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A non-asymptotic model of dynamics of honeycomb lattice-type plates

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Abstract

Lightweight structures, consisted of special composite material systems like sandwich plates, are often used in aerospace or naval engineering. In composite sandwich plates, the intermediate core is usually made of cellular structures, e.g. honeycomb micro-frames, reinforcing static and dynamic properties of these plates. Here, a new non-asymptotic continuum model of honeycomb lattice-type plates is shown and applied to the analysis of dynamic problems. The general formulation of the model for periodic lattice-type plates of an arbitrary lay-out was presented by Cielecka and Jędrysiak [Journal of Theoretical and Applied Mechanics 40 (2002) 23–46]. This model, partly based on the tolerance averaging method developed for periodic composite solids by Woźniak and Wierzbicki [Averaging techniques in thermomechanics of composite solids, Wydawnictwo Politechniki Częstochowskiej, Częstochowa, 2000], takes into account the effect of the length microstructure size on the dynamic plate behaviour. The shown method leads to the model equations describing the above effect for honeycomb lattice-type plates. These equations have the form similar to equations for isotropic cases. The dynamic analysis of such plates exemplifies this effect, which is significant and cannot be neglected. The physical correctness of the obtained results is also discussed.

1. Introduction

In this paper, the formulation and application of a non-asymptotic continuum model to the study of dynamics of linear-elastic lattice-type plates having a honeycomb structure in $0x_1x_2$-plane (cf. Fig. 1) are shown. The presented model is a special case of the model derived by Cielecka and Jędrysiak [1] for lattice-type plates having a periodic structure of an arbitrary lay-out. Plates of this kind consist of a very large number of small cells separated by prismatic, slender beams in plane $0x_1x_2$. The length dimensions of the cell are assumed to be small in comparison with the minimum characteristic length dimension of the whole plate. The effect of the cell length size, which is related to the periodic structure of the lattice-type plate and also called also *the length-scale effect*, is very interesting to describe in the dynamic behaviour of such plate.

The mass distribution of this plate is assumed in the form of concentrated masses, and inertia moments assigned to every nodal joint of a lattice. By this means, the considered lattice-type plate is represented by a

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Nomenclature

a, b superscripts, which run over $1,\ldots,n$
A superscripts, which run over $1,\ldots,N$
$\mathscr{A}, \mathscr{E}, \mathscr{K}, \mathscr{W}$ overall energy, the strain energy, the kinetic energy, the external loading work of the lattice-type plate, respectively
b, h width, the mean height of beams
B^A prismatic linear-elastic beams in the undeformed representative element, with axes situated on the plane $0x_1x_2$, $A = 1,\ldots,N$; cf. Figs. 2 and 3
E^A Young's modulus of beam B^A
$f^a(\mathbf{z},t)$ resultant external force applied to the joint $\mathscr{J}^a$ in a cell with a centre $\mathbf{z} \in \mathscr{L}$
F^A cross-section area of beam B^A
G^A Kirchhoff's (shear) modulus of beam B^A
$h^{a\alpha}$, $\mathbf{g}^{a\alpha}$ systems of numbers called shape parameters, describing forms of oscillations in the periodicity cell Δ
i, j, k, l subscripts, which run over 1, 2, related to the system $0x_1x_2$
I^A moment of inertia of beam B^A
I^A_o central moment of inertia of beam B^A
$\mathscr{J}^a$ rigid joints with centres at points $\mathbf{x} = \mathbf{x}^a = (x_1^a, x_2^a)$, connecting beams B^A, $a = 1,.., n$; cf. Figs. 2 and 3
$\mathbf{J}^a$ second-order tensor of the rotational moment of inertia of joint $\mathscr{J}^a$
k wavenumber, $k = 2\pi/L$
$\mathbf{K}$, $\mathbf{L}$ tensors with components $K_{\cdot ij}$, $L_{kl\cdot}$ (dots are the possible sequences of subscripts)
$\mathbf{K}\otimes\mathbf{L}$, $\mathbf{K}\cdot\mathbf{L}$, $\mathbf{K}:\mathbf{L}$ objects having components $K_{\cdot ij}L_{kl\cdot}$, $K_{\cdot ij}L_{jk\cdot}$, $K_{\cdot ij}L_{ij\cdot}$, respectively
l *the microstructure length parameter*—characteristic length of cell Δ, defined as the square root of microstructure length parameter—the area $|\Delta|$ of the cell, $l \equiv \sqrt{|\Delta|}$ and assumed that $l/L_{\min} \ll 1$
$L_{\min}$ smallest characteristic length dimension of Ξ, cf. Fig. 1
L span of the honeycomb lattice-type plate along $x = x_1$ axis
$\mathscr{L}$ set of all periodically situated points on $0x_1x_2$ being centres of all mutually disjoined cells constituting the region Ξ
$\mathbf{m}^a(\mathbf{z},t)$ resultant external couples applied to the joint $\mathscr{J}^a$ in a cell with a centre $\mathbf{z} \in \mathscr{L}$
M^a total concentrated mass assigned to joint $\mathscr{J}^a$
n number of rigid joints in a representative element
N number of prismatic linear-elastic beams in a representative element
N_a number of beams for which a rigid joint $\mathscr{J}^a$ is the end
$PL(T)$ set of periodic-like functions in the problem under consideration
$PL^\mu(T)$ set of oscillating functions with a positive valued periodic weight-function μ in the problem under consideration
q dimensionless wavenumber
Q^α, $\mathbf{R}^\alpha$ extra kinematic unknowns called the internal kinematic variables, related to the deflection and the rotations, respectively
s^A span of beam B^A, cf. Figs. 2 and 3
$SV(T)$ set of slowly varying functions in the problem under consideration
t time coordinate
$\mathbf{t}^A$, $\mathbf{n}^A$ unit vectors assigned to every beam B^A, cf. Fig. 2
$T = (\mathtt{F}, \varepsilon(\cdot))$ pair called the tolerance system
v^a, $\mathbf{r}^a$ fluctuating parts of deflection and rotations, respectively
$\mathbf{V}$ new internal variable replacing the variable $\mathbf{R}^1$
$w^a(\mathbf{z},t)$ deflection vector of the joint $\mathscr{J}^a$, belonging to a cell with the centre $\mathbf{z}$, $\mathbf{z} \in \mathscr{L}$, at an arbitrary instant t
W, $\mathbf{\Phi}$ averaging parts of the deflection and rotations, called the macrodeflection and macrorotations, respectively
$\mathbf{x} = (x_1,x_2)$ points on the $0x_1x_2$-plane
$0x_1x_2$ Cartesian orthogonal coordinate system on the plane
α, β superscripts, which run over $1,\ldots,n-1$
$\varepsilon(\cdot)$ mapping $\mathtt{F} \ni f \to \varepsilon_f \mathbb{R}^+$
$\tilde{\varepsilon}^A, \kappa^A, \tilde{\kappa}^A$ strain components related to B^A
ε_f a constant tolerance parameter for a continuous real valued function f defined on $\bar{\Xi}$
σ^A strain energy assigned to beam B^A
ρ^A mass density of beam B^A
ν^A Poisson's ratio of beam B^A
∂_i partial differentiation of x_i, also denoted by $(\cdot)_{,i}$
$\partial\Xi$ boundary of a periodic structure, cf. Fig. 1

$\boldsymbol{\varphi}^a(\mathbf{z},t)$	rotation vector of the joint j^a, belonging to a cell with the centre $\mathbf{z}$, $\mathbf{z} \in \mathscr{L}$, at an arbitrary instant t
φ	angle denoted in Fig. 3
ω_-	lower ("fundamental") free vibration frequency
ω_+	higher free vibration frequency, related to a periodic plate structure
Ω_-, Ω_+	non-dimensional frequency parameters for a lower and a higher free vibration frequency, respectively
Δ	parallelogram on the $0x_1x_2$-plane constituting a cell representative of a whole periodic lattice-type plate, cf. Fig. 1
Ξ	region on $0x_1x_2$-plane being an interior of a union of all closures of repeated cells, cf. Fig. 1
$\nabla \mathbf{K}$	gradient of arbitrary smooth field $\mathbf{K}(\cdot)$ (with components $\partial_i K_{kl\cdot}$)
$\nabla \cdot \mathbf{K}$	divergence of the field $\mathbf{K}(\cdot)$ (with components $\partial_i K_{ij\cdot}$)

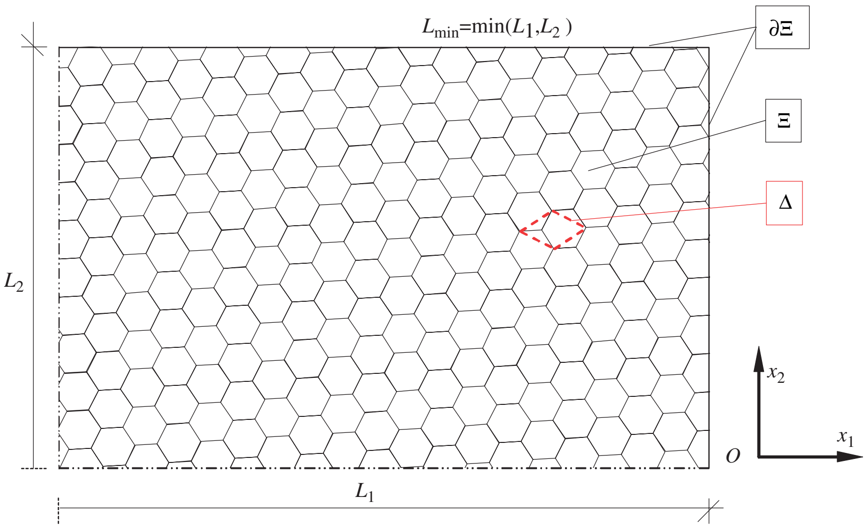

Fig. 1. A fragment of periodic lattice-type plate having honeycomb structure with the periodicity cell Δ, the interior region of the plate Ξ, the boundary of the plate $\partial\Xi$, the smallest characteristic length dimension of the plate $L_{\min}$.

certain plane periodic system of mutually interacted rigid joints. A review of discrete models describing dynamics of periodic systems of concentrated masses was discussed by Brillouin [2], where the effect of the cell length size was shown as additional higher-order frequencies. However, the direct approach to dynamics of periodic systems consisting of a very large number of rigid bodies meets computational difficulties (e.g. necessity of solving a large number of differential equations). Hence, in order to simplify the analysis of special problems, different averaged continuum models were proposed. It can be mentioned those, related to the frame-type lattice structures, described by governing equations with couple stresses and summarized by Woźniak [3]; models for cellular media, e.g. Gibson et al. [4]. Some in-plane problems were analysed by Lewiński [5–8], where Rogula–Kunin's approach (cf. Ref. [9]) was applied as a tool of modelling to hexagonal gridworks.

A typical mathematical modelling approach for periodic lattice-type structures is based on the asymptotic procedures of the homogenization theory, cf. Refs. [10,11], where a small parameter describing the size of the periodic cell is introduced. Continuum models using asymptotic homogenization methods were applied e.g. by Cioranescu and Saint Paulin [12,13]. Using these approaches, the periodic solids or structures are approximated by certain equivalent homogeneous media. The obtained so-called effective modulae describe the averaged properties and the overall behaviour of the medium can be analysed. Unfortunately, in homogenization methods, all length dimensions of the cell tend to zero and a number of the cells tends to infinity, hence derived model equations neglect the effect of the cell size on the global behaviour of periodic systems. Thus many interesting problems cannot be investigated; for example, those related to the effects of the cell size appeared in the dynamics of composite materials, e.g. higher-order vibrations related to periodic structure.

Certain non-asymptotic approaches to analyse periodic structures were proposed e.g. by Mead [14], Engels and Meirovitch [15]. In Ref. [14], a general theory of harmonic wave propagation in periodic systems of one or two dimensions was shown. Equations of motion formulated for a periodic element within a multi-coupled periodic system describe the motion in the element, which was qualified by a set of generalized coordinates, associated with a real wave mode. The combined analysis, based on the application of this theory to periodic structures and the finite element model to the unit cell, was presented for the two-dimensional periodic cellular structures e.g. by Langley et al. [16] (to beam grillages) and by Ruzzene et al. [17] (to grids with hexagonal cells). In these works, the characteristics of wave propagation for those structures were investigated. However, also in the aforementioned papers, problems of the length-scale effect related to periodic structure, e.g. higher-order vibrations, were not analysed. Although, it seems that employing the above approaches, these problems can be investigated.

