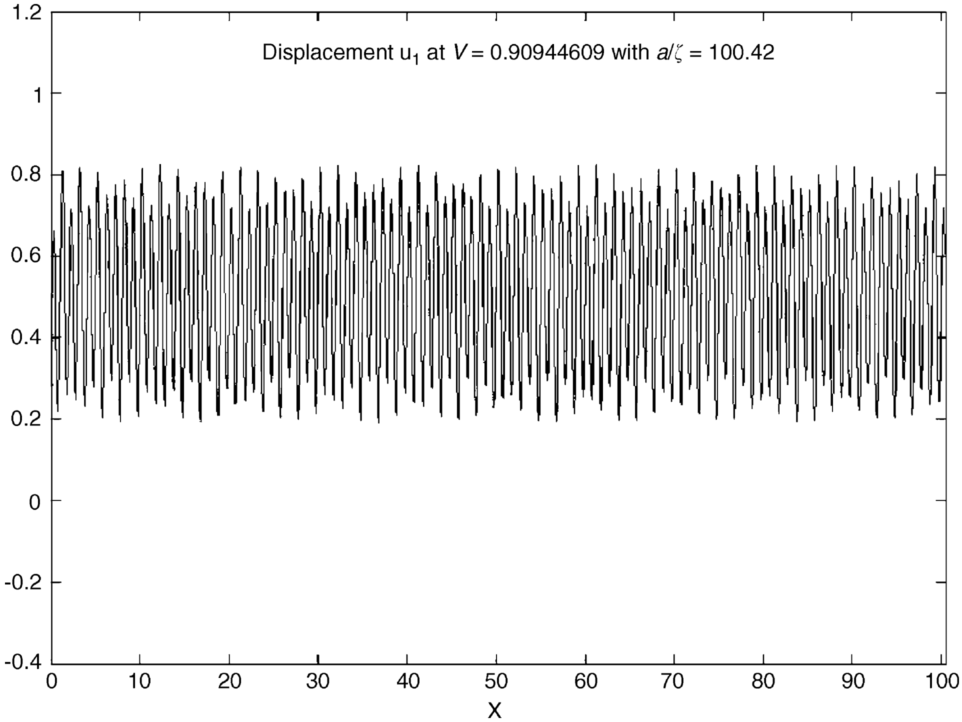

Fig. 4. Displacement u_1 of a strip with normalized phase velocity $v = 0.9094$ and $a/\zeta = 100.42$ designated as A in Fig. 3. This is a predominately transverse mode.

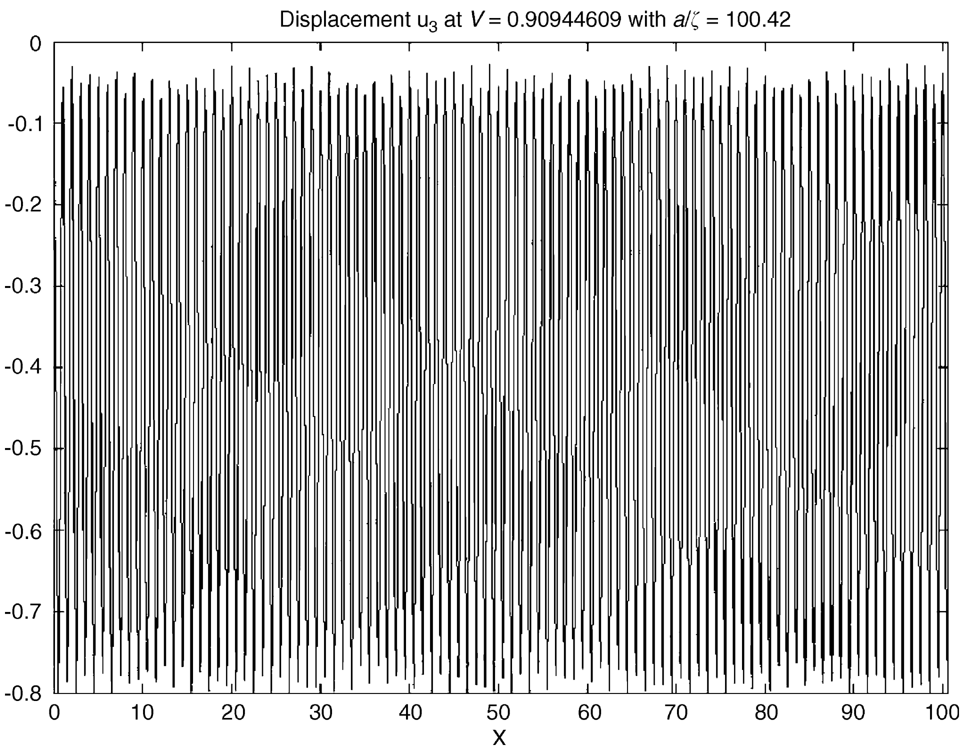

Fig. 5. Displacement u_3 of a strip with normalized phase velocity $v = 0.9094$ and $a/\zeta = 100.42$ designated as A in Fig. 3. This is a predominately transverse mode.

