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Stabilization of vibration systems via delayed state difference feedback

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Abstract

This paper presents a study of the stabilization problem, via delayed state feedback of Pyragas type, for unstable vibration systems that have even number of characteristic roots with positive real parts. On the basis of stability switches with respect to the time delay, a new stabilization criterion is established, and an effective procedure for determining the admissible values of the feedback gains and the delay is given. Two examples are given in detail to demonstrate the efficiency of the theory.

1. Introduction

The problem of active control has drawn much attention over the past decades. A general description of the control problem of vibration systems is to determine whether there is a control $\mathbf{u}$ such that the trivial solution $\mathbf{x} = \mathbf{0}$ of the controlled system described by

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{C}\dot{\mathbf{x}} + \mathbf{K}\mathbf{x} = \mathbf{B}\mathbf{u}, \quad \mathbf{x} \in \mathbb{R}^n \tag{1}$$

is asymptotically stable, or to meet some other demands, where $\mathbf{M}$ and $\mathbf{K}$ are positive definite matrices. Among the existing control schemes, the delayed state difference feedback, which was originally proposed in Ref. [1] for controlling chaos of nonlinear systems, becomes one of most powerful control techniques due to its flexibility in applications. It has also been widely applied in vibration control. For example, it was used in Ref. [2] to improve the stability of periodic vibro-impact processes, as well as in Ref. [3] to stabilize the periodic motion of vibration systems of multiple degrees of freedom. In the theory of machine tool vibration, the regenerative stiff force and the regenerative damping force are also in the form of delayed feedback of Pyragas type [4]. Then there comes a problem: can a delayed feedback of Pyragas type always stabilize an unstable motion of a vibration system?

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An *affirmative* answer to this question was given in Ref. [5]. It tells that if the number of characteristic roots with positive real parts of a dynamic system is *even* and if they are not associated with the origin of the complex plane, then there is a delayed feedback control of Pyragas type that stabilizes the unstable motion. More precisely, let us consider in general the *n*th order linear controllable system of the form

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}, \quad \mathbf{x} \in \mathbb{R}^n \tag{2}$$

and let the controller $\mathbf{u}$ be in the form of Pyragas type

$$\mathbf{u} = -\mathbf{K}(\mathbf{x}(t) - \mathbf{x}(t-\tau)) \tag{3}$$

then the closed-loop system is described by a delay differential equation (DDE)

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) - \mathbf{B}\mathbf{K}(\mathbf{x}(t) - \mathbf{x}(t-\tau)). \tag{4}$$

In the literature, most of results assume the stability at $\tau = 0$. Here $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$ is assumed to have *even* number of characteristic roots with positive real parts, so the closed-loop system (4) is unstable if $\tau > 0$ is small enough. In particular, for the vibration system (1) of our interest, the number of characteristic roots with positive real parts of a unstable vibration system must be even. Two key steps are involved in establishing the main result of Ref. [5] for the stabilization problem. For example, for the case of scalar input, one needs firstly to prove that the system (2) can be stabilized by the derivative feedback

$$u = -\mathbf{K}^{\mathrm{T}}\dot{\mathbf{x}}(t) \tag{5}$$

for proper $\mathbf{K}$ with $\mathbf{K}^{\mathrm{T}}\mathbf{B} = 0$, and then to prove that it can be stabilized by

$$u = -\mathbf{K}^{\mathrm{T}}\frac{1}{\tau}(\mathbf{x}(t) - \mathbf{x}(t-\tau)) \tag{6}$$

for a certain small $\tau > 0$. Thus, from the viewpoint of control, the stabilization problem has been well-solved.

On the other hand, the PD control and PID control are most widely used in applications, and an accelerative feedback is not popular in vibration control. When we come back to the vibration systems, the derivative feedback (5) implies usually an accelerative feedback, and a small τ leads to large feedback gains due to (6). More importantly, Ref. [5] does not provide an estimation or estimating routines for the admissible delay values. This motives us to develop an alternative approach that is easier and computational tractable for the stabilization problem.

The stability of the controlled system, described by a DDE, can be carried out by investigating the root location of the characteristic roots. The trivial solution $\mathbf{x} = \mathbf{0}$ is asymptotically stable if and only if all the characteristic roots have negative real parts. If all the system parameter are fully *known*, then the Hassard theorem [6] and the Nyquist plot [7] are preferable for the stability test. When the delay effect on the stability is addressed, the concept of *stability switches* [8,9] is helpful for the stabilization problem, which describes a phenomena of the stability interchanges of the trivial solution from being unstable to being stable, or from being stable to being unstable, as τ varies from zero to infinite. As is well-known, a system undergoes stability switches only if a characteristic root appears on or crosses the imaginary axis, so the key steps in the stability analysis are to determine when the system is marginally stable and to determine the changing direction of the characteristic roots at the critical points. On the other hand, the fundamental frequency is an important index of vibration systems, it would be helpful to engineers if the criterion for the stabilization problem is associated with the vibration frequencies. In Ref. [10], on the basis of stability switches, some results of this kind for an undamped vibration system of single degree of freedom (sdof) with delayed feedback have been established. The main advantage of paper [10] is that the stability criteria are simply in terms of the fundamental frequencies.

As a followup work of Ref. [10], this paper presents an alternative study of the stabilization problem of Eq. (2) that has even number of characteristic roots with positive real part, a problem that has been studied in Ref. [5]. In the design phase, the feedback gains are to be determined, and the exact value of the delay is usually not known. So the Hassard criterion and the Nyquist criterion are not applicable in general for the stabilization problem. By means of the stability switches, the changing direction of the characteristic roots at the critical roots can be determined fully by the system parameters, this will be discussed in Section 2 for the closed-loop system (4). In order to make the exposition as simple as possible, we begin in Section 3 with the case when the uncontrolled system has exactly one pair of characteristic roots with positive real part and the characteristic function of the closed-loop is in a relatively simple form, then a new stabilization criterion will be established. The ideas can be easily extended to the general case, so in Section 4, only a rough description is given for the case when the plant has at least two pairs of characteristic roots with positive real parts. Finally in Section 5, some concluding remarks are drawn from the discussion.

2. The changing direction of characteristic roots at critical points

Let us study the stability of the closed-loop system, described by

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) - \mathbf{BK}(\mathbf{x}(t) - \mathbf{x}(t-\tau)). \tag{7}$$

The characteristic quasi-polynomial $p(\lambda, \tau)$ now reads $p(\lambda, \tau) = \det(\lambda \mathbf{I} - \mathbf{A} + \mathbf{BK} - \mathbf{BK}e^{-\lambda\tau})$, or in the form

$$P(\lambda, z) = \sum_{k=0}^{m} q_k(\lambda) z^k \quad (z = e^{-\lambda\tau}), \tag{8}$$

where $\deg[q_0(\lambda)] > \deg[q_i(\lambda)]$ $(i \geqslant 1)$, $\lambda = 0$ is not a root of $p(\lambda, \tau) = 0$, and the $q_i(\lambda)$'s have no common pure imaginary roots. Assume that the system is marginally stable, namely $p(\lambda, \tau)$ has a pair of pure imaginary roots: $\lambda = \pm i\omega$ with $\omega > 0$ for some values of τ, then the vibration frequency ω satisfies a polynomial equation $F(\omega) = 0$, which can be found out by following Ref. [11].