In the last decade, an alternative non-asymptotic modelling method was proposed and applied to the analysis of dynamic phenomena for both discrete and continuous periodic structures and also periodic composites in a series of papers: Baron [18], Cielecka [19], Cielecka et al. [20,21], Cielecka and Jędrysiak [1], Jędrysiak [22,23], Mazur-Śniady et al. [24], Michalak [25], Wierzbicki and Woźniak [26] and others. The aforementioned approach was called *the tolerance averaging method* and summarized in the book by Woźniak and Wierzbicki [27]. In the modelling procedure of this averaging technique, the concept of tolerance related to the accuracy of the performed calculations is introduced. For vibrations and wave propagation problems of periodic solids, this concept leads to additional unknowns, usually called kinematic internal variables, describing together with the averaged displacement fields the dynamic behaviour of the solid [27]. The main advantage of the tolerance averaging method is the wide variety of special problems of the overall behaviour of periodic composite media, which can be analysed, e.g. *the length-scale effects*.

The main aim of this contribution is to investigate the dynamic behaviour of a periodic lattice-type plate having a honeycomb structure, taking into account the length-scale effect on vibrations of the plate. On account of the mentioned above problem, the governing equations of the new non-asymptotic model of dynamics for periodic lattice-type plates will be applied, which were derived in Ref. [1]. These equations have been obtained by means of the modelling procedure being a certain adaptation of the tolerance averaging method [27] to these plates. For periodic lattice-type plates, this procedure leads to equations with constant coefficients, which involve additional kinematic internal variables [1,27]. These additional kinematic unknowns are treated as certain amplitudes of displacement fluctuations in the periodicity cell. It will be shown that the non-asymptotic model equations for honeycomb-type plates have the form similar to equations for isotropic cases, i.e. these equations have isotropic coefficients, cf. Ref. [21]. Thus, the overall response of a honeycomb structure of the lattice-type plate is isotropic, similarly as in honeycomb cellular media, cf. Ref. [7]. In order to make the paper self-contained, the foundations of the tolerance averaging method and the internal variable model, cf. Ref. [1], are outlined in the subsequent section. The obtained continuum model takes into account the length-scale effect on the global behaviour of plate under consideration and gives the possibility of analysing additional higher-order vibrations related to a periodic plate structure. As an application of the model, the additional higher-order free vibration frequencies of the plate band with honeycomb lattice structure will be derived at an example. The proposed model is useful to the analysis of long wave propagation problems. Moreover, a comparison between obtained results and the "exact" solutions, calculated similar to Ref. [2], will be shown and treated as a certain physical correctness of the model.

2. Modelling approach

2.1. Preliminaries

Denote by i, j, k, l subscripts, which run over 1, 2 and are related to Cartesian orthogonal coordinates x_1, x_2 in the $0x_1x_2$-plane. Let indices a, b and A run over $1,\ldots,n$ and $1,\ldots,N$, respectively; indices α, β take the values

1,…, $n-1$. Summation convention holds for all the aforementioned indices unless otherwise stated. Points on the $0x_1x_2$-plane are denoted by $\mathbf{x} = (x_1, x_2)$ and t is the time coordinate. For tensors $\mathbf{K}$, $\mathbf{L}$ (with components $K_{\cdot ij}$, $L_{kl\cdot}$—dots are the possible sequences of subscripts) by $\mathbf{K}\otimes\mathbf{L}$, $\mathbf{K}\cdot\mathbf{L}$ and $\mathbf{K}$:$\mathbf{L}$, objects having components $K_{\cdot ij}L_{kl\cdot}$, $K_{\cdot ij}L_{jk\cdot}$ and $K_{\cdot ij}L_{ij\cdot}$, respectively, will be defined. The gradient of arbitrary smooth field $\mathbf{K}(\cdot)$ is denoted by $\nabla\mathbf{K}$ (with components $\partial_i K_{kl\cdot}$). Symbol $\nabla\cdot\mathbf{K}$ is the divergence of the field $\mathbf{K}(\cdot)$ (with components $\partial_i K_{ij\cdot}$) and ∂_i is the partial differentiation of x_i, also denoted by $(\cdot)_{,i}$.

By Δ denote a parallelogram on the $0x_1x_2$-plane constituting a cell representative of a whole periodic lattice-type plate, cf. Fig. 1. It can be observed that Δ contains the representative structural element for the plate. In general, this element can include one, two or several periodicity cells. The choice of this element is not unique and is dependent on the class of motions, which are investigated. The undeformed representative element is assumed to be made of N prismatic linear-elastic beams B^A (cf. Fig. 2), $A = 1,\ldots, N$, with axes situated on the plane $0x_1x_2$. For the smallest cell of the honeycomb-type plate, it is $N = 3$, cf. Figs. 1 and 3. The beams B^A in the representative cell are interconnected by rigid joints j^a (cf. Fig. 2), $a = 1,\ldots, n$, with centres at points $\mathbf{x} = \mathbf{x}^a = (x_1^a, x_2^a)$ on the plane $0x_1x_2$. The length dimension in this plane of every rigid joint is assumed to be negligibly small as compared with the spans of interconnecting beams. Moreover, the plane $0x_1x_2$ is a symmetry plane both for every beam and every rigid joint, treated as certain spatial (three-dimensional) elements. It is assumed that the beams are bent and twisted in planes perpendicular to $0x_1x_2$-plane and the rigid joints rotate in these planes; centres of joints displace in the direction normal to $0x_1x_2$-plane.

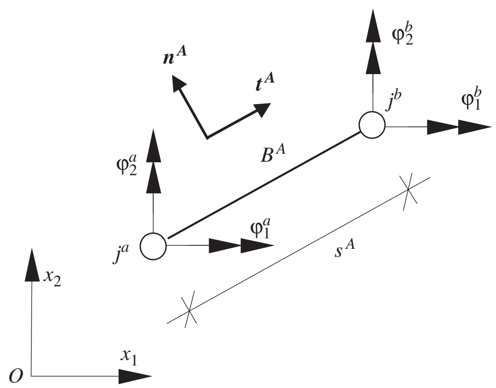

Fig. 2. Unit vectors $\mathbf{n}^A$, $\mathbf{t}^A$ and rigid joints j^a, j^b assigned to a beam B^A.

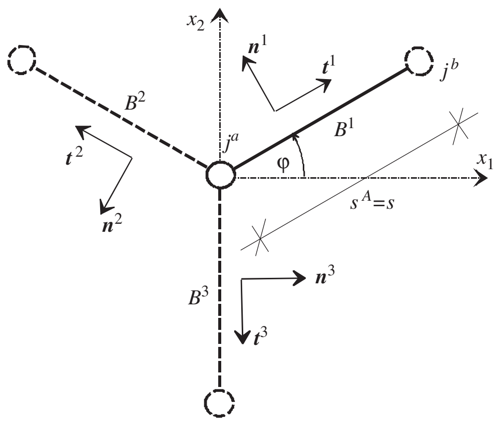

Fig. 3. The periodicity cell of honeycomb lattice-type plate.

Let Ξ denote a region on $0x_1x_2$-plane being an interior of a union of all closures of repeated cells, cf. Fig. 1. It should be emphasized that the periodic structure of the whole lattice-type plate can be disturbed in the structural elements situated near the boundary $\partial\Xi$ (cf. Fig. 1) of Ξ. These considerations are related to the so-called bulk region and therefore the effect of a boundary layer is ignored. Denote by $L_{\min}$ the smallest characteristic length dimension of Ξ and by l the characteristic length of cell Δ, which is defined as the square root of the area $|\Delta|$ of cell, $l \equiv \sqrt{|\Delta|}$. It will be assumed that $l/L_{\min} \ll 1$. Hence, the length l will be called *the microstructure length parameter* of the lattice-type plate.

Properties of beam B^A are described by the flexural stiffness $E^A I^A$, the torsional stiffness $G^A I_o^A$, the span s^A, the mass density ρ^A, the Poisson's ratio v^A and the cross-section area F^A. It is assumed that the mass of beam B^A is equally distributed at the beam ends (joints) as two equal concentrated masses. Moreover, the rotational moment of inertia of joint j^a is represented by the second-order tensor $\mathbf{J}^a$. Let unit vectors $\mathbf{t}^A$, $\mathbf{n}^A$ be assigned to every beam B^A, cf. Fig. 2. In order to describe a kinetic energy of a beam by velocities of deflections and rotations of its ends [1], the total concentrated mass M^a assigned to joint j^a is given by

$$M^a = \frac{1}{2}\sum_{A=1}^{N_a} \rho^A F^A s^A, \tag{1}$$

and the tensor $\mathbf{J}^a$ of the rotational moment of inertia of joint j^a can be taken in the form [1]

$$\mathbf{J}^a = \frac{1}{2}\sum_{A=1}^{N_a} \rho^A s^A [I^A(\mathbf{n}^A \otimes \mathbf{n}^A) + I_0^A(\mathbf{t}^A \otimes \mathbf{t}^A)], \tag{2}$$

where N_a is the number of beams ending in a rigid joint j^a. For honeycomb plates it is $N_a = 3$.

Let $\mathscr{L}$ be a set of all periodically situated points on $0x_1x_2$ being centres of all mutually disjoined cells constituting the region Ξ. Then, let a deflection and rotation vector of the joint j^a, belonging to a cell with the centre $\mathbf{z}$, $\mathbf{z} \in \mathscr{L}$, at an arbitrary instant t, be denoted by $w^a(\mathbf{z},t)$ and $\boldsymbol{\varphi}^a(\mathbf{z},t)$, respectively. All external loads are assumed to be applied exclusively to the centres of rigid joints. The resultant external force and external couples applied to the joint j^a in a cell with a centre $\mathbf{z} \in \mathscr{L}$ are denoted by $f^a(\mathbf{z},t)$ and $\mathbf{m}^a(\mathbf{z},t)$, respectively. It is also assumed that every beam B^A, interconnecting rigid joints j^a and j^b, is considered in the framework of the Euler–Bernoulli beam theory. The strain components related to B^A will be taken in the following form (no summation over A in Eqs. (3)–(5)):

$$\tilde{\varepsilon}^A \equiv (w^b - w^a)/s^A + 0.5(\boldsymbol{\varphi}^a + \boldsymbol{\varphi}^b)\cdot \mathbf{n}^A, \quad \kappa^A \equiv (\boldsymbol{\varphi}^b - \boldsymbol{\varphi}^a)\cdot \mathbf{n}^A, \quad \tilde{\kappa}^A \equiv (\boldsymbol{\varphi}^b - \boldsymbol{\varphi}^a)\cdot \mathbf{t}^A. \tag{3}$$

Using notations

$$\tilde{\Lambda}^A \equiv 12E^A I^A (s^A)^{-1}, \quad K^A \equiv E^A I^A (s^A)^{-1}, \quad \tilde{K}^A \equiv G^A I_o^A (s^A)^{-1}, \tag{4}$$

the strain energy σ^A assigned to beam B^A is equal to

$$\sigma^A = \tfrac{1}{2}[\tilde{\Lambda}^A(\tilde{\varepsilon}^A)^2 + K^A(\kappa^A)^2 + \tilde{K}^A(\tilde{\kappa}^A)^2]. \tag{5}$$

It is necessary to emphasize that all denotations and formulae mentioned above are related to an arbitrary but fixed repeated element of the periodic lattice-type plate (except some elements situated near boundary $\partial\Xi$ of Ξ).