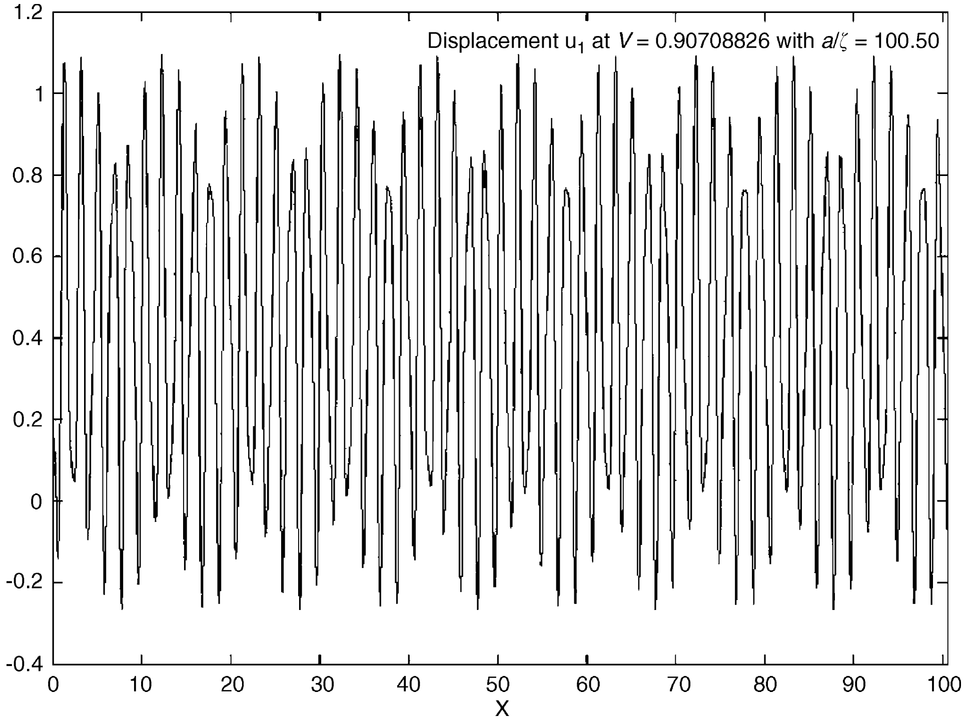

Fig. 6. Displacement u_1 of a strip with normalized phase velocity $v = 0.9071$ and $a/\zeta = 100.50$ designated as B in Fig. 3. This is a predominately longitudinal mode.

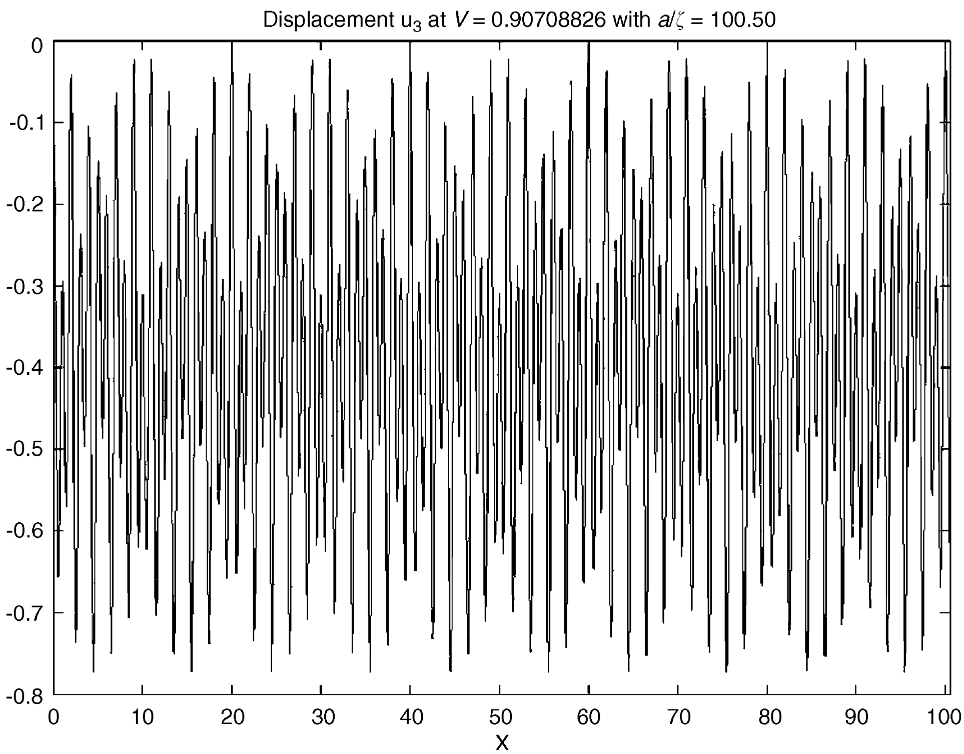

Fig. 7. Displacement u_3 of a strip with normalized phase velocity $v = 0.9071$ and $a/\zeta = 100.50$ designated as B in Fig. 3. This is a predominately longitudinal mode.

The applications of the two-dimensional theory can be found in many fields in which precise solutions of surface acoustic wave propagating are required in a simple and efficient manner. By implementing this theory with anisotropic materials and the consideration of complication factors like extra layer on the top, design and analysis of electrical devices based on surface acoustic wave phenomena can be simplified with accurate and efficient equations and solutions. The clear exhibition of the overtone modes in the velocity spectra is of great importance in possible utilization of the higher-order modes. In addition, it shows that the general principle of two-dimensional expansion and integration pioneered by Mindlin can be applied to elastic wave propagation problems in other natures for new equations based on the dimension reduction and simplification of the three-dimensional theory for emerging applications.

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