In fact, we define

$$P^{(1)}(\lambda, z) = q_0(-\lambda) P(\lambda, \tau) - q_m(\lambda) z^m P(-\lambda, 1/z). \tag{9}$$

It is easy to see that $P^{(1)}(\lambda, \tau)$ is independent of z^m and thus it can be written as

$$P^{(1)}(\lambda, z) = \sum_{k=0}^{m-1} q_k^{(1)}(\lambda) z^k,$$

where $q_0^{(1)}(\lambda) = q_0(\lambda) q_0(-\lambda) - q_m(\lambda) q_m(-\lambda)$. Repeating this procedure yields

$$P^{(j)}(\lambda, z) = \sum_{k=0}^{m-j} q_k^{(j)}(\lambda) z^k \quad (j = 1, 2, \ldots, m). \tag{10}$$

It is easy to see that $P^{(m)}(\lambda, z) = q_0^{(m)}(\lambda)$ is independent of z.

If $p(\lambda, \tau) = 0$ has a root $\lambda = i\omega$ for some $\tau > 0$, then $(\lambda, z) = (i\omega, e^{-i\omega\tau})$ is a root of $P(\lambda, z) = 0$, and so is it a root of $P(-\lambda, 1/z) = 0$ since the coefficients are real. Moreover, $(\lambda, z) = (i\omega, e^{-i\omega\tau})$ is a common root of $P^{(j)}(\lambda, z) = 0$ and $P^{(j)}(-\lambda, 1/z) = 0$ for all $j = 1, 2, \ldots, m$. Hence, one has $q_0^{(m)}(\pm i\omega) = 0$, which can be simplified as a polynomial equation with respect to ω that has *even*-order terms only:

$$F(\omega) = 0. \tag{11}$$

If $p(\lambda, \tau) = q_0(\lambda) + q_1(\lambda)e^{-\lambda\tau}$, for example, we have $F(\omega) = q_0(i\omega)q_0(-i\omega) - q_1(i\omega)q_1(-i\omega) = |q_0(i\omega)|^2 - |q_1(i\omega)|^2$, and obviously $F(\omega) = 0$ if and only if $F(-\omega) = 0$. For convenience, we call $F(\omega)$ the *critical function* hereafter. Once a root ω_* of $F(\omega) = 0$ is in hand, many routines, say Ref. [12], are available to find the critical values τ_k of the delay τ such that $p(i\omega_*, \tau_k) = 0$.

Remark 1. The condition $F(\omega) = 0$ can be derived in different ways. In Ref. [12], for instance, the real and imaginary parts of $p(i\omega, \tau) = 0$ are firstly converted into two polynomial equations

$$\begin{aligned} a_0(\omega)y^m + a_1(\omega)y^{m-1} + \cdots + a_m(\omega) = 0, \\ b_0(\omega)y^m + b_1(\omega)y^{m-1} + \cdots + b_m(\omega) = 0, \end{aligned} \tag{12}$$

respectively, by using the transformation $e^{-i\omega\tau} = (1 + iy)/(1 - iy)$. Let $x_0 = 1, x_1 = y, x_2 = y^2, \ldots, x^{2m-1} = y^{2m-1}$, and let $\mathbf{x} = [x_{2m-1}\ x_{2m-2}\ \cdots\ x_1\ x_0]^{\mathrm{T}}$, then the two nonlinear equations can be transformed into a *linear* equation in matrix form: $\boldsymbol{\Delta}\mathbf{x} = \mathbf{0}$, where $\boldsymbol{\Delta}$ is the Sylvester matrix of the two polynomials in Eq. (12)

with respect to y

$$\mathbf{\Delta} = \begin{bmatrix} a_0 & a_1 & a_2 & \cdots & a_{m-1} & a_m & & & \\ & a_0 & a_1 & a_2 & \cdots & a_{m-1} & a_m & & \\ & & & & \vdots & \vdots & & & \\ & & & & a_0 & a_1 & a_2 & \cdots & a_m \\ b_0 & b_1 & b_2 & \cdots & b_{m-1} & b_m & & & \\ & b_0 & b_1 & b_2 & \cdots & b_{m-1} & b_m & & \\ & & & & \vdots & \vdots & & & \\ & & & & b_0 & b_1 & b_2 & \cdots & b_m \end{bmatrix}.$$

A necessary condition for $p(\mathrm{i}\omega, \tau) = 0$ is the Sylvester resultant, namely the determinant of $\mathbf{\Delta}$, vanishes. This is the same as $F(\omega) = 0$ except for a non-zero factor. This approach can not only yield the condition $F(\omega) = 0$ but also provide an effective method for determining the corresponding critical values τ_k, by solving linear equations, at each real root ω of $F(\omega) = 0$.

To complete the stability analysis, it is required to determine the changing direction of the characteristic roots as τ passes through the critical values, namely to determine the sign of $S:=\mathrm{Re}(\mathrm{d}\lambda/\mathrm{d}\tau)|_{(\omega^*,\tau_k)}$ at each pair of critical values (ω^*, τ_k) that satisfy $p(\mathrm{i}\omega^*, \tau_k) = 0$. The system increases (decreases) two *new* characteristic roots with positive real parts as τ passes through τ_k from the left to the right if $S>0$ $(S<0)$. Let

$$\Pi(\lambda) = q_0^{(1)}(\lambda) \cdots q_0^{(m-1)}(\lambda), \tag{13}$$

where $q_0^j(\lambda)$'s are defined in Eq. (10). For simplicity, we call $\Pi(\lambda)$ the *coefficient function*. In general, the changing direction depends not only on the critical function $F(\omega)$, but also on the coefficient function. More precisely, one has

Theorem 1. *The following equation holds*:

$$\mathrm{sgn}\,\mathrm{Re}\left[\frac{\mathrm{d}\lambda}{\mathrm{d}\tau}\right]_{(\omega^*,\tau_k)} = \begin{cases} \mathrm{sgn}[\Pi(\mathrm{i}\omega^*)F'(\omega^*)], & m>1, \\ \mathrm{sgn}[F'(\omega^*)], & m = 1. \end{cases} \tag{14}$$

Proof. When $m = 1$, Eq. (14) can be proved simply as follows [9]. In fact, by differentiating $p(\lambda, \tau) = 0$ with respect to τ, we have

$$\frac{\mathrm{d}\tau}{\mathrm{d}\lambda} = \left(\frac{\mathrm{d}\lambda}{\mathrm{d}\tau}\right)^{-1} = -\frac{q_0'(\lambda)}{\lambda q_0(\lambda)} + \frac{q_1'(\lambda)}{\lambda q_1(\lambda)} - \frac{\tau}{\lambda}. \tag{15}$$