Introduce the Hamiltonian $\mathscr{A} = \mathscr{E} - \mathscr{K} - \mathscr{W}$, where

$$\begin{aligned}
\mathscr{E} &= \frac{1}{2}\sum_{\mathbf{z}\in\mathscr{L}}\sum_{A=1}^{N}\left\{\tilde{\Lambda}^A\left[\tilde{\varepsilon}^A(\mathbf{z},t)\right]^2 + K^A\left[\kappa^A(\mathbf{z},t)\right]^2 + \tilde{K}^A\left[\tilde{\kappa}^A(\mathbf{z},t)\right]^2\right\},\\
\mathscr{K} &= \frac{1}{2}\sum_{\mathbf{z}\in\mathscr{L}}\sum_{a=1}^{n}\{M^a[\dot{w}^a(\mathbf{z},t)]^2 + [\dot{\boldsymbol{\varphi}}^a(\mathbf{z},t)\otimes \dot{\boldsymbol{\varphi}}^a(\mathbf{z},t)] : \mathbf{J}^a\},\\
\mathscr{W} &= \sum_{\mathbf{z}\in\mathscr{L}}\sum_{a=1}^{n}[f^a(\mathbf{z},t)\, w^a(\mathbf{z},t) + \mathbf{m}^a(\mathbf{z},t)\cdot \boldsymbol{\varphi}^a(\mathbf{z},t)].
\end{aligned} \tag{6}$$

Taking into account formulae (3) from the principle of stationary action, the system of equations for $w^a(\mathbf{z},t)$, $\boldsymbol{\varphi}^a(\mathbf{z},t)$, $\mathbf{z}\in\mathscr{L}$, $a=1,\ldots,n$, $i=1$, 2, representing a discrete model of a periodic lattice-type plate, can be derived. However, these equations are not convenient for investigations of the plate overall dynamic behaviour because the number of points $\mathscr{L}$ is very large and therefore relations (3),(4), (6) together with assumptions formulated in the subsequent section will be treated only as a basis for deriving the governing equations of a continuum model of the lattice-type plate under consideration.

2.2. The tolerance averaging method

Below, some concepts related to the tolerance averaging method and defined in Ref. [27] will be reminded.

A concept of *a tolerance* is introduced as the binary relation (denoted by "$\approx$") defined on a certain non-empty set $\mathtt{A}$, which is reflexive, symmetric and not transitive. Assuming that $\mathtt{A}$ is a set of real numbers $\mathbb{R}$ with a unit measure and ε is a positive number determining the accuracy of computations of elements $\mathtt{A}$, for every a_1, $a_2\in\mathtt{A}$ it holds $a_1 \overset{\varepsilon}{\approx} a_2 \Leftrightarrow |a_1-a_2|\leqslant\varepsilon$, where ε is a constant *tolerance parameter*.

Denote by $\mathtt{F}$ the set of all continuous real valued functions f defined on $\bar{\Xi}\equiv\Xi\cup\partial\Xi$ (and their derivatives) in the problem under consideration, which values are determined within the known tolerance and satisfying the following condition: for every points $\mathbf{x}$, $\mathbf{y}\in\bar{\Xi}$, it holds $f(\mathbf{x})\overset{\varepsilon_f}{\approx}f(\mathbf{y})\Leftrightarrow|f(\mathbf{x})-f(\mathbf{y})|\leqslant\varepsilon_f$. It is a domain of the mapping $\mathtt{F}\ni f\to\varepsilon_f\in\mathbb{R}^+$, denoted by $\varepsilon(\cdot)$. The pair $T=(\mathtt{F},\varepsilon(\cdot))$ is called *the tolerance system*, cf. Ref. [27].

For the known certain tolerance system $T=(\mathtt{F},\varepsilon(\cdot))$ and a cell Δ the concepts of *a slowly varying* function and a *periodic-like* function are introduced. Let $\Phi\in\mathtt{F}$ be a continuous function defined on $\bar{\Xi}$. The function Φ will be called *a slowly varying* function, if for every points $\mathbf{x},\mathbf{y}\in\bar{\Xi}$ it holds the following condition $||\mathbf{x}-\mathbf{y}||\leqslant l\Rightarrow\Phi(\mathbf{x})\overset{\varepsilon_\Phi}{\approx}\Phi(\mathbf{y})$, where $||\mathbf{x}–\mathbf{y}||$ is a distance between points $\mathbf{x}$, $\mathbf{y}$. If Φ with all derivatives (also time derivatives) are slowly varying functions it will be written as $\Phi\in SV(T)$.

Let $\phi\in\mathtt{F}$ be a continuous function and for every $\mathbf{x}\in\Xi$, a symbol $\phi_\mathbf{x}$ be a certain continuous periodic function. A continuous function $\phi\in\mathtt{F}$ will be called *a periodic-like* function if for every $\mathbf{x}\in\Xi$ such a $\phi_\mathbf{x}\in\mathtt{F}$ exists that for every $\mathbf{y}\in\Xi$ it holds $||\mathbf{x}-\mathbf{y}||\leqslant l\Rightarrow\phi(\mathbf{y})\overset{\varepsilon_\phi}{\approx}\phi_\mathbf{x}(\mathbf{y})$. If derivatives of ϕ hold similar conditions, it will be written as $\phi\in PL(T)$. The function $\phi_\mathbf{x}$ is called a periodic approximation of ϕ. Moreover, if ϕ is a periodic-like function and the condition $\langle\mu\phi\rangle(\mathbf{x})\overset{\varepsilon}{\approx}0$ is satisfied for every $\mathbf{x}\in\Xi$, where μ is a positive valued periodic function, ϕ will be called *an oscillating* function, $\phi\in PL^\mu(T)$.

The above concepts with lemmas formulated and proved in Ref. [27] are the mathematical background of *the tolerance averaging method*, applied to analyse higher-order vibrations for continuous periodic structures and composites [18,22–26]. In several papers, the tolerance averaging method has been adapted to dynamics of discrete periodic structures such as plane periodic lattice-type structures [19], two-dimensional periodic cellular media [20,21] and periodic lattice-type plates of an arbitrary lay-out [1]. In this paper, the modelling of honeycomb lattice-type plates and an analysis of higher-order vibrations of the plates will be proposed.

2.3. The tolerance averaging model

Now, a passage from the discrete model of the periodic lattice-type plate to a certain non-asymptotic continuum model, based on the tolerance averaging will be presented [27].

The main assumption of the tolerance averaging is *the Conformability Hypothesis*. It is assumed that the deflection $w^a(\mathbf{z},t)$ and the rotations $\boldsymbol{\varphi}^a(\mathbf{z},t)$ of the rigid joint J^a in a cell with the centre $\mathbf{z}$, $\mathbf{z}\in\mathscr{L}$, have to be conformable to a periodic structure of the lattice-type plate, i.e. these are periodic-like functions for any time t, $w^a(\cdot,t)$, $\boldsymbol{\varphi}^a(\cdot,t)\in PL(T)$.

In order to derive the governing equations of the non-asymptotic continuum model [1], the applied modelling procedure can be shown in five steps.

(1) The plate deflection $w^a(\cdot,t)$ and rotations $\boldsymbol{\varphi}^a(\cdot,t)$ are decomposed:

$$w^a(\mathbf{z},t)=W(\mathbf{x}^a,t)+\mathrm{v}^a(\mathbf{x}^a,t),\quad \boldsymbol{\varphi}^a(\mathbf{z},t)=\boldsymbol{\Phi}(\mathbf{x}^a,t)+\mathbf{r}^a(\mathbf{x}^a,t),\quad \mathbf{z}\in\mathscr{L},\tag{7}$$

where W and $\mathbf{\Phi} \in SV(T)$ are averaging parts of the deflection and rotations, called *the macrodeflection* and *macrorotations*, respectively, and defined as

$$W(\mathbf{z},t) = M^{-1}M^a w^a(\mathbf{z},t), \quad \mathbf{\Phi}(\mathbf{z},t) = \mathbf{J}^{-1} \cdot \boldsymbol{\varphi}^a(\mathbf{z},t) \cdot \mathbf{J}^a, \quad \mathbf{z} \in \mathscr{L}$$

with denotations $M = \sum_{a=1}^{n} M^a$, $\mathbf{J} = \sum_{a=1}^{n} \mathbf{J}^a$; functions v^a, $\mathbf{r}^a \in PL^\mu(T)$ are *the fluctuating parts of deflection* and *rotations*, which hold the normalizing conditions $M^a\ \mathrm{v}^a(\mathbf{z},t) = 0, \mathbf{J}^a \cdot \mathbf{r}^a(\mathbf{z},t) = 0,\ \mathbf{z} \in \mathscr{L}$, in dynamic problems. Because these considerations are related to the bulk region, the effect of a boundary layer in Eq. (7) is ignored.

(2) A certain *periodic problem* on the periodicity cell Δ is formulated for periodic functions $\mathrm{v}^a_\mathbf{x}$ and $\mathbf{r}^a_\mathbf{x}$, being local periodic approximations of the fluctuations of deflection v^a and rotations $\mathbf{r}^a$ at $\mathbf{x} \in \Xi$. Solutions to this problem are being looked for in the form of finite series:

$$\mathrm{v}^a \cong \mathrm{v}^a_\mathbf{x} \simeq l h^{a\alpha} Q^\alpha(\mathbf{x}^a,t), \quad \mathbf{r}^a \cong \mathbf{r}^a_\mathbf{x} \simeq l \mathbf{g}^{a\alpha} \cdot \mathbf{R}^\alpha(\mathbf{x}^a,t), \tag{8}$$

where $h^{a\alpha}$, $\mathbf{g}^{a\alpha}$ are *systems of numbers*, also called *shape parameters*, satisfying conditions:

$$\sum_{a=1}^{n} M^a h^{a\alpha} = 0, \quad \sum_{a=1}^{n} \mathbf{J}^a \cdot \mathbf{g}^{a\alpha} = 0, \tag{9}$$

$\alpha = 1, \ldots, n-1$; $a = 1, \ldots, n$; and l is the given a priori microstructure length parameter of the periodic lattice-type plate under consideration. It should be emphasized that the systems $h^{a\alpha}$, $\mathbf{g}^{a\alpha}$ are not uniquely determined but their choice will be irrelevant, and functions Q^α and $\mathbf{R}^\alpha$ are extra kinematic unknowns. Because functions $\mathrm{v}^a_x(\cdot,t)$ and $\mathbf{r}^a_\mathbf{x}(\cdot,t)$ are periodic approximations of $\mathrm{v}^a(\cdot,t)$ and $\mathbf{r}^a(\cdot,t)$ in $\Delta(\mathbf{x})$ and $\mathrm{v}^a_\mathbf{x}(\cdot,t)$, $\mathbf{r}^a(\cdot,t) \in PL^\mu(T)$, then from Eq. (8) it follows that $Q^\alpha(\cdot,t)$, $\mathbf{R}^\alpha(\cdot,t) \in SV(T)$.

It has to be emphasized that the research of solutions to the periodic problem on the periodicity cell Δ in the form (8) in the framework of the tolerance averaging method is different than in the well-known methods of the asymptotic homogenization, where these solutions are looked for in the form of power series of a small parameter describing the length size of the periodicity cell.

In the proposed model, the shape parameters $h^{a\alpha}$, $\mathbf{g}^{a\alpha}$ are interpreted as forms of oscillations in the periodicity cell Δ, while the additional unknowns Q^α, $\mathbf{R}^\alpha$ are treated as amplitudes of these oscillations, i.e. of the fluctuations of deflection v^a and rotations $\mathbf{r}^a$.

(3) Finite differences of the basic unknowns $W(\cdot,t)$, $\mathbf{\Phi}(\cdot,t)$, $Q^\alpha(\cdot,t)$, $\mathbf{R}^\alpha(\cdot,t)$, being slowly varying functions within every cell Δ, are approximated by the values of their appropriate derivatives and increments of these unknowns inside the cell are neglected in calculation of averages over this cell. Moreover, finite sums over $\mathscr{L}$ in Eq. (6) are approximated by the integrals over Ξ.

Defining $\bar{h}^{A\alpha} \equiv h^{b\alpha} - h^{a\alpha}$, $\bar{\mathbf{g}}^{A\alpha} \equiv 0.5\,(\mathbf{g}^{a\alpha} + \mathbf{g}^{b\alpha})$, $\hat{\mathbf{g}}^{A\alpha} \equiv \mathbf{g}^{b\alpha} - \mathbf{g}^{a\alpha}$, $\lambda^A \equiv l/s^A$, and setting

$$\mathbf{\Psi}(\mathbf{x},t) \equiv \nabla W(\mathbf{x},t) + \boldsymbol{\varepsilon} : \mathbf{\Phi}(\mathbf{x},t), \quad \mathbf{x} \in \Xi,$$

where $\boldsymbol{\varepsilon}$ stands for the Ricci symbol, and using the aforementioned approximations, formulae (3) take the following form (no summation over A!)

$$\begin{aligned}
\tilde{\varepsilon}^A(\mathbf{z},t) &\cong \mathbf{t}^A \cdot \mathbf{\Psi}(\mathbf{z},t) + \lambda^A \bar{h}^{A\alpha} Q^\alpha(\mathbf{z},t) + l\, \mathbf{n}^A \cdot \bar{\mathbf{g}}^{A\alpha} \cdot \mathbf{R}^\alpha(\mathbf{z},t),\\
\kappa^A(\mathbf{z},t) &\cong s^A(\mathbf{n}^A \otimes \mathbf{t}^A) : \nabla\mathbf{\Phi}(\mathbf{z},t) + l\, \mathbf{n}^A \cdot \hat{\mathbf{g}}^{A\alpha} \cdot \mathbf{R}^\alpha(\mathbf{z},t),\\
\tilde{\kappa}^A(\mathbf{z},t) &\cong s^A(\mathbf{t}^A \otimes \mathbf{t}^A) : \nabla\mathbf{\Phi}(\mathbf{z},t) + l\, \mathbf{t}^A \cdot \hat{\mathbf{g}}^{A\alpha} \cdot \mathbf{R}^\alpha(\mathbf{z},t), \quad \mathbf{z} \in \mathscr{L}.
\end{aligned} \tag{10}$$

It is visible that the derived strain components for the tolerance averaging model involve spatial derivatives of averaging parts of the deflection $W(\mathbf{z},t)$ and the rotations $\mathbf{\Phi}(\mathbf{z},t)$, but do not involve derivatives of the extra kinematic unknowns $Q^\alpha(\mathbf{z},t)$ and $\mathbf{R}^\alpha(\mathbf{z},t)$, describing fluctuating parts of a deflection and rotations in an arbitrary but fixed cell Δ of the periodic lattice-type plates under consideration.

Substituting the right-hand sides of Eqs. (10) and (7)-(8) into Eq. $(6)_{1,2}$ the strain and kinetic energy densities obtained are $\bar{\mathscr{E}} \equiv \mathscr{E}/|\Delta|$, and $\bar{\mathscr{K}} \equiv \mathscr{K}/|\Delta|$, respectively, as the quadratic symmetric and positive-definite forms $\bar{\mathscr{E}} \equiv \bar{\mathscr{E}}(\mathbf{\Psi}, \nabla\mathbf{\Phi}, Q^\alpha, \mathbf{R}^\alpha)$, $\bar{\mathscr{K}} \equiv \bar{\mathscr{K}}(\dot{W}, \dot{\mathbf{\Phi}}, \dot{Q}^\alpha, \dot{\mathbf{R}}^\alpha)$, where the cell area is defined by $|\Delta|$. The exact integral formulae can be found in Ref. [1] and these are reminded in Appendix A.

(4) Regarding the external loading on the periodic lattice-type plate, the condition is imposed, namely, it is assumed that continuous slowly varying functions $f(\cdot,t)$, $f^\alpha(\cdot,t)$, $\mathbf{m}(\cdot,t)$, $\mathbf{m}^\alpha(\cdot,t)$ (defined on Ξ for every t) exist, such that the conditions

$$f(\mathbf{z},t) = |\Delta|^{-1}\sum_{a=1}^{n} f^a(\mathbf{z},t), \quad f^\alpha(\mathbf{z},t) = |\Delta|^{-1}\sum_{a=1}^{n} f^a(\mathbf{z},t)\, h^{a\alpha},$$
$$\mathbf{m}(\mathbf{z},t) = h^{-1}\,|\Delta|^{-1}\sum_{a=1}^{n}\mathbf{m}^a(\mathbf{z},t), \quad \mathbf{m}^\alpha(\mathbf{z},t) = h^{-1}\,|\Delta|^{-1}\sum_{a=1}^{n}\mathbf{g}^{a\alpha}\cdot\mathbf{m}^a(\mathbf{z},t), \tag{11}$$

hold for every $\mathbf{z}\in\mathscr{L}$; where h stands for a mean height of the beams in the direction normal to $0x_1x_2$-plane. Thus, from $(6)_3$ the integral form of the density of the external loading work $\bar{\mathscr{W}} \equiv \mathscr{W}/|\Delta|$ is obtained, cf. Ref. [1]; Appendix A.

(5) At last, from the Hamiltonian for the lattice-type periodic plates shown in Ref. [1] in the integral form and reminded here in Appendix A, employing the principle of stationary action the after-mentioned equations for the macrodeflection W, the macrorotations $\mathbf{\Phi}$ and the extra kinematic unknowns Q^α, $\mathbf{R}^\alpha$ ($\alpha = 1,\ldots, n-1$) together with the constitutive equations are derived [1]:

(i) Equations of motion

$$\nabla\cdot\mathbf{P} - \mu\ddot{W} + f = 0,$$
$$\nabla\cdot\mathbf{M} + \boldsymbol{\varepsilon} : \mathbf{P} - h^2\boldsymbol{\chi}\cdot\ddot{\mathbf{\Phi}} + h\mathbf{m} = \mathbf{0}, \tag{12a}$$

(ii) Dynamic evolution equations

$$l^2\mu^{\alpha\beta}\ddot{Q}^\beta + S^\alpha - lf^\alpha = 0,$$
$$h^2l^2\boldsymbol{\chi}^{\alpha\beta}\cdot\ddot{\mathbf{R}}^\beta + \mathbf{H}^\alpha - hl\mathbf{m}^\alpha = \mathbf{0}, \tag{12b}$$

(iii) Constitutive equations

$$\mathbf{P} = \frac{\partial\bar{\mathscr{E}}}{\partial\nabla W}, \quad \mathbf{M} = \frac{\partial\bar{\mathscr{E}}}{\partial\nabla\mathbf{\Phi}}, \quad S^\alpha = \frac{\partial\bar{\mathscr{E}}}{\partial Q^\alpha}, \quad \mathbf{H}^\alpha = \frac{\partial\bar{\mathscr{E}}}{\partial\mathbf{R}^\alpha}, \tag{12c}$$

where

$$\mu \equiv |\Delta|^{-1}\sum_{a=1}^{n} M^a, \quad \mu^{\alpha\beta} \equiv |\Delta|^{-1}\sum_{a=1}^{n} M^a h^{a\alpha}h^{a\beta},$$
$$\boldsymbol{\chi} \equiv h^{-2}|\Delta|^{-1}\sum_{a=1}^{n}\mathbf{J}^a, \quad \boldsymbol{\chi}^{\alpha\beta} \equiv h^{-2}|\Delta|^{-1}\sum_{a=1}^{n}\mathbf{g}^{a\alpha}\cdot\mathbf{J}^a\cdot\mathbf{g}^{a\beta}.$$

These equations have to be satisfied for every t in the region Ξ of $0x_1x_2$ and represent a continuum non-asymptotic model of the periodic lattice-type plate of an arbitrary lay-out. This continuum model called the internal variable model [1] will be called further as *the tolerance averaging model.*

Eqs. (12a)–(12c) have physical sense for unknowns W, $\mathbf{\Phi}$, Q^α, $\mathbf{R}^\alpha$ being slowly varying functions for every t, cf. Refs. [1,27], i.e.

$$W(\cdot,t),\ \mathbf{\Phi}(\cdot,t),\ Q^\alpha(\cdot,t),\ \mathbf{R}^\alpha(\cdot,t) \in SV(T). \tag{13}$$

Hence, this model is useful to analyse long wave propagation problems, in which a typical wavelength is sufficiently large comparing to the length size of the periodicity cell.

It should be emphasized that for the extra kinematic unknowns Q^α, $\mathbf{R}^\alpha$ the obtained ordinary differential equation (12b) involve exclusively their time-derivatives, while the macrodeflection W and the macrorotations $\mathbf{\Phi}$ are governed by the partial differential equations (12a). Derived equations (12b) do not involve spatial derivatives of unknowns Q^α, $\mathbf{R}^\alpha$ because they do not appear during calculations as a consequence of the assumed averaging approximations (see Section 2.3, p. 3). Hence, in general for Q^α, $\mathbf{R}^\alpha$ boundary conditions cannot be formulated and that is why these are called *internal kinematic variables*.

For more detailed discussion of this model, the reader is referred to Ref. [1].

3. Governing equations for honeycomb lattice-type plates

In this section, the general equations of the tolerance averaging model (12a–c) will be applied to derive governing equations for lattice-type plates having a honeycomb structure in $0x_1x_2$-plane and it will be shown that the equations take the form similar to equations of isotropic cases.

For honeycomb lattice-type plates, the smallest periodicity cell Δ can be fixed in the form shown in Fig. 1. The cell (cf. Fig. 3) consists of three identical beams B^1, B^2, B^3 and two identical rigid joints j^1, j^2, therefore $N=3$ and $n=2$. All material and geometrical properties of beams are assumed to be constant ($A=1, 2, 3$), i.e. a Young's modulus $E^A=E$, a Kirchhoff's modulus $G^A=G$, a Poisson's ratio $\nu^A=\nu$, a mass density $\rho^A=\rho$ and a length of beam $s^A=s$, an area of beam cross-section $F^A=F$ and moments of inertia of beam cross section: $I^A=I$, $I^A_o=I_o$. The area $|\Delta|$ of the cell and the square of the microstructure length parameter l are identical: $|\Delta|=l^2=3\sqrt{3}s^2/2$. The assumptions mentioned above indicate that stiffnesses given by Eq. (4) are the same and equal to

$$\tilde{\Lambda}^A=\tilde{\Lambda}=12EIs^{-1},\quad K^A=K=EIs^{-1},\quad \tilde{K}^A=\tilde{K}=GI_os^{-1},$$

concentrated masses M^a of joints (1) and terms of the tensor $\mathbf{J}^a$ ($a=1, 2$) of the rotational moment of inertia (2) are also equal to

$$M^a=M=\tfrac{3}{2}\rho Fs,\quad J^a_{ij}=\delta_{ij}J=\tfrac{3}{4}\rho s(I+I_o),\quad i,j=1,2. \tag{14}$$

Because $n-1=1$, it will be dealt with one internal kinematic variable $Q\equiv Q^1$ and one vector kinematic variable $\mathbf{R}^1$. From Eq. (14) and the conditions (9) for shape parameters h^{a1}, $\mathbf{g}^{a1}$, it follows that the parameters are equal to

$$h^{11}=-h^{21}=1,\quad g^{11}_{ij}=-g^{21}_{ij}=1,\quad i,j=1,2. \tag{15}$$

Let us introduce the following denotations:

$$\begin{gathered}\vartheta\equiv\tfrac{3}{2}\tilde{\Lambda}l^{-2},\quad \gamma\equiv 12\tilde{\Lambda}s^{-2},\quad \kappa\equiv 6(\tilde{K}+K)l^{-2},\quad \eta\equiv\tfrac{3}{2}(\tilde{K}-K)sl^{-3},\\ \zeta\equiv\tfrac{3}{8}Ks^2l^{-2},\quad \tilde{\zeta}\equiv\tfrac{3}{8}\tilde{K}s^2l^{-2},\quad \mu=2Ml^{-2},\quad \chi=2Jh^{-2}l^{-2}.\end{gathered} \tag{16}$$