Let $\bar{z}$ be the conjugate of complex z, $\mathrm{Re}(z)$ and $\mathrm{Im}(z)$ stand for the real and imaginary parts of z, then the simple fact $\mathrm{Re}[1/(a+b\mathrm{i})] = \mathrm{Re}(a-b\mathrm{i})/(a^2+b^2) = \mathrm{Re}(a+b\mathrm{i})$ yields

$$\begin{aligned} \mathrm{sgn}\,\mathrm{Re}\left[\frac{\mathrm{d}\lambda}{\mathrm{d}\tau}\right]_{(\omega^*,\tau_k)} &= \mathrm{sgn}\,\mathrm{Re}\left[-\frac{q_0'(\mathrm{i}\omega^*)}{\mathrm{i}\omega^* q_0(\mathrm{i}\omega^*)} + \frac{q_1'(\mathrm{i}\omega^*)}{\mathrm{i}\omega^* q_1(\mathrm{i}\omega^*)}\right] \\ &= -\,\mathrm{sgn}\,\mathrm{Im}\left[\frac{q_0'(\mathrm{i}\omega^*)}{\omega^* q_0(\mathrm{i}\omega^*)} - \frac{q_1'(\mathrm{i}\omega^*)}{\omega^* q_1(\mathrm{i}\omega^*)}\right] \\ &= -\,\mathrm{sgn}\,\mathrm{Im}[q_0'(\mathrm{i}\omega^*)\bar{q_0}(\mathrm{i}\omega^*) - q_1'(\mathrm{i}\omega^*)\bar{q_1}(\mathrm{i}\omega^*)] \\ &= \mathrm{sgn}\,[\mathrm{Re}q_0(\mathrm{i}\omega^*)\mathrm{Re}q_0'(\mathrm{i}\omega^*) + \mathrm{Im}q_0(\mathrm{i}\omega^*)\mathrm{Im}q_0'(\mathrm{i}\omega^*) \\ &\quad - \mathrm{Re}q_1(\mathrm{i}\omega^*)\mathrm{Re}q_1'(\mathrm{i}\omega^*) - \mathrm{Im}q_1(\mathrm{i}\omega^*)\mathrm{Im}q_1'(\mathrm{i}\omega^*)] \\ &= \mathrm{sgn}\,F'(\omega^*). \end{aligned}$$

The proof is completed for $m = 1$. For a proof of the general case, it is referred to [11]. □

3. The stabilization problem for a simple case

We begin with the case when **A** has exactly one pair of conjugate eigenvalues with positive real part but the other eigenvalues stay in the left half open complex plane. Or equivalently, the plant has one pair of characteristic roots with positive real part when $\tau = 0$. In addition, we assume that the characteristic quasi-polynomial is in the form

$$p(\lambda, \tau) = q_0(\lambda) + q_1(\lambda)\mathrm{e}^{-\lambda\tau}. \tag{16}$$

A general discussion will be given in Section 4.

3.1. Stabilization criteria

Now the critical function $F(\omega)$ may have no real roots, may have exactly one pair of real roots, or may have two/more pairs of real roots. If $F(\omega)$ has no real roots, the system does not change its stability, the trivial solution $\mathbf{x} = \mathbf{0}$ is unstable for all given delay. If $F(\omega)$ has exactly *one* pair of (simple) real roots $\pm\omega^*$, a series of critical values τ_k of delay can be determined, satisfying

$$\tau_k = \tau_0 + \frac{2k\pi}{\omega^*}, \quad (k = 0, 1, 2, \ldots), \tag{17}$$

where τ_0 is the minimal critical delay, see, for example, Ref. [8]. Because $F(\omega)$ has exactly one positive zero ω^*, and $F(\infty) \to \infty$, we must have $F'(\omega^*) > 0$. It follows that

$$\operatorname{sgn} \operatorname{Re}\left[\frac{\mathrm{d}\lambda}{\mathrm{d}\tau}\right]_{(\omega^*, \tau_k)} = \operatorname{sgn} F'(\omega^*) = 1.$$

It implies that as τ passes through each τ_k, the system increases one *new* pair of characteristic roots with positive real part. Thus, the trivial solution is unstable for all given delay.

Therefore, in order to stabilize the unstable motion, it is necessary that $F(\omega)$ has two or more different pairs of real roots. For simplicity, it is convenient to consider the case when $F(\omega)$ has exactly two different pairs of real roots: $\pm\omega_1$ and $\pm\omega_2$ with $0 < \omega_2 < \omega_1$. Let $\tau_{j,0}$ be the minimal critical delay corresponding to ω_j ($j = 1, 2$), then the two series of critical delays $\tau_{j,n}$ satisfy

$$\tau_{j,n} = \tau_{j,0} + \frac{2n\pi}{\omega_j}, \quad (j = 1, 2, \; n = 0, 1, 2, \ldots) \tag{18}$$

and

$$\tau_{1,k+1} - \tau_{1,k} = \frac{2\pi}{\omega_1} < \frac{2\pi}{\omega_2} = \tau_{2,k+1} - \tau_{2,k}, \quad (k \geqslant 0). \tag{19}$$

Moreover, we have

Theorem 2. *The following two claims hold*:

$$\operatorname{sgn} \operatorname{Re}\left[\frac{\mathrm{d}\lambda}{\mathrm{d}\tau}\right]_{(\omega_1, \tau_{1,k})} = \operatorname{sgn} F'(\omega_1) = 1,$$

$$\operatorname{sgn} \operatorname{Re}\left[\frac{\mathrm{d}\lambda}{\mathrm{d}\tau}\right]_{(\omega_2, \tau_{2,k})} = \operatorname{sgn} F'(\omega_2) = -1.$$

Proof. By the definition of derivative, we have

$$F'(\omega_i) = \lim_{\omega \to \omega_i} \frac{F(\omega) - F(\omega_i)}{\omega - \omega_i}, \quad (i = 1, 2).$$

Thus, the two inequalities $F'(\omega_1) > 0$ and $F'(\omega_2) < 0$ hold due to the facts: $F(\omega) > F(\omega_1) = 0$, $(\forall \omega > \omega_1 > 0)$, $F(\omega) < F(\omega_1) = 0$, $(\forall \omega_2 < \omega < \omega_1)$, and $F(\omega) > F(\omega_2) = 0$, $(\forall 0 < \omega < \omega_2)$. This completes the proof. □

Therefore, we have the following necessary and sufficient conditions for the stabilization problem.

Theorem 3. *Let the feedback gain matrix* $\mathbf{K}$ *be chosen such that* $F(\omega)$ *has exactly two different positive roots* $\omega_1 > \omega_2 > 0$. *Then in order that the unstable solution* $\mathbf{x} = \mathbf{0}$ *of the closed-loop, namely Eq.* (7), *is asymptotically stable, it is necessary that the two corresponding minimal critical values* $\tau_{1,0}$, $\tau_{2,0}$ *of the time delay satisfy the following condition*:

$$\tau_{2,0} < \tau_{1,0}. \tag{20}$$

Conversely, if $\tau_{2,0} < \tau_{1,0}$ *holds, a delayed feedback* $\mathbf{u} = -\mathbf{K}(\mathbf{x}(t) - \mathbf{x}(t-\tau))$ *with any* $\tau \in (\tau_{2,0}, \tau_{1,0})$ *stabilize the unstable solution of the control plant.*

Proof. Theorem 2 tells that as τ increases from zero to infinite, the system increases one new pair of conjugate characteristic roots with real parts being from negative to positive when τ passes through each $\tau_{1,k}$ from the left to the right, or decreases one new pair of conjugate characteristic roots with real parts being from positive to negative when τ passes through each $\tau_{2,k}$ from the left to the right. Because of the condition (19), the number of characteristic roots with real parts being from positive to negative is less than that of characteristic roots with real parts being from negative to positive. Let the critical values $\tau_{1,k}$ and $\tau_{2,k}$ of delay be rearranged as a list from small to large, then as τ increases, the system changes its stability from being unstable to being stable, or alternatively from being stable to being unstable, and the change ends if in the list of critical values of delay, some $\tau_{1,k}$ is followed immediately by $\tau_{1,k+1}$.