After calculation of Eqs. (10) and (6), the constitutive equations for $\mathbf{P}=[P_i]$, $\mathbf{M}=[M_{ij}]$ and S^1, $\mathbf{H}^1=[H_i{}^1]$, $i, j=1, 2$, given by the general formulae (12c), have after-mentioned forms

$$\begin{aligned}&\mathbf{P}=\mathbf{A}(\nabla W+\boldsymbol{\varepsilon}:\boldsymbol{\Phi}), && \mathbf{M}=\mathbb{C}:\nabla\boldsymbol{\Phi}+l^2\mathbb{B}^1:\mathbf{R}^1,\\ &S^1=A^{11}Q^1, && \mathbf{H}^1=l^2\mathbb{B}^1:\nabla\boldsymbol{\Phi}+l^2\mathbf{A}^{11}\cdot\mathbf{R}^1,\end{aligned} \tag{17}$$

where

$$\begin{aligned}&A_{ij}=\delta_{ij}\vartheta,\quad A^{11}=\gamma,\quad A^{11}_{ij}=\delta_{ij}\kappa,\\ &C_{iiii}=3\tilde{\zeta}+\zeta,\quad C_{iijj}=C_{ijji}=\tilde{\zeta}-\zeta,\quad C_{ijij}=\tilde{\zeta}+3\zeta,\quad i\neq j,\\ &B^1_{122}=B^1_{221}=B^1_{212}=-B^1_{111}=-\eta\cos 3\varphi,\quad B^1_{121}=B^1_{211}=B^1_{112}=-B^1_{222}=\eta\sin 3\varphi,\end{aligned}$$

and φ is an angle denoted in Fig. 3. Now, instead of $\mathbf{R}^1$, a new internal variable $\mathbf{V}$ will be introduced, which is defined as

$$V_{11}=-V_{22}=R^1_1\cos 3\varphi+R^1_2\sin 3\varphi,\quad V_{12}=V_{21}=R^1_1\sin 3\varphi-R^1_2\cos 3\varphi. \tag{18}$$

Because the unknown $\mathbf{R}^1$ is a slowly varying function, cf. Eq. (13), the new unknown $\mathbf{V}$ is also slowly varying, $\mathbf{V}(\cdot,t)\in SV(T)$. Taking into account the obtained constitutive equations (17), the introduced new internal variable $\mathbf{V}$ (18), applying the general equations (12a,b), may be received after quite simple calculations, which result in the following system of equations:

$$\vartheta(\nabla\otimes\nabla)W-\mu\ddot{W}+\vartheta\boldsymbol{\varepsilon}:(\nabla\cdot\boldsymbol{\Phi})+f=0,$$

$$\vartheta\boldsymbol{\varepsilon}:\nabla W+\vartheta\boldsymbol{\varepsilon}:(\boldsymbol{\varepsilon}:\boldsymbol{\Phi})+2(\tilde{\zeta}-\zeta)\nabla(\nabla\cdot\boldsymbol{\Phi})+(\tilde{\zeta}+3\zeta)(\nabla\cdot\nabla)\boldsymbol{\Phi}-h^2\chi\ddot{\boldsymbol{\Phi}}+l^2\eta\nabla\cdot\mathbf{V}+h\mathbf{m}=\mathbf{0},$$

$$l^2\mu\ddot{Q}+\gamma Q-lf^1=0,$$

$$l^2h^2\chi\ddot{\mathbf{V}}+l^2\kappa\mathbf{V}+l^2\eta[\nabla\boldsymbol{\Phi}+\nabla\boldsymbol{\Phi}^{\mathrm{T}}-\mathbf{1}\cdot(\nabla\cdot\boldsymbol{\Phi})]-lh\mathbf{m}^1=\mathbf{0}. \tag{19}$$

Eqs. (19) with denotations (16) represent the tolerance averaging model for honeycomb lattice-type plates under consideration. The basic unknowns of the model are the macrodeflection W, the macrorotation $\boldsymbol{\Phi}$ and the internal kinematic variables Q, $\mathbf{V}$.

The above equations have constant coefficients and some of which involve the microstructure length parameter l. Thus, Eqs. (19) describe the length-scale effect in dynamical problems of honeycomb lattice-type plates and make it possible to investigate higher-order vibrations related to the periodic plate structure. The main feature of these equations is their form, which is similar to equations of isotropic cases, because coefficients in Eqs. (19) are independent of the angle φ, i.e. these are isotropic, cf. Ref. [21]. Thus, the dynamical response of lattice-type plates with the internal honeycomb structure is isotropic for unknowns: the macrodeflection W, the macrorotation $\boldsymbol{\Phi}$, the internal kinematic variables Q and the *new* internal variables $\mathbf{V}$, defined by relations (18). It should be emphasized that Eqs. (19) can be applied in the analysis of long wave propagation problems, what corresponds with the condition that all unknowns in this model have to be slowly varying functions, cf. Eq. (13).

In order to evaluate obtained results, a certain continuous asymptotic model will be presented. The governing equations of that model can be derived from the equations of the tolerance averaging model (12a–c) by the asymptotic procedure, in which the microstructure length parameter l is scaled down. At the same time, it is assumed that the length parameter h, being the mean height of beams, i.e. a dimension of the direction normal to $0x_1x_2$-plane, tends towards zero much faster than the parameter l, therefore $h=o(l)$. After applying this procedure (cf. Ref. [1] and Appendix B), the following equation of motion is obtained:

$$(\tilde{\zeta}+3\zeta)(\nabla\otimes\nabla):(\nabla\otimes\nabla)W+\mu\ddot{W}-f=0, \tag{20}$$

which represents an asymptotic model of the periodic lattice-type plate under consideration, called *the local model*. The only unknown in this model is the macrodeflection W, which has to be a slowly varying function for every t in the region Ξ of $0x_1x_2$.

4. Free vibrations of a simply supported lattice-type plate band with honeycomb structure

As an example, let us consider free vibrations of a honeycomb lattice-type plate band along $x=x_1$ axis, which is simply supported on the opposite edges $x=0$ and $x=L$. Hence, all loadings subjected to this plate will be neglected, i.e. $f=f^1=0$, $\mathbf{m}=\mathbf{m}^1=\mathbf{0}$, and the rotation $\Phi_1=0$. Denoting $\Phi=\Phi_2$, $V=V_{12}$, $S=V_{11}=-V_{22}$, Eqs. (19) of the tolerance averaging model uncouple on the system of three equations:

$$\begin{aligned}&\vartheta W_{,11}-\mu\ddot{W}+\vartheta\Phi_{,1}=0,\\&-\vartheta W_{,1}+(\tilde{\zeta}+3\zeta)\Phi_{,11}-\vartheta\Phi-h^2\chi\ddot{\Phi}+l^2\eta V_{,1}=0,\\&\eta\Phi_{,1}+h^2\chi\ddot{V}+\kappa V=0,\end{aligned} \tag{21}$$

and two independent equations

$$l^2\mu\ddot{Q}+\gamma Q=0,\quad h^2\chi\ddot{S}+\kappa S=0. \tag{22}$$

Solutions to Eq. (21) and (22) will be looked for in the form

$$\begin{aligned}&W(x,t)=A_W\sin kx\cos\omega t,\quad \Phi(x,t)=A_\Phi\cos kx\cos\omega t,\quad V(x,t)=A_V\sin kx\cos\omega t,\\&Q(x,t)=A_Q\sin kx\cos\omega t,\quad S(x,t)=A_S\sin kx\cos\omega t,\end{aligned} \tag{23}$$

satisfying boundary conditions of the simply supported plate band, where ω is a frequency of vibrations and $k = 2\pi/L$ is a wavenumber.

Substituting the right-hand sides of Eq. $(23)_{1,2,3}$ into Eqs. (21), the system of three linear algebraic equations for A_W, A_Φ, A_V is obtained. Non-trivial solutions to these equations can be derived, provided that the determinant of the system is equal to zero. Then, the characteristic equation for the free vibration frequencies derived from Eqs. (21) has the following form:

$$\bar{a}\omega^6 - \bar{b}\omega^4 + \bar{c}\omega^2 - \bar{d} = 0, \tag{24}$$

where coefficients $\bar{a}, \bar{b}, \bar{c}, \bar{d}$, using notations (16), are defined as

$$\begin{aligned} &\bar{a} \equiv h^4\mu\chi^2, \quad \bar{b} \equiv h^2\chi\{\mu(\vartheta+\kappa) + [h^2\vartheta\chi + \mu(3\zeta+\tilde{\zeta})]k^2\}, \quad \bar{d} \equiv \vartheta[\kappa(3\zeta+\tilde{\zeta}) - l^2\eta^2]k^4, \\ &\bar{c} \equiv h^2\vartheta\chi(3\zeta+\tilde{\zeta})k^4 + [\kappa\mu(3\zeta+\tilde{\zeta}) + h^2\vartheta\kappa\chi - \mu\eta^2]k^2 + \vartheta\kappa\mu. \end{aligned} \tag{25}$$

It can be shown that coefficients (25) are positive. Introducing additional notations

$$\bar{\alpha} \equiv 27\bar{d}\bar{a}^2 + 2\bar{b}^3 - 9\bar{a}\bar{b}\bar{c}, \quad \bar{\beta} \equiv 3\bar{a}\bar{c} - \bar{b}^2,$$

from characteristic equation (24) the following formulae for free vibration frequencies can be derived:

$$\begin{aligned} &\omega_{-1} = \sqrt{(3\bar{a})^{-1}\left[\bar{b} - (\sqrt[3]{2})^{-1}\left(\mathrm{Re}\sqrt[3]{\bar{\alpha} + \mathrm{i}\sqrt{-\bar{\alpha}^2 - 4\bar{\beta}^3}} + \sqrt{3}\,\mathrm{Im}\sqrt[3]{\bar{\alpha} + \mathrm{i}\sqrt{-\bar{\alpha}^2 - 4\bar{\beta}^3}}\right)\right]}, \\ &\omega_{-2} = \sqrt{(3\bar{a})^{-1}\left[\bar{b} - (\sqrt[3]{2})^{-1}\left(\mathrm{Re}\sqrt[3]{\bar{\alpha} + \mathrm{i}\sqrt{-\bar{\alpha}^2 - 4\bar{\beta}^3}} - \sqrt{3}\,\mathrm{Im}\sqrt[3]{\bar{\alpha} + \mathrm{i}\sqrt{-\bar{\alpha}^2 - 4\bar{\beta}^3}}\right)\right]}, \\ &\omega_{+3} = \sqrt{(3\bar{a})^{-1}(\bar{b} + \sqrt[3]{4}\,\mathrm{Re}\sqrt[3]{\bar{\alpha} + \mathrm{i}\sqrt{-\bar{\alpha}^2 - 4\bar{\beta}^3}})}. \end{aligned} \tag{26}$$

Now, substituting the right-hand sides of Eqs. $(23)_{4,5}$ into Eqs. (22) two independent linear algebraic equations for A_Q, A_S are obtained and then two frequencies are derived:

$$\omega_{+1} = l^{-1}\sqrt{\gamma\mu^{-1}}, \quad \omega_{+2} = h^{-1}\sqrt{\kappa\chi^{-1}}. \tag{27}$$

Thus, in the framework of the tolerance averaging model for the honeycomb lattice-type plate band, free vibration frequencies defined by Eqs. (26) and (27) are obtained. It should be emphasized that all these frequencies are the consequence of the application of the simplest model in which the representative element contains one periodicity cell (cf. Fig. 1) and the form of oscillations of the cell is described by shape parameters assumed as Eq. (15). More accurate models are based on representative elements including two or several periodicity cells.

Within the local model, free vibrations of the plate under consideration are described by the equation obtained from Eq. (20), i.e. by

$$(\tilde{\zeta} + 3\zeta)W_{,1111} + \mu\ddot{W} = 0,$$

for which a solution will be assumed as Eq. $(23)_1$. Substituting this solution into the above equation, after simple transformations, the following formula for free vibration frequency is derived

$$\omega = k^2\sqrt{(\tilde{\zeta} + 3\zeta)\mu^{-1}}. \tag{28}$$

Thus, in the framework of the local model, only one frequency given by Eq. (28) can be analysed.

On the basis of the above results, it can be observed that in the framework of both the models, the tolerance averaging and the local models, the basic lowest free vibration frequency can be derived (formula (26) or the simple form (28)). However, higher frequencies related to periodic structure of the honeycomb lattice-type plate band cannot be obtained using the local model and also the local model cannot describe the length-scale effects, related to a periodic structure, cf. Ref. [27]. The proposed tolerance averaging model, being a non-asymptotic model, makes it possible to analyse higher-order vibrations; at the presented example higher-order free vibration frequencies ω_{+1}, ω_{+2} and ω_{+3} are given by formulae (27) and $(26)_3$, respectively.

5. Calculational results

An application of formulae for free vibration frequencies presented in the previous section will be illustrated by a calculational example. Let us introduce the following dimensionless parameters:

$$\xi \equiv bh^{-1}, \quad \delta \equiv hs^{-1}, \quad q \equiv kl, \tag{29}$$

where b is the width of the beam; h is the height of the beam; l is the microstructure length parameter; s is the length of the beam (cf. Fig. 3); $k = 2\pi/L$ is the wavenumber. The parameter q is called the dimensionless wavenumber.

Let us introduce non-dimensional frequency parameters defined for free vibration frequencies obtained from the tolerance averaging model and of the local model, given by Eqs. (26), (27) and (28), respectively, in the following form:

$$\Omega_{-1} \equiv \iota\omega_{-1}, \quad \Omega_{-2} \equiv \iota\omega_{-2}, \quad \Omega_{+1} \equiv \iota\omega_{+1}, \quad \Omega_{+2} \equiv \iota\omega_{+2}, \quad \Omega_{+3} \equiv \iota\omega_{+3}, \quad \Omega \equiv \iota\omega, \tag{30}$$

where $\iota = l(\rho/E)^{1/2}$; ρ is the mass density of the material of the beam; E is the Young's modulus.

Calculational results for the non-dimensional frequency parameters (30) are presented as diagrams in Figs. 4–6. These plots are made for different values of parameters ξ, δ, q, defined by Eqs. (29), and for the Poisson's ratio $\nu = 0.3$. In Fig. 4a, there are plots of the relations between the first basic (lower) frequencies Ω_{-1}, Ω and the dimensionless wavenumber q $(29)_3$. Diagram of the second basic (lower) frequency as the relation between Ω_{-2} and q and of the higher frequency as the relation between Ω_{+3} and q are presented in Fig. 4b. The dimensionless wavenumber q is a number belonging to the interval [0, 0.1]. In Fig. 5, the diagrams of relations between non-dimensional frequency parameters and the dimensionless parameter δ $(29)_1$ are shown. These plots of the first lower frequencies Ω_{-1} and Ω are shown in Fig. 5a; in Fig. 5b—the second lower frequency Ω_{-2} and for the third higher frequency Ω_{+3}; and in Fig. 5c—the first higher frequency Ω_{+1}. In Fig. 6, diagrams of relations between non-dimensional frequency parameters and the dimensionless parameter ξ $(29)_1$ are presented. In Fig. 6a, there are shown plots of the first lower frequencies Ω_{-1} and Ω; in Fig. 6b the curves of the second lower frequency Ω_{+2} (related to the macrorotations about x_2-axis), and of higher frequencies Ω_{+2} and Ω_{+3} (related to the fluctuating parts of rotations about x_1- and x_2-axis) can be seen; moreover, in Fig. 6c, there is an enlarged fragment of Fig. 6b.

It has to be observed that using the transformation of frequencies to non-dimensional frequency parameters (30), the quantity Ω_{+1} describing the first higher frequency ω_{+1} is independent of the parameter $\xi = b/h$ and

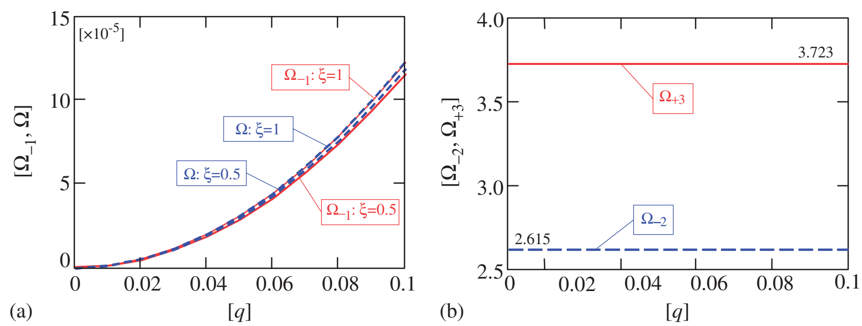

Fig. 4. Diagrams of relations between non-dimensional frequency parameters and the dimensionless wavenumber q: (a) for lower frequencies Ω_{-1}, Ω ($\delta = 0.1$); (b) for lower frequency Ω_{-2} and higher frequency Ω_{+3} ($\delta = 0.1$, $\xi = 0.5$); parameters ξ, δ, q, are defined by Eq. (30); Poisson's ratio $\nu = 0.3$.

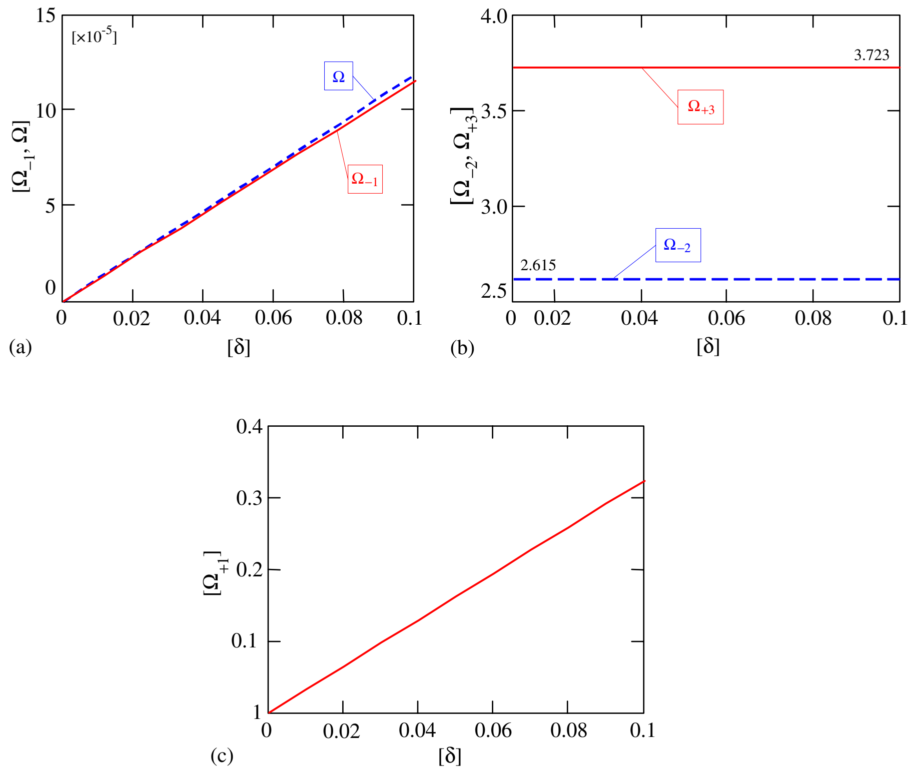

Fig. 5. Diagrams of relations between non-dimensional frequency parameters and the dimensionless parameter δ ($q = 0.1$, $\xi = 0.5$; Poisson's ratio $\nu = 0.3$): (a) for lower frequencies Ω_{-1}, Ω; (b) for lower frequency Ω_{-2} and higher frequency Ω_{+3}; and (c) for higher frequency Ω_{+1}; parameters ξ, δ, q, are defined by Eq. (30).

the quantity Ω_{+2} related to the second higher frequency ω_{+2} is independent of the parameter $\delta = h/s$. Both the aforesaid parameters are independent of the dimensionless wavenumber q.

Analysing diagrams shown in Figs. 4–6, some comments can be formulated:

1. values of the first lower free vibration frequencies calculated within the tolerance averaging model are smaller than those from the local model (cf. Figs. 4a,5a and 6a), but differences between them are very small (cf. Fig. 4a), and for parameter $\xi \in [1, 1.5]$, these values are very close (cf. Fig. 6a);
2. higher free vibration frequencies related to the periodic structure of the honeycomb lattice-type plate, whose plots are shown in Figs. 4b, 5b,c, 6b,c can be obtained only within the tolerance averaging model;
3. values of the first higher frequency Ω_{+1} are bigger than values of the first (basic) lower frequency Ω_{-1}, but smaller than the second one Ω_{-2}, cf. Figs. 5a and b;
4. it exists such a value $\hat{\xi} > 0$ that for $\xi < \hat{\xi}$, values of the second higher frequency Ω_{+2} (related to the fluctuating parts of rotations about x_1-axis) are very close to values of the second lower frequency Ω_{-2} (related to the macrorotations about x_2-axis), and for $\xi \geqslant \hat{\xi}$, values of the second higher frequency Ω_{+2} are very close to values of the third higher frequency Ω_{+3} (related to the fluctuating parts of rotations about x_2-axis), cf. Figs. 6b and c.

An example of the general model of periodic lattice-type plates based on the tolerance averaging method was presented in Ref. [1]; unfortunately diagrams of spectral lines in Fig. 4 in Ref. [1] were made for improper parameters and hence, values of frequencies obtained in the above-mentioned paper, in particular of the first higher frequency (denoted by Ω_2 in Ref. [1]), are much bigger than those calculated here. However, the comments and conclusions in Ref. [1] were correctly stated.

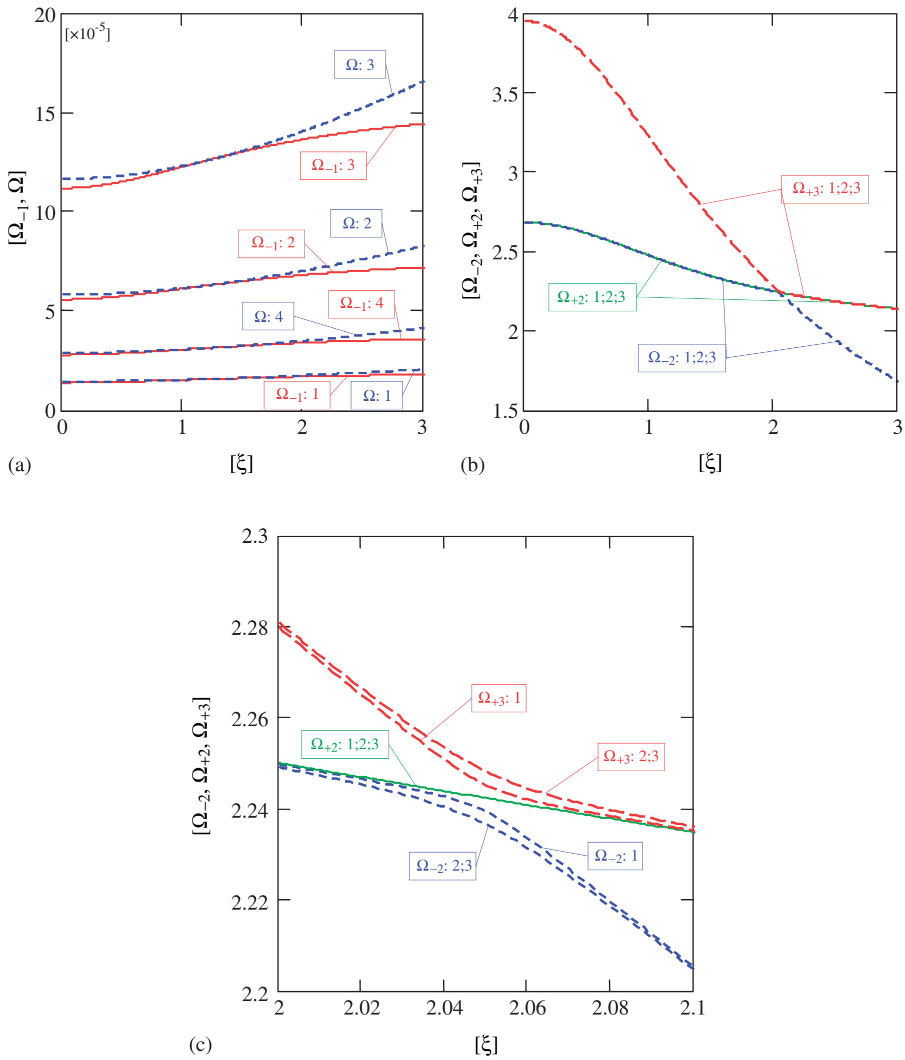

Fig. 6. Diagrams of relations between non-dimensional frequency parameters and the dimensionless parameter ξ: (a) for the first lower frequencies Ω_{-1}, Ω; (b) for the second lower frequency Ω_{-2} and for higher frequencies Ω_{+2}, Ω_{+3}; and (c) for lower frequency Ω_{-2} and for higher frequencies Ω_{+2}, Ω_{+3} (zoom of a fragment of Fig. b)); parameters ξ, δ, q, are defined by Eq. (30); lines: 1—$(q,\delta) = (0.05,0.05)$, 2—$(q,\delta) = (0.1,0.05)$, 3—$(q,\delta) = (0.1,0.1)$, 4—$(q,\delta) = (0.05,0.1)$.