Now, if on the contrary, $\tau_{1,0} \leqslant \tau_{2,0}$, then one has $\tau_{1,0} < \tau_{1,1} < \tau_{2,0} < \cdots$, or $\tau_{1,0} < \tau_{2,0} < \tau_{1,1} < \cdots$, or $\tau_{10} = \tau_{20} < \tau_{11} < \tau_{21} < \cdots$. In each of the three cases, the above analysis shows that the trivial solution $x = 0$ is unstable for all given τ. Therefore, it is necessary that $\tau_{2,0} < \tau_{1,0}$ for the stabilization problem.

Conversely, if $\tau_{2,0} < \tau_{1,0}$ holds, then the trivial solution of Eq. (7) is unstable for $\tau \in [0, \tau_{2,0})$, and it is asymptotically stable for $\tau \in (\tau_{2,0}, \tau_{1,0})$. The proof is completed. □

Remark 2. In the design phase, the feedback gains are usually not known, so the critical function $F(\omega)$ involves some unknown parameters. In this case, it is usually difficult to determine for what values of gains, the critical function $F(\omega)$ has exactly two different positive roots. However, it is convenient to apply the theory of complete discrimination system for polynomials [13] that is similar to the classical Sturm sequence theory, as done in Ref. [8].

Remark 3. This paper addresses on the stability criteria for the controlled systems, it does not provide any discussion on the achievable performance of the controllers, another important problem beyond the scope of this paper. Roughly speaking, the stability near the critical points $\tau_{1,0}, \tau_{2,0}$ is usually very poor, because the maximal real part of the characteristic roots is negative but is close to zero. To achieve a better performance, say, the maximal real part of the characteristic roots is less than a given number $-\alpha < 0$, the routines of this paper can be followed if λ is replaced by $s - \alpha$, since $\mathrm{Re}(s) < 0$ if and only if $\mathrm{Re}(\lambda) = \mathrm{Re}(s - \alpha) < -\alpha$.

3.2. Stabilization to a sdof vibration system

To demonstrate the proposed approach, let us consider the following typical sdof linear vibration system $m\ddot{x} + c\dot{x} + kx = 0$ or in the non-dimensional form

$$\ddot{x} + 2\xi\dot{x} + x = 0. \tag{21}$$

In the literature, it is usually assumed that $c > 0$ namely $\xi > 0$ such that the trivial solution $x = 0$ is asymptotically stable. Here in this paper, we assume that the system has a *negative* damping ($\xi < 0$) such that $x = 0$ is unstable, and we want to determine if there are u, v and a positive T such that $x = 0$ of the closed-loop system under delayed feedback of Pyragas type

$$\ddot{x} + 2\xi\dot{x} + x = -u[x(t) - x(t-T)] - v[\dot{x}(t) - \dot{x}(t-T)] \tag{22}$$

is asymptotically stable. Such a problem is encountered in the stabilization problem of unstable periodic solutions, see, for example Refs. [2,3,10]. As is well-known, the trivial solution of Eq. (22) is asymptotically stable if and only if the characteristic quasi-polynomial

$$p(\lambda, \tau) = \lambda^2 + 2\,\xi\,\lambda + 1 + u\,(1 - \mathrm{e}^{-\lambda\,\tau}) + v\,\lambda\,(1 - \mathrm{e}^{-\lambda\,\tau}) \tag{23}$$

has roots staying in the open left-half complex plane only.

By separating the real and imaginary parts of $p(\mathrm{i}\omega, \tau) = 0$, we have

$$\mathrm{Re}[p(\mathrm{i}\omega, \tau)] := -\,\omega^2 + 1 + u\,(1 - \cos(\omega\,\tau)) - v\,\omega\,\sin(\omega\,\tau) = 0,$$
$$\mathrm{Im}[p(\mathrm{i}\omega, \tau)] := 2\,\xi\,\omega + u\,\sin(\omega\,\tau) + v\,\omega\,(1 - \cos(\omega\,\tau)) = 0.$$

Solving for $\sin(\omega\tau), \cos(\omega\tau)$ gives

$$\cos(\omega\,\tau) = \frac{-\omega^2\,u + u + u^2 + v^2\,\omega^2 + 2\,v\,\omega^2\,\xi}{u^2 + v^2\,\omega^2}, \quad \sin(\omega\,\tau) = -\frac{\omega\,(2\,u\,\xi + v\,\omega^2 - v)}{u^2 + v^2\,\omega^2}$$

providing that $u^2 + v^2\omega^2 \neq 0$. Then, $p(\mathrm{i}\omega, \tau) = 0$ yields

$$F(\omega) := \omega^4 + (-2\,u + 4\,v\,\xi - 2 + 4\,\xi^2)\,\omega^2 + 2\,u + 1 = 0 \tag{24}$$

since $\sin^2(\omega\tau) + \cos^2(\omega\tau) = 1$. In order to stabilize the unstable motion of the sdof vibration system, it is necessary that $F(\omega) = 0$ has exactly two pairs of real roots.

Now, assume that $F(\omega) = \omega^4 + p\,\omega^2 + q$, given in (24), has two *different* pairs of real roots: $\pm\omega_1$ and $\pm\omega_2$ with $0 < \omega_2 < \omega_1$, this is true if the following three conditions hold

$$p < 0, \quad q \geqslant 0, \quad p^2 - 4q \geqslant 0. \tag{25}$$

Then the critical delays $\tau_{j,n}$ corresponding to ω_j satisfy $\tau_{j,n} = \tau_{j,0} + 2n\pi/\omega_j, \quad (j = 1, 2,\ n = 0, 1, 2, \ldots)$ and $\tau_{1,k+1} - \tau_{1,k} = 2\,\pi/\omega_1 < 2\,\pi/\omega_2 = \tau_{2,k+1} - \tau_{2,k}, \quad (k \geqslant 0)$. Moreover, we have

$$\operatorname{sgn}\mathrm{Re}\left[\frac{\mathrm{d}\lambda}{\mathrm{d}\tau}\right]_{(\omega_1, \tau_{1,k})} = \operatorname{sgn} F'(\omega_1) = 1, \quad \operatorname{sgn}\mathrm{Re}\left[\frac{\mathrm{d}\lambda}{\mathrm{d}\tau}\right]_{(\omega_2, \tau_{2,k})} = \operatorname{sgn} F'(\omega_2) = -1.$$

Theorem 4. *In order that the unstable trivial solution $x = 0$ of Eq.* (21) *under a delayed feedback of Pyragas type $-u[x(t) - x(t-T)] - v[\dot{x}(t) - \dot{x}(t-T)]$ is asymptotically stable, it is necessary that $p < 0, q \geqslant 0, p^2 - 4q \geqslant 0$ such that $F(\omega)$ has exactly two different positive real roots $\omega_1 > \omega_2 > 0$, and that the minimal critical values $\tau_{1,0}$, $\tau_{2,0}$ of the time delay satisfy the following condition $\tau_{2,0} < \tau_{1,0}$. Conversely, if the condition $\tau_{2,0} < \tau_{1,0}$ holds, then the delayed feedback with any $\tau \in (\tau_{2,0}, \tau_{1,0})$ stabilizes the unstable trivial solution of Eq.* (21).

In general, it is not obvious to see whether the condition $\tau_{2,0} < \tau_{1,0}$ holds or not. But it is easy to find out the admissible controllers. For example, if $u = 0$, then the critical stability condition yields

$$\cos(\omega\,\tau) = \frac{v + 2\,\xi}{v}, \quad \sin(\omega\,\tau) = -\frac{\omega^2 - 1}{v\,\omega} \tag{26}$$

the critical function $F(\omega)$ reads simply

$$F(\omega) = \omega^4 + (-2 + 4\,v\,\xi + 4\,\xi^2)\,\omega^2 + 1$$

and it has exactly two different pairs of real roots if and only if $4\,v\,\xi - 2 + 4\,\xi^2 < 0$ and $0 < 16\,\xi\,(\xi + v)\,(v\,\xi - 1 + \xi^2)$, namely, $\xi < 0$, $4\,v\,\xi - 2 + 4\,\xi^2 < 0$ and $0 < \xi + v$, or simply

$$\xi < 0, \quad 0 < \xi + v. \tag{27}$$

Thus, for each $v > -\xi$, the admissible values of the delay can be found easily.

Example 1. $\xi = -0.5$. Taking $u = 0$ and $v = 0.8 > -\xi$, for instance, gives two real zeros $\omega_2 = 0.6851, \omega_1 = 1.4597$ of $F(\omega) = \omega^4 - 2.60\,\omega^2 + 1$. Solving Eq. (26) gives the corresponding minimal critical delay values $\tau_{1,0} = 3.0553$ and $\tau_{2,0} = 2.6617$, respectively, $\tau_{2,0}$ is smaller than $\tau_{1,0}$ as expected. Moreover, the two sequences $\tau_{1,n}$ and $\tau_{2,n}$, as defined in Eq. (18), are found to be

$$\tau_{1,0} = 3.0553, \quad \tau_{1,1} = 7.3598, \quad \tau_{1,2} = 12.4026, \quad \tau_{1,3} = 16.8756, \ldots$$
$$\tau_{2,0} = 2.6617, \quad \tau_{2,1} = 11.8331, \quad \tau_{2,2} = 19.6572, \quad \tau_{2,3} = 28.4833, \ldots\ .$$

The critical values of τ can be ranked as follows

$$\tau_{2,0} < \underline{\tau_{1,0} < \tau_{1,1}} < \tau_{1,2} < \tau_{2,1} < \cdots. \tag{28}$$

As a result, the unstable trivial solution $x = 0$ can be stabilized if the delay τ is chosen in (2.661, 3.055). Of course, the stability is poor if $\tau \in (2.661, 3.055)$ is chosen close to the endpoints, since the maximal real part of the characteristic roots is close to zero.

The numerical simulation checks the results well. The initial value is taken to be $x(t) = 0.6, \dot{x}(t) = 0$ ($\forall t \in [-\tau, 0]$), the step-size of integration by using Runge–Kutta method is 0.01. As shown in Fig. 1, the simulation results are in very good agreement with the theoretical prediction.

Example 2. $\xi = -0.3$. In this case, we chose $u = 0$ and $v = 0.5 > -\xi$, then $F(\omega) = \omega^4 - 2.24\,\omega^2 + 1$ and it has two real zeros: $\omega_1 = 0.7846, \omega_2 = 1.2745$. As done above, the two sequences $\tau_{1,n}, \tau_{2,n}$ are found to be

$$\tau_{1,0} = 3.5394, \quad \tau_{1,1} = 8.4693, \quad \tau_{1,2} = 13.3992, \quad \tau_{1,3} = 18.3290, \ldots,$$
$$\tau_{2,0} = 2.2586, \quad \tau_{2,1} = 10.2666, \quad \tau_{2,2} = 18.2746, \quad \tau_{2,3} = 26.2826, \ldots .$$

The critical values of τ can be ranked as follows:

$$\tau_{2,0} < \underline{\tau_{1,0} < \tau_{1,1}} < \tau_{2,1} < \tau_{1,2} < \cdots. \tag{29}$$

As a result, the unstable solution $x = 0$ is stabilized under the delayed feedback control if the delay τ is chosen in (2.2586, 3.5394).

Moreover, under the condition (27), $F(\omega)$ has exactly two different positive roots: $0 < \omega_2 < \omega_1$, and the critical values of time delay satisfy

$$\cos(\omega_1\,\tau_{1,k}) = \cos(\omega_2\,\tau_{2,k}), \quad \omega_1\,\omega_2 = 1. \tag{30}$$

The condition $\cos(\omega_1\,\tau_{1,k}) = \cos(\omega_2\,\tau_{2,k})$ yields

$$\omega_2\,\tau_{2,0} = 2\,\pi - \omega_1\,\tau_{1,0}, \quad -\frac{\omega_1^2 - 1}{v\,\omega_1} = \frac{\omega_2^2 - 1}{v\,\omega_2}. \tag{31}$$

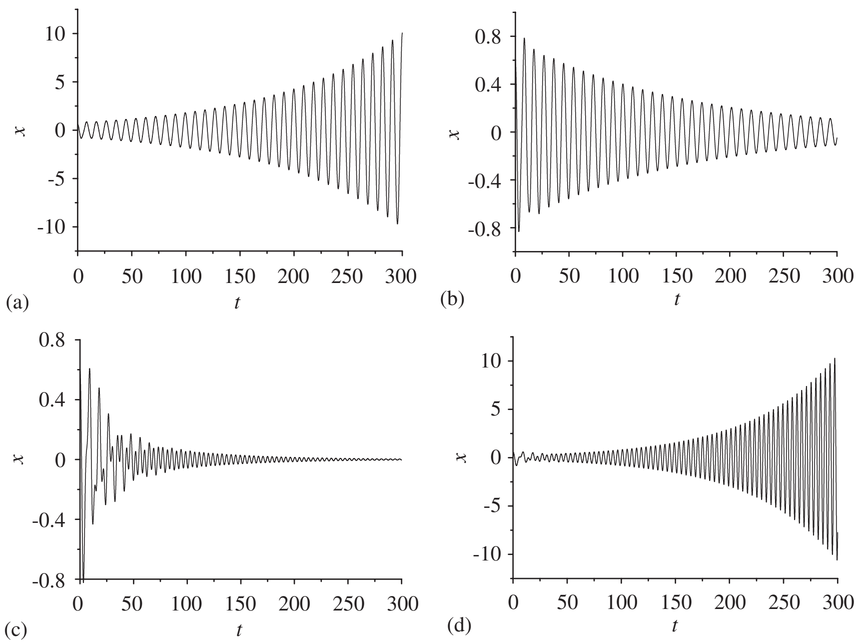

Fig. 1. The time histories of Eq. (22) with $u = 0, v = 0.8 > 0.5 = -\xi$ for (a) $\tau = 2.6$; (b) $\tau = 2.7$; (c) $\tau = 3.0$; and (d) $\tau = 3.1$.