6. Physical correctness of the tolerance averaging model

The length-size effect related to the periodic structure of the honeycomb lattice-type plate is manifested first of all in *additional higher-order frequencies* obtained in the framework of the tolerance averaging model. Thus, in order to justify the proposed model a travelling wave along the x-axis through an unbounded lattice-type plate with honeycomb structure subject the cylindrical bending will be analysed. This problem will be investigated in the framework of the new tolerance averaging model, of the local model and of the discrete model. The x-axis will be parallel to an arbitrary but fixed family of beams of the lattice-type plate under consideration.

6.1. Travelling wave by the tolerance averaging and the local model

All material and geometrical properties are assumed as in Section 3. Hence, within the tolerance averaging model, a travelling wave of this unbounded plate can be analysed using Eqs. (21) and (22). In these equations, let us neglect the rotational inertia terms, i.e. terms with coefficient χ, defined by Eq. $(16)_8$. Thus, the following differential equations are obtained:

$$\vartheta W_{,11} - \mu \ddot{W} + \vartheta \Phi_{,1} = 0, \quad -\vartheta W_{,1} + (\tilde{\zeta} + 3\zeta - l^2\eta^2\kappa^{-1})\Phi_{,11} - \vartheta\Phi = 0, \quad l^2\mu\ddot{Q} + \gamma Q = 0$$

Assuming solutions to the above equations in the form

$$W(x,t) = A_W \exp \mathrm{i}(kx - \omega t), \quad \Phi(x,t) = A_\Phi \exp \mathrm{i}(kx - \omega t), \quad Q(x,t) = A_Q \exp \mathrm{i}(kx - \omega t),$$

where A_W, A_Φ, A_Q are amplitudes, ω is a frequency and $k = 2\pi/L$ is the wavenumber, after some transformations, in the framework of *the tolerance averaging model*, we arrive at formulae of *a lower* ω_- and *a higher frequency* ω_+ for the plate under consideration

$$\omega_- = k^2\sqrt{\frac{\tilde{\zeta} + 3\zeta - l^2\eta^2\kappa^{-1}}{\mu[(\tilde{\zeta} + 3\zeta - l^2\eta^2\kappa^{-1})\vartheta^{-1}k^2 + 1]}}, \quad \omega_+ = l^{-1}\sqrt{\gamma\mu^{-1}}. \tag{31}$$

The lower frequency ω_- corresponds to the first basic (lower) frequency ω_{-1} given by Eq. $(26)_1$ and the higher frequency ω_+ is identical with the first higher frequency ω_{+1} $(27)_1$. The similar procedure within *the local model* leads to the only one frequency ω, which is described by the formula identical with Eq. (28).

6.2. Travelling wave by the "exact" discrete model

The above problem of a travelling wave of the lattice-type plate with the honeycomb structure can be also considered within *the "exact" discrete model*, which is similar to the one proposed and applied by Brillouin in Ref. [2] to analyse longitudinal vibrations of the one-dimensional diatomic structure. In the aforementioned paper, solutions for that model are treated as the "exact" solutions for the structure under consideration. Let us consider cells being repeated elements of the honeycomb lattice-type plate, which are numbered $m-1$, m, $m+1$. The length of repeated elements measured along the x-axis is equal to $l_o = 3s/2$.

Denoting by $w_m^{j^a}, \varphi_m^{j^a}$ and $w_m^{j^b}, \varphi_m^{j^b}$ deflections and rotations of joints j^a and j^b at the cell m, respectively, and using the known formulae of structural mechanics for transversal forces, bending and torsion moments in each one of three beams constituted the cell m, the equations of motion can be written for both the joints j^a, j^b belonging to the cell m. These equations involve unknown deflections $w_m^{j^a}, w_m^{j^b}$ and rotations $\varphi_m^{j^a}, \varphi_m^{j^b}$ of joints j^a, j^b belonging to the cell m and also deflections and rotations of joints j^b and j^a belonging to the cell $m-1$ and to the cell $m+1$, respectively, i.e. $w_{m-1}^{j^b}, \varphi_{m-1}^{j^b}$ and $w_{m+1}^{j^a}, \varphi_{m+1}^{j^a}$. Afterwards, the known procedure of investigations leads to the characteristic equation, which can be written in the form

$$\breve{\alpha} l_o^6 \varpi^4 - \breve{\beta} l_o^3 \varpi^2 + \breve{\delta} = 0, \tag{32}$$

where coefficients are defined by

$$\breve{\alpha} \equiv \tfrac{2}{243}M^2\left[31 + 21\tfrac{GI_o}{EI} + 2\left(3\tfrac{GI_o}{EI} - 2\right)\cos(kl_o)\right], \quad \breve{\beta} \equiv 2MEI\left[19 + 18\tfrac{GI_o}{EI} + \left(8 + 9\tfrac{GI_o}{EI}\right)\cos(kl_o)\right],$$
$$\breve{\delta} \equiv 27(EI)^2\left(1 + 3\tfrac{GI_o}{EI}\right)[3 + \cos(2kl_o) - 4\cos(kl_o)].$$

Solutions to Eq. (32) being the "exact" formulae of frequencies for the travelling wave in the framework of *the discrete model* have the following forms:

$$\varpi_- = \sqrt{\left(\breve{\beta} - \sqrt{\breve{\beta}^2 - 4\breve{\alpha}\breve{\delta}}\right)\left(2\breve{\alpha}l_o^3\right)^{-1}}, \quad \varpi_+ = \sqrt{\left(\breve{\beta} + \sqrt{\breve{\beta}^2 - 4\breve{\alpha}\breve{\delta}}\right)\left(2\breve{\alpha}l_o^3\right)^{-1}}, \tag{33}$$

where ϖ_- and ϖ_+ are *lower* and *higher frequencies*, respectively. The above model takes into account the inertia forces related to concentrated masses M at all joints.

6.3. Calculational results and remarks

Using formulae similar to Eqs. (30), non-dimensional frequency parameters, describing frequencies for the travelling wave (given by Eqs. (31), (28), (33)), can be assumed and denoted for the tolerance averaging model by Ω_-, Ω_+, for the local model by Ω and for the discrete model by Θ_-, Θ_+. In Fig. 7, plots of these parameters

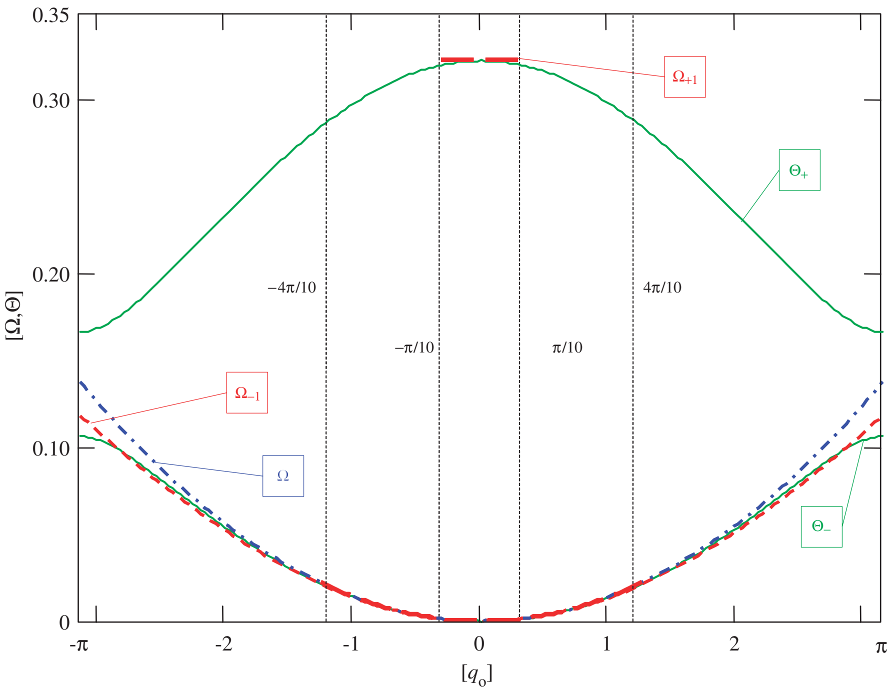

Fig. 7. Diagrams of relations between non-dimensional frequency parameters and the dimensionless wavenumber q_o ($\xi = 0.8$, $\delta = 0.1$; Poisson's ratio $\nu = 0.3$): for lower frequencies Ω_{-1} (the tolerance averaging model), Ω (the local model), Θ_{-} (the discrete model—the "exact" solution); for higher frequencies Ω_{+1} (the tolerance averaging model), Θ_{+} (the discrete model—the "exact" solution); parameters ξ, δ are defined by Eq. (30).

versus the dimensionless wavenumber $q_o \equiv kl_o$, $q_o \in [-\pi, \pi]$ are shown (the modes of vibrations are constructed from two waves propagating in opposite directions; thus, one wave has positive wavenumbers, but the other has negative ones). These diagrams are made for the parameters $\delta = h/s = 0.1$, $\xi = b/h = 0.8$ (h is the height of beams, b is the width of beams, s is the span of beams) and the Poisson's ratio $\nu = 0.3$. From results shown in Fig. 7, it can be observed that

1. lower frequencies of the travelling wave obtained within both the tolerance averaging and the local models are very close frequencies calculated from the discrete model within a wide scope of the dimensionless wavenumber q_o, $q_o \in [-0.4\pi, 0.4\pi]$; and
2. differences between higher frequencies for long wave propagation problems, $q_o \in [-0.1\pi, 0.1\pi]$, obtained within the discrete model and the tolerance averaging model are very small.

7. Conclusions

In this paper, the new averaged continuum model of periodic lattice-type plates for the analysis of dynamic problems is presented. The model makes it possible to investigate the effect of the cell length size on the overall dynamic behaviour of these plates. The proposed model is based on the tolerance averaging (summarized for periodic composites in Ref. [27]) and hence is called the tolerance averaging model. The effect of the cell length size is manifested e.g. by higher-order vibrations. Using the new model, higher free vibration frequencies can be investigated, which are related to a periodic plate structure. This problem has been shown here at the example of honeycomb lattice-type plates. Below, the general conclusions are formulated.

1. The governing equations of the tolerance averaging model describe the effect of the cell length size, because these involve the microstructure length parameter l.