Now, if $|\xi|$ $(= -\xi)$ is *small* and $v > -\xi$, then we have

$$\cos(\omega\tau) = \frac{v + 2\xi}{v} \approx 1, \quad \sin(\omega\tau) = -\frac{\omega^2 - 1}{v\omega} \approx 0. \tag{32}$$

It follows that $\omega_1 \approx 1$, $\cos(\omega_1\tau_{1,0}) > 0$, $\sin(\omega_1\tau_{1,0}) < 0$, $\cos(\omega_2\tau_{2,0}) > 0$ and $\sin(\omega_2\tau_{2,0}) > 0$. Therefore the condition $\tau_{2,0} < \tau_{1,0}$ is governed:

$$\omega_1\tau_{1,0} \in \left(\frac{3}{2}\pi, 2\pi\right), \quad \omega_2\tau_{2,0} \in \left(0, \frac{\pi}{2}\right) \tag{33}$$

and

$$\tau_{2,0} = (2\pi - \omega_1\tau_{1,0})\omega_1 < \tau_{1,0}. \tag{34}$$

This means that if $|\xi|$ is small, a delayed velocity feedback of Pyragas type can always stabilize the unstable motion. It is similar to the observation in Ref. [14], where it has proved that a delayed velocity feedback $v\dot{x}(t-\tau)$ can always stabilize the unstable motion of a small perturbation of undamped sdof vibration system, by means of the averaging technique for time-delay equations of slow variables.

When $|\xi|$ is *not small*, Eq. (26) holds too, if v $(> -\xi)$ is large enough. However, a large v may result in a large ω so that Eq. (34) may not be guaranteed. In this case, we assume that $v = 0, u \neq 0$, and see if the condition $\tau_{2,0} < \tau_{1,0}$ holds or not.

For $u \neq 0$ and $v = 0$, the function $F(\omega) = \omega^4 + (-2 + 4\xi^2 - 2u)\omega^2 + 1 + 2u$ has two different positive real roots if and only if $u > \max\{-1/2, 2\xi^2 - 1, 2\xi^2 - 2\xi\}$. The two positive roots $\omega_1 > \omega_2$ satisfy

$$\omega_1^2 + \omega_2^2 = 2u + 2 - 4\xi^2, \quad \omega_1^2\omega_2^2 = 2u + 1. \tag{35}$$

Corresponding to ω_1, ω_2, the critical values of delay are determined from

$$\cos(\omega\tau) = \frac{u^2 - u\omega^2 + u}{u^2}, \quad \sin(\omega\tau) = -2\frac{\xi\omega}{u}. \tag{36}$$

Thus, for sufficient large $u > 0$, we note that $\omega_1 \approx \sqrt{2u+1}$ and $\omega_2 \approx 1$, so we have $\omega_1\tau_{1,0} \in (\pi/2, \pi)$, $\omega_1\tau_{1,0} \approx \pi$, $\omega_2\tau_{2,0} \in (0, \pi/2)$ and $\omega_2\tau_{2,0} \approx 0$. As a result, there is a large u such that

$$\tau_{1,0} \approx \frac{\pi}{\omega_1} > \frac{\pi}{\sqrt{2u+1}} > \tau_{2,0}. \tag{37}$$

This means that a delayed feedback $-u[x(t) - x(t-\tau)]$ with large u can always stabilize the unstable motion of the sdof vibration system.

Example 3. $\xi = -1.3$. Firstly we chose $u = 0$ and $v = 1.7 > -\xi$, then $F(\omega) = \omega^4 - 4.08\,\omega^2 + 1$ and it has two real zeros: $\omega_1 = 0.5118, \omega_2 = 1.9540$. As done above, we have $2.1261 = \tau_{1,0} < \tau_{2,0} = 4.1595$. So no $\tau > 0$ exists to stabilize the unstable solution $x = 0$ of the sdof system. Similar results can be obtained for larger v. When $u \neq 0, v = 0$, however, the trivial solution can be stabilized for large u. In fact, we have $u > 5.98$ to guarantee the existence of two different positive roots ω_1, ω_2 of $F(\omega)$. If $u = 7$, then $F(\omega) = \omega^4 - 9.24\,\omega^2 + 15$ has two different positive roots: $\omega_2 = 1.4495$, $\omega_1 = 2.6719$. The two minimal critical values of delay are found to be $\tau_{1,0} = 0.5417$, $\tau_{2,0} = 0.3922$, respectively, so the trivial solution is stabilized if $\tau \in (0.3922, 0.5417)$. If we increase u to a large number 70, then corresponding to the two roots $\omega_2 = 1.0251, \omega_1 = 11.584$ of $F(\omega) = \omega^4 - 135.24\,\omega^2 + 141$, the two minimal critical values of delay are found to be $\tau_{1,0} = 0.2328$, $\tau_{2,0} = 0.0372$, respectively. It means that if $\tau \in (0.0372, 0.2328)$, then $x = 0$ can be stabilized by the delayed velocity feedbacks.

4. The stabilization problem for the general case

The proposed stabilization scheme can also be extended to the case when the control plant has more than two characteristic roots with positive real parts. As done before, the first step is to determine for what values of delay, the system is marginally stable, which results in the critical function. The second step is to determine the changing direction of the characteristic roots as the delay passes through the critical values.

We first assume that the characteristic function of the closed-loop system has the form $p(\lambda,\tau) = q_0(\lambda) + q_1(\lambda)e^{-\lambda\tau}$ and assume that the plant has two pairs of conjugate complex characteristic roots with positive real parts, then we chose the feedback gains such that the critical function $F(\omega)$ have four pairs of roots: $\pm i\,\omega_j$ $(j = 1,2,3,4)$, with $\omega_1 > \omega_2 > \omega_3 > \omega_4$, and the four corresponding minimal critical values of delay are denoted by $\tau_{j,0}$ $(j = 1,2,3,4)$, respectively. Then the unstable solution is stabilized only if $\max\{\tau_{4,0},\tau_{2,0}\} < \min\{\tau_{3,0},\tau_{1,0}\}$, and the trivial solution of the controlled system is asymptotically stable for all $\tau \in (\max\{\tau_{4,0},\tau_{2,0}\}, \min\{\tau_{3,0},\tau_{1,0}\})$. If the control plant has $2N$ characteristic roots with positive real parts, the feedback gains can be chosen in this way such that the critical function $F(\omega)$ has exactly $2N$ positive zeros: $\omega_1 > \omega_2 > \cdots > \omega_{2N-1} > \omega_{2N} > 0$. Corresponding to each ω_k, we find out the minimal critical delay value $\tau_{k,0}$. Then the unstable solution is stabilized only if

$$\max\{\tau_{2N,0},\tau_{2N-2,0},\ldots,\tau_{2,0}\} < \min\{\tau_{2N-1,0},\tau_{2N-3,0},\ldots,\tau_{1,0}\} \tag{38}$$

and the trivial solution of the controlled system is asymptotically stable for all $\tau \in (\max\{\tau_{2N,0},\tau_{2N-2,0},\ldots,\tau_{2,0}\}, \min\{\tau_{2N-1,0},\tau_{2N-3,0},\ldots,\tau_{1,0}\})$.

To see this, let us consider the stability of the following simple system:

$$\begin{aligned}
\ddot{x}_1(t) - c_1\dot{x}_1 + \Omega_1^2 x_1 &= -\alpha(x_1(t) - x_1(t-\tau)) - \beta(x_2(t) - x_2(t-\tau)),\\
\ddot{x}_2(t) - c_2\dot{x}_2 + \Omega_2^2 x_2 &= -\alpha(x_1(t) - x_1(t-\tau)) - \beta(x_2(t) - x_2(t-\tau)),
\end{aligned} \tag{39}$$

where $c_1 > 0$ and $c_2 > 0$ to ensure that at $\tau = 0$, the system has two pairs of conjugate characteristic roots with positive real parts. We want to chose proper α, β, τ such that the trivial solution of Eq. (39) is asymptotically stable.

For simplicity, let $\beta = \alpha$, then straightforward computation yields the characteristic function $p(\lambda,\tau)$

$$\begin{aligned}
p = {} & \lambda^4 - (c_2 + c_1)\lambda^3 + (2\,\alpha + c_1 c_2 + \Omega_2^2 + \Omega_1^2)\lambda^2\\
& - (c_1\Omega_2^2 + \alpha\, c_2 + \Omega_1^2 c_2 + \alpha\, c_1)\lambda + (\Omega_1^2\Omega_2^2 + \alpha\,\Omega_1^2 + \alpha\,\Omega_2^2)\\
& + [-2\,\alpha\,\lambda^2 + (\alpha\, c_1 + \alpha\, c_2)\lambda - \alpha\,\Omega_2^2 - \alpha\,\Omega_1^2]e^{-\lambda\,\tau}
\end{aligned}$$

and the critical function $F(\omega)$

$$\begin{aligned}
F = {} & \omega^8 + (c_1^2 - 2\,\Omega_1^2 - 2\,\Omega_2^2 - 4\,\alpha + c_2^2)\Omega^6 + [6\,\alpha(\Omega_1^2 + \Omega_2^2) + 4\,\Omega_1^2\Omega_2^2\\
& - 2\,c_2^2\Omega_1^2 - 2\,c_1^2\Omega_2^2 - 2\,\alpha(c_1^2 + c_2^2) + c_1^2 c_2^2 + \Omega_2^4 + \Omega_1^4]\omega^4\\
& + [-8\,\alpha\,\Omega_1^2\Omega_2^2 + \Omega_1^4 c_2^2 - 2\,\Omega_2^4\Omega_1^2 - 2\,\alpha(\Omega_1^4 + \Omega_2^4) + c_1^2\Omega_2^4 - 2\,\Omega_1^4\Omega_2^2\\
& + 2\,\alpha(c_2^2\Omega_1^2 + c_1^2\Omega_2^2)]\omega^2 + (2\,\Omega_1^4\Omega_2^2\alpha + 2\,\Omega_1^2\Omega_2^4\alpha + \Omega_1^4\Omega_2^4)
\end{aligned}$$

which depends on the feedback gain α.

Let $\Omega_1 = 1, \Omega_2 = 4, c_1 = 0.1, c_2 = 0.2$, then

$$\begin{aligned}
F = {} & \omega^8 - (33.95 + 4\alpha)\omega^6 + (320.6004 + 101.90\alpha)\omega^4\\
& - (641.60\alpha + 541.40)\omega^2 + 256 + 544\alpha.
\end{aligned}$$

Numerical simulation (or using the complete discrimination system for polynomials [13]) shows that α should be greater than 0.7429 so that $F(\omega)$ has four positive roots: $\omega_1 > \omega_2 > \omega_3 > \omega_4$. The graph of $\partial F/\partial\omega = 0$ divides the (α,ω)-plane into several regions, and in each region $\partial F/\partial\omega$ has definite sign. As shown in Fig. 2, in the regions that the curves of $\omega_4(\alpha)$ and $\omega_2(\alpha)$ stay in, we have $\partial F/\partial\omega < 0$, while in the regions that the curves of $\omega_3(\alpha)$ and $\omega_1(\alpha)$ stay in, we have $\partial F/\partial\omega > 0$.

Now let $\alpha = 1 > 0.7429$, then $\omega_4 = 1.0029, \omega_3 = 1.6500, \omega_2 = 4.0413, \omega_1 = 4.2296$, and the four minimal critical delays are $\tau_{4,0} = 0.1002$, $\tau_{2,0} = 0.2181$, $\tau_{3,0} = 1.7665$ and $\tau_{1,0} = 0.5638$, respectively. Since $\max\{\tau_{4,0},\tau_{2,0}\} < \min\{\tau_{3,0},\tau_{1,0}\}$, so the unstable motion is stabilized if we chose $\alpha = 1$ and $\tau \in (0.2181, 0.5638)$. It is in very good agreement with the numerical simulation as shown in Fig. 3, by using Runge–Kutta method. Here, the initial conditions are given by $x_1(t) = 0.1, \dot{x}_1(t) = 0, x_2(t) = 0.1, \dot{x}_2(t) = 0$ $(\forall t \in [-\tau, 0])$, and the step-size of integration is taken as 0.01. Similarly, if we increase the value of α to 20, then $F(\omega)$ has four zeros $\omega_4 = 1.0005, \omega_3 = 2.7949, \omega_2 = 3.9967, \omega_1 = 9.4427$, and the corresponding minimal critical delay values are

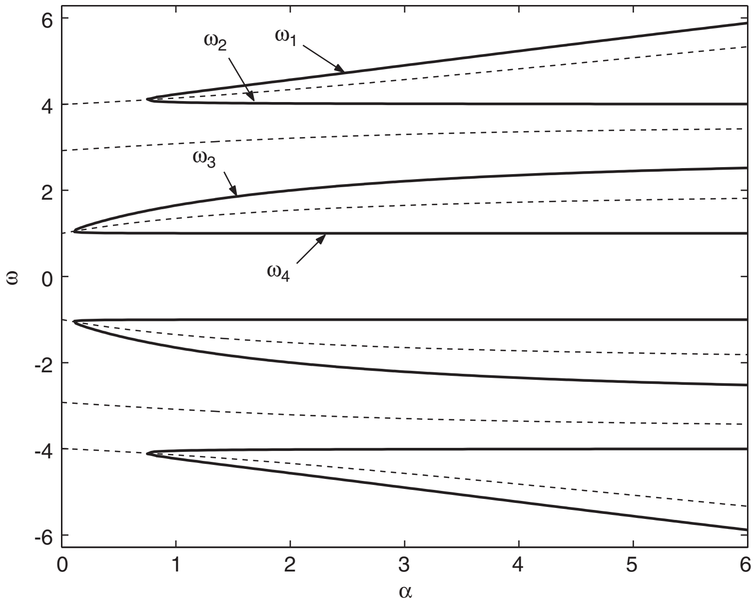

Fig. 2. The roots of the critical function $F(\omega)$ of Eq. (39) for $\Omega_1 = 1, \Omega_2 = 4, c_1 = 0.1, c_2 = 0.2$. Here $F(\omega)$ has four positive roots if $\alpha > 0.7429$, and the dotted curves are plotted by $dF/d\omega = 0$.

found to be $\tau_{4,0} = 0.0050, \tau_{3,0} = 0.7520, \tau_{2,0} = 0.0100, \tau_{1,0} = 0.3287$, respectively. Thus the trivial solution of the closed-loop system is asymptotically stable if $\tau \in (0.0100, 0.3287)$. Numerical simulation shows that the larger the value of α, the smaller the admissible value of the delay.