2. It should be emphasized that for lattice-type plates with the honeycomb structure, the governing equations of the tolerance averaging model have the form similar to equations for isotropic cases, i.e. all coefficients of these equations are isotropic, cf. Ref. [21].
3. Contrary to the local (homogenized) model, using the new model, it is possible to analyse the effects of the cell size (in the meaning related to a periodic plate structure, cf. Ref. [27]) and calculate higher free vibration frequencies in lattice-type plates with the honeycomb structure.
4. The tolerance averaging model can be formulated on different levels of accuracy, cf. Ref. [27]; on every level, the systems of real numbers $h^{a\alpha}$, $\mathbf{g}^{a\alpha}$ ($a = 1,\ldots,n$; $\alpha = 1,\ldots,n-1$) have to be assumed a priori as a description of the class of motions in the periodicity cell Δ being investigated. The simplest model is based on the smallest repeated cell and has a minimum number of internal kinematic variables. In order to analyse higher-order motions, e.g. higher-order frequencies, we have to assume that the basic cell is composed of two or more repeated elements, hence, a model may involve even a large number of kinematic internal variables, cf. Refs. [21,27].
5. The tolerance averaging model yields the a posteriori applicability conditions for solutions to the model equations, i.e. all unknown functions (a macrodeflection, macrorotations and internal variables) are assumed to be slowly varying functions, cf. Eq. (13). Thus, the model is mainly limited to the analysis of long wave propagation problems (i.e. problems, in which the wavelength is large in comparison with the microstructure length parameter l), cf. Refs. [21,27].
6. The benchmark problem shown in Section 6 stands a certain justification of the tolerance averaging model. Obtained results within the new model constitute an approximation of exact solutions calculated using the known discrete model. Hence, the new model has defined physical meaning for long wave propagation problems properly.
7. From the aforementioned example, it is visible that the tolerance averaging model allows to obtain certain limits for frequencies, which can appear in considered periodic lattice-type plates, i.e. the inferior limit of lower frequencies and the superior limit of higher frequencies for long waves, $q_o \in [-0.1\pi, 0.1\pi]$, cf. Fig. 7.

Appendix A. The integral form of Hamiltonian for lattice-type periodic plates

Introduce the notations

$$\mathbf{A} \equiv |\Delta|^{-1}\sum_{A=1}^{N}\tilde{\Lambda}^{A}\mathbf{t}^{A}\otimes\mathbf{t}^{A}, \quad A^{\alpha\beta} \equiv |\Delta|^{-1}\sum_{A=1}^{N}(\lambda^{A})^{2}\tilde{\Lambda}^{A}\bar{h}^{A\alpha}\bar{h}^{A\beta},$$

$$\mathbb{C} \equiv |\Delta|^{-1}\sum_{A=1}^{N}(s^{A})^{2}[K^{A}(\mathbf{n}^{A}\otimes\mathbf{n}^{A}\otimes\mathbf{t}^{A}\otimes\mathbf{t}^{A}) + \tilde{K}^{A}(\mathbf{t}^{A}\otimes\mathbf{t}^{A}\otimes\mathbf{t}^{A}\otimes\mathbf{t}^{A})],$$

$$\mathbf{A}^{\alpha\beta} \equiv |\Delta|^{-1}\sum_{A=1}^{N}[\tilde{\Lambda}^{A}\bar{\mathbf{g}}^{A\alpha}\cdot(\mathbf{n}^{A}\otimes\mathbf{n}^{A})\cdot\bar{\mathbf{g}}^{A\beta} + K^{A}\hat{\mathbf{g}}^{A\alpha}\cdot(\mathbf{n}^{A}\otimes\mathbf{n}^{A})\cdot\hat{\mathbf{g}}^{A\beta} + \tilde{K}^{A}\hat{\mathbf{g}}^{A\alpha}\cdot(\mathbf{t}^{A}\otimes\mathbf{t}^{A})\cdot\hat{\mathbf{g}}^{A\beta}],$$

$$\mathbb{B}^{\alpha} \equiv |\Delta|^{-1}\sum_{A=1}^{N}(\lambda^{A})^{-1}[K^{A}(\mathbf{n}^{A}\otimes\mathbf{n}^{A}\otimes\mathbf{t}^{A})\cdot\hat{\mathbf{g}}^{A\alpha} + \tilde{K}^{A}(\mathbf{t}^{A}\otimes\mathbf{t}^{A}\otimes\mathbf{t}^{A})\cdot\hat{\mathbf{g}}^{A\alpha}],$$

$$\mathbf{D}^{\alpha} \equiv |\Delta|^{-1}\sum_{A=1}^{N}\lambda^{A}\tilde{\Lambda}^{A}\bar{h}^{A\alpha}\mathbf{t}^{A}, \quad \hat{\mathbf{D}}^{\alpha} \equiv |\Delta|^{-1}\sum_{A=1}^{N}\tilde{\Lambda}^{A}(\mathbf{t}^{A}\otimes\mathbf{n}^{A})\cdot\bar{\mathbf{g}}^{A\alpha},$$

$$\mathbf{D}^{\alpha\beta} \equiv |\Delta|^{-1}\sum_{A=1}^{N}\lambda^{A}\tilde{\Lambda}^{A}\bar{h}^{A\alpha}\mathbf{n}^{A}\cdot\bar{\mathbf{g}}^{A\beta},$$

$$\mu \equiv |\Delta|^{-1}\sum_{a=1}^{n}M^{a}, \quad \mu^{\alpha\beta} \equiv |\Delta|^{-1}\sum_{a=1}^{n}M^{a}h^{a\alpha}h^{a\beta},$$

$$\boldsymbol{\chi} \equiv h^{-2}|\Delta|^{-1}\sum_{a=1}^{n}\mathbf{J}^{a}, \quad \boldsymbol{\chi}^{\alpha\beta} \equiv h^{-2}|\Delta|^{-1}\sum_{a=1}^{n}\mathbf{g}^{a\alpha}\cdot\mathbf{J}^{a}\cdot\mathbf{g}^{a\beta}, \tag{A.1}$$

where h stands for the mean height of beams in the direction normal to $0x_1x_2$-plane. After substituting to Eq. (6), the right-hand sides of formulae (10) and (11) and taking into account the tolerance averaging approximations (cf. Section 2.3, p. 3), as well as the conditions (9) and the notations (A.1), we arrive at the integral form of Hamiltonian $\bar{\mathscr{A}} = \bar{\mathscr{E}} - \bar{\mathscr{K}} - \bar{\mathscr{W}}$, where

$$\begin{aligned}\bar{\mathscr{E}} &= \int_{\Xi}(\tfrac{1}{2}\mathbf{\Psi}\cdot\mathbf{A}\cdot\mathbf{\Psi} + \tfrac{1}{2}\nabla\mathbf{\Phi}:\mathbb{C}:\nabla\mathbf{\Phi} + \tfrac{1}{2}A^{\alpha\beta}Q^{\alpha}Q^{\beta} + \tfrac{1}{2}l^2\mathbf{R}^{\alpha}\cdot\mathbf{A}^{\alpha\beta}\cdot\mathbf{R}^{\beta} \\ &\quad + l^2\mathbf{R}^{\alpha}\cdot(\mathbb{B}^{\alpha}:\nabla\mathbf{\Phi}) + \mathbf{D}^{\alpha}\cdot\mathbf{\Psi}Q^{\alpha} + l\mathbf{\Psi}\cdot\mathbf{D}^{\alpha}\cdot\mathbf{R}^{\alpha} + l\mathbf{D}^{\alpha\beta}\cdot\mathbf{R}^{\beta}Q^{\alpha})\,\mathrm{d}\mathbf{x}, \\ \bar{\mathscr{K}} &= \frac{1}{2}\int_{\Xi}(\mu\dot{W}\dot{W} + l^2\mu^{\alpha\beta}\dot{Q}^{\alpha}\dot{Q}^{\beta} + h^2\dot{\mathbf{\Phi}}\cdot\boldsymbol{\chi}\cdot\dot{\mathbf{\Phi}} + h^2l^2\dot{\mathbf{R}}^{\alpha}\cdot\boldsymbol{\chi}^{\alpha\beta}\cdot\dot{\mathbf{R}}^{\beta})\,\mathrm{d}\mathbf{x}, \\ \bar{\mathscr{W}} &= \int_{\Xi}(fW + lf^{\alpha}Q^{\alpha} + h\mathbf{m}\cdot\mathbf{\Phi} + hl\mathbf{m}^{\alpha}\cdot\mathbf{R}^{\alpha})\,\mathrm{d}\mathbf{x}.\end{aligned} \quad \text{(A.2)}$$

It is visible that the Hamiltonian does not involve spatial derivatives of extra kinematic unknowns Q^{α}, $\mathbf{R}^{\alpha}$ describing displacement fluctuations in the periodicity cell.

Appendix B. Passage from the tolerance averaging to the local model

The continuum model called the local model will be derived from Eqs. (19) by the asymptotic procedure in which the microstructure length parameter l is scaled down. At the same time, it is assumed that the length parameter h (being the mean height of beams) tends towards zero much faster than the parameter l, i.e., $h = o(l)$.

Taking into account definitions (A.1), it is visible that all coefficients but $\mathbb{C}$ will be constant under the above re-scaling. Neglecting the terms involving h and setting $l \to 0$ in governing equations $(19)_{1,2,3}$, we arrive at the following equations:

$$\begin{aligned}&\vartheta(\nabla\otimes\nabla)W - \mu\ddot{W} + \vartheta\boldsymbol{\varepsilon}:(\nabla\cdot\mathbf{\Phi}) + f = 0, \\ &\vartheta\boldsymbol{\varepsilon}:(\nabla W + \boldsymbol{\varepsilon}:\mathbf{\Phi}) + 2(\tilde{\zeta} - \zeta)\nabla(\nabla\cdot\mathbf{\Phi}) + (\tilde{\zeta} + 3\zeta)(\nabla\cdot\nabla)\mathbf{\Phi} = \mathbf{0}\end{aligned} \quad \text{(B.1)}$$

and $Q = 0$.

According to denotations (16), the coefficients $\zeta, \tilde{\zeta}$ are of an order of l^2 and hence can be written in the form $\zeta = l^2\hat{\zeta}$, $\tilde{\zeta} = l^2\bar{\zeta}$. The coefficients $\hat{\zeta}, \bar{\zeta}$ are constant under the limit passage $l \to 0$. Hence Eq. $(\text{B.1})_2$ yields

$$\vartheta\boldsymbol{\varepsilon}:(\nabla W + \boldsymbol{\varepsilon}:\mathbf{\Phi}) + l^2[2(\bar{\zeta} - \hat{\zeta})\nabla(\nabla\cdot\mathbf{\Phi}) + (\bar{\zeta} + 3\hat{\zeta})(\nabla\cdot\nabla)\mathbf{\Phi}] = \mathbf{0}. \quad \text{(B.2)}$$

Under limit passage $l \to 0$, we obtain

$$\nabla W + \boldsymbol{\varepsilon}:\mathbf{\Phi} = \mathbf{0}. \quad \text{(B.3)}$$

Hence, we have

$$\mathbf{\Phi} = -\boldsymbol{\varepsilon}:(\nabla W). \quad \text{(B.4)}$$

Substituting to $(\text{B.1})_1$ the right-hand side of (B.4), we obtain finally Eq. (20)

$$(\tilde{\zeta} + 3\zeta)(\nabla\otimes\nabla):(\nabla\otimes\nabla)W + \mu\ddot{W} - f = 0. \quad \text{(B.5)}$$

The obtained Eq. (B.5) represents an asymptotic model of the periodic lattice-type plate under consideration, which can be called the local model. The only unknown in this model is a deflection W, which has to be a slowly varying function for every t in the region Ξ of $0x_1x_2$.

Limiting considerations to free vibrations in the cylindrical bending case along $x = x_1$ axis Eq. (B.5) takes the form

$$(\tilde{\zeta} + 3\zeta)W_{,1111} + \mu\ddot{W} = 0. \quad \text{(B.6)}$$

For a plate in the cylindrical bending, simply supported on the edges $x = 0$, $x = L$, the solution is assumed in the form $(23)_1$, i.e. $W(x,t) = A_W \sin kx \cos \omega t$. Substituting this solution to (B.6), we obtain the characteristic equation of the local model

$$(\tilde{\zeta} + 3\zeta)k^4 - \mu\,\omega^2 = 0. \tag{B.7}$$

Solution to Eq. (B.7) takes the form (28):

$$\omega = k^2\sqrt{(\tilde{\zeta} + 3\zeta)\mu^{-1}}.$$

Thus, in the framework of the local model only one frequency can be analysed.

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