In general, if the characteristic quasi-polynomial is in the form

$$p(\lambda, \tau) = \sum_{k=0}^{m} q_k(\lambda) e^{-k\lambda\tau}, \quad (m \geqslant 2) \tag{40}$$

and if the control plant has $2N$ characteristic roots with positive real parts, we chose the feedback gains in this way such that the critical function $F(\omega)$ has exactly $2N$ positive zeros: $\omega_1, \omega_2, \ldots, \omega_{2N-1}, \omega_{2N}$, and compute the minimal critical delay value $\tau_{k,0}$ corresponding to each ω_k. In this case, the sign of $F'(\omega_k)$ is not enough to determine the changing direction of the characteristic roots as the delay varies, the contribution of the sign of $\Pi(i\omega) = q_0^{(1)}(i\omega) \cdots q_0^{(m-1)}(i\omega)$ must be taken into consideration due to Eq. (14). The unstable solution is stabilized only if there are exactly N vibration frequencies, say $\omega_1, \omega_2, \ldots, \omega_N$ without loss of generality, that satisfy $\Pi(i\omega_k)F'(\omega_k) < 0$, and the other N vibration frequencies $\omega_{N+1}, \omega_{N+2}, \ldots, \omega_{2N}$ satisfy $\Pi(i\omega_k)F'(\omega_k) > 0$. The trivial solution of the controlled system is asymptotically stable for all $\tau \in (\max\{\tau_{1,0}, \tau_{2,0}, \ldots, \tau_{N,0}\}, \min\{\tau_{N+1,0}, \tau_{N+2,0}, \ldots, \tau_{2N,0}\})$.

5. Conclusions

In this paper, the stabilization problem under delayed feedback of Pyragas type is discussed for the linear dynamical systems that have even number of characteristic roots with positive real part. From the viewpoint of stability switches, as the delay increases from zero to infinity, the closed-loop system experiences stability changes from being unstable to being stable, or from being stable to being unstable, and eventually to being unstable. When the delay passes from a critical delay value, the system increases or decreases *two* new conjugate characteristic roots with positive real parts, this implies that a delayed feedback control *cannot* stabilize the unstable motion of a system that has *odd* number of characteristic roots with positive real parts. If the control plant has $2N$ characteristic roots with positive real parts, the first step of our stabilization

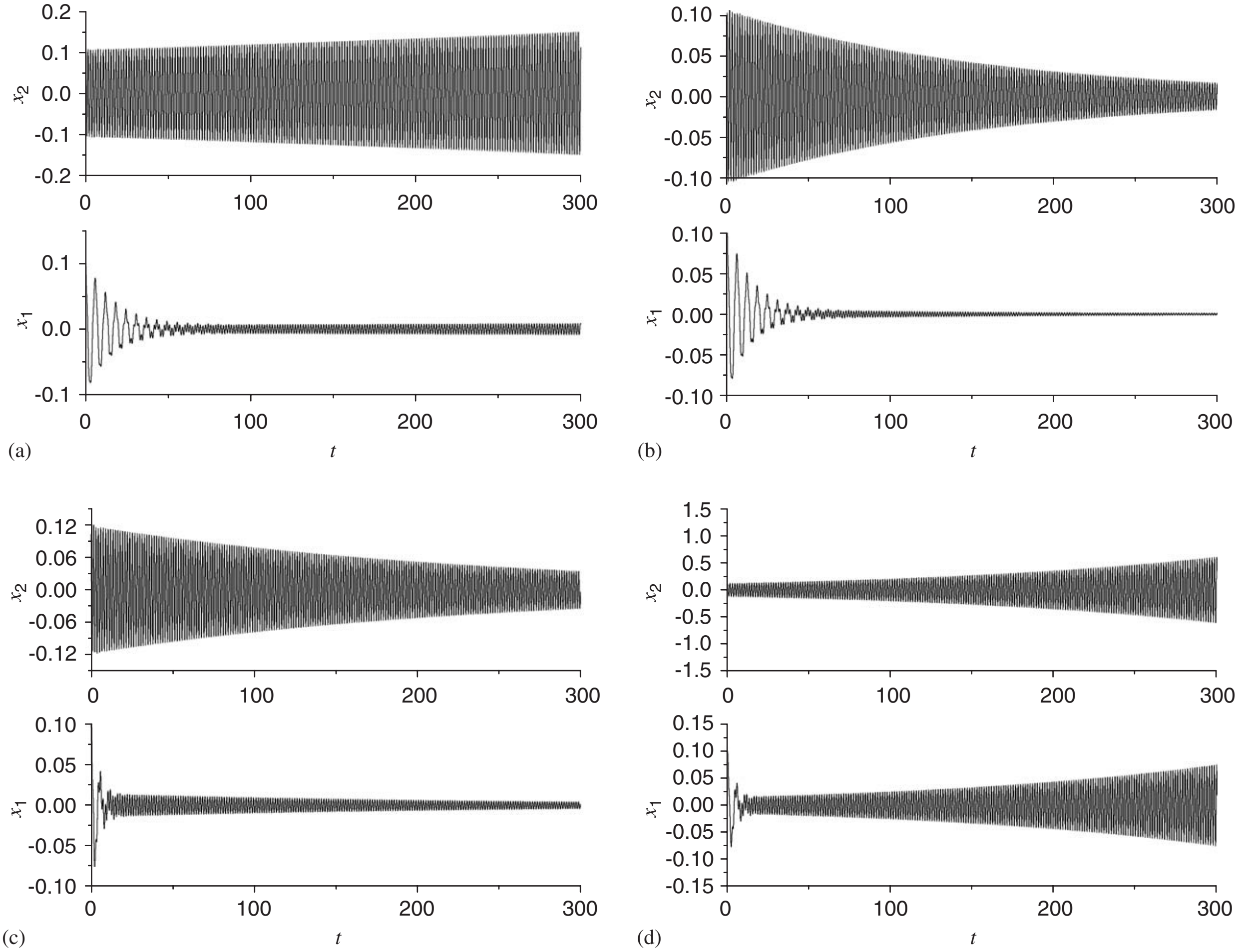

Fig. 3. The time histories of Eq. (39) for: $\alpha = 1$; (a) $\tau = 0.21$; (b) $\tau = 0.23$; (c) $\tau = 0.55$; and (d) $\tau = 0.57$.

procedure is to determine the admissible feedback gains such that the critical function $F(\omega)$ has exactly $2N$ positive roots (vibration frequencies), and then to find out all the minimal critical values of the delay corresponding to all the vibration frequencies. The second step is to determine the changing direction of the characteristic roots as the delay passes through the critical values. In general, the changing direction depends not only on the critical function $F(\omega)$, but also on the coefficient function $\Pi(\mathrm{i}\omega)$. The system can be stabilized only if the delay firstly passes through the N minimal critical delay values corresponding to the N vibration frequencies, at which the characteristic root goes through the imaginary axis of the complex plane from the right to the left. The peculiarity of this paper is that the present stabilization method is constructive, easier, and computational tractable in finding out the admissible feedback gains and the admissible values of the delay, as shown in the two demonstrative examples.

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