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Vibration of an oscillator with random damping: Analytical expression for the probability density function

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Abstract

This study deals with the impact of probabilistic viscous and hysteretic damping upon the dynamic response of a linear single degree-of-freedom oscillator. Assuming that damping is governed by an unspecified probability density function (PDF), an analytical expression of the output PDF of the oscillator's transfer function is provided in terms of the input damping PDF. The instance of a uniform PDF is thoroughly treated for both viscous and hysteretic damping cases. The methodology also yields to analytical expressions of all moments of the output law of probability. A definition of a new concept of envelopes is then introduced and their analytical expression is derived. Exact analytical results are compared with Monte Carlo simulations for an academic case study. An identification procedure of a predefined input damping PDF is proposed allowing one to obtain probabilistic parameters from experimental frequency response data.
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1. Introduction

A lot of work related to the treatment of parameters uncertainty within mechanical systems exists in the literature. This field of research which is referring to *uncertain parameters* is from a general point of view studying the influence of uncertainty brought by intrinsic parameters upon the system's behavior, rather than the uncertainty associated to external excitation.

In recent studies [1,2] which cover a broad range of engineering problems, numerous methods have been investigated that treat parameters uncertainty: Monte Carlo approach [3], method of chaotic polynomials [2], method of intervals [4], Taguchi method [5], perturbation method or still sensitivity method [6]. Most of these approximate techniques apply to the investigation of the response of a linear problem with respect to a random parameter and may be linked to finite element methods for predictions of complex systems. Unfortunately, each previously mentioned method can only provide an approximate solution and inherits of its own limitations. First drawback is bound to convergence problems. For example, the perturbation method exhibits strong divergence problems around resonances frequencies. High-order chaotic polynomials may be required for reducing oscillating behavior and accurately converging towards genuine solutions of the

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Nomenclature

c	viscous damping coefficient	M_k	moment of order k
$E(\cdot)$	expected value of $\cdot$	$P(\cdot)$	probability of $\cdot$
$f\cdot$	probability density function of $\cdot$	η	loss factor
H	transfer function	$\sigma\cdot$	standard deviation of $\cdot$
k	stiffness	ω	driving frequency
m	mass	$(\bar{\cdot})$	mean value of $\cdot$
		$(\hat{\cdot})$	Fourier transform of $\cdot$

problem [2]. Inner limitations relate to polynomial or series truncations in the case of perturbation method or chaotic polynomials. Limitations can also have a numerical origin as in Monte Carlo simulations for which convergence requires a high number of computational drawings. Alternatively, Soize proposed a non-parametric model to treat uncertainty [7–9]. Although the construction of non-parametric model is mathematically more complex, the main interest is that it only involves basic and direct calculations.

Additionally, obtaining the system response law of probability is made difficult for all previous methods which only provide the first moments of the law. A possibility is nevertheless offered in the case of the method of intervals to build the transfer function envelope by using a modified Rump algorithm [4]. Again, analytical definition of envelopes is out of reach for all aforementioned methods.

The present study aims at analytically solving the problem of an idealized spring mass damped oscillator, whose damping is assumed to be probabilistic. Both viscous and hysteretic damping are considered here. To the best of the authors' knowledge, such a direct analytical resolution has seldom been tried. In Refs. [10–12], authors propose an analytical method to only estimate mean value and standard deviation of the steady-state displacement response of such system. Moreover, introducing an analytical treatment of damping's randomness quickly leads to the investigation of a class of problems in which the system response is nonlinearly dependent on intrinsic parameters. Here though the simple spring–mass response is linear with respect to driving displacement amplitude, its dependance is nonlinear according to driving frequency of excitation and damping. Analytical developments described hereafter also emphasize some of the difficulties inherent to probabilistic modeling.

The paper is organized as follows: basic definitions and probabilistic description of the problem are recalled in Section 1. Random viscous damping is treated in Section 2 and random hysteretic damping in Section 3. A new definition of the envelope is introduced in Section 4 and a comparison with numerical simulations achieved by applying the sensitivity method is given in Section 5. An identification method is finally proposed in Section 6.

2. Description of the problem

2.1. Description of the mechanical system

Consider a single degree-of-freedom system, consisting of a mass, a viscous damping and a spring. The forced equation of motion of this system is

$$m\ddot{x}(t) + 2c\dot{x}(t) + kx(t) = f(t), \tag{1}$$

where t is the time variable, x is the displacement, k is the spring stiffness, m is the mass, c is the viscous damping coefficient and f the external forcing.

Eq. (1) may be written in the Fourier domain as

$$[-m\omega^2 + 2ci\omega + k]\hat{x} = \hat{f}, \tag{2}$$

where ω stands for the driving frequency and $\hat{X}$ denotes the Fourier transform of X.

Alternatively to Eq. (2), a hysteretic damping η—also called loss factor—may be introduced in the Fourier domain. From a physical point of view, hysteretic damping may be defined as the ratio of energy dissipated to energy stored per cycle. Then Eq. (2) becomes:

$$[-m\omega^2 + k(1 + i\eta)]\hat{x} = \hat{f}. \tag{3}$$

The transfer function of the simply damped oscillator defined as ($\hat{H} = \hat{x}/\hat{f}$) is classically given by

$$\hat{H}(\omega) = \frac{1}{-m\omega^2 + 2ic\omega + k} \tag{4}$$

for viscous damping and

$$\hat{H}(\omega) = \frac{1}{-m\omega^2 + k(1 + i\eta)} \tag{5}$$

for hysteretic damping.

In following sections, the probabilistic parameter is assumed to be either the viscous damping coefficient c or the loss factor η. It can be noted that the mass and stiffness are assumed to be deterministic. The transfer function $\hat{H}$ is nonlinear with respect to the probabilistic parameters.

2.2. Probabilistic description

Let v denote a probabilistic parameter of the previous mechanical problem, whose density is governed by a strictly positive measurable function f_v belonging to Lebesgue space $L^2(\mathbb{R})$ called density of v and such that:

$$\int_{-\infty}^{+\infty} f_v(t)\,\mathrm{d}t = 1, \tag{6}$$

$$\int_{-\infty}^{+\infty} Q(t) f_v(t)\,\mathrm{d}t < +\infty, \tag{7}$$

where $Q(t)$ is a polynomial function in the variable t. If a density f_v satisfies Eqs. (6) and (7), the parameter v is referred to a *random variable*.

The probability that v lies in the interval $[v_1, v_2]$ is denoted $P(v_1 \leqslant v \leqslant v_2)$, and is given by

$$P(v_1 \leqslant v \leqslant v_2) = \int_{v_1}^{v_2} f_v(t)\,\mathrm{d}t. \tag{8}$$

The mean value (written $\bar{v}$) or expected value (written E) of a random variable v is defined by

$$\bar{v} = E(v) = \int_{-\infty}^{+\infty} t f_v(t)\,\mathrm{d}t. \tag{9}$$

The standard deviation of the random variable v (written σ_v) is defined by the square root of $E((v - E(v))^2)$ namely:

$$\sigma_v^2 = E((v - E(v))^2) = E(v^2) - E^2(v) = \int_{-\infty}^{+\infty} t^2 f_v(t)\,\mathrm{d}t - \bar{v}^2. \tag{10}$$

If the first two moments exist, v can be re-normalized to the standardized random variable ξ with an average of 0 and a standard deviation of 1. Thus, the standardized random variable is defined by

$$\xi = \frac{v - E(v)}{\sigma_v}. \tag{11}$$

Define $f_\xi(t) = \sigma_v f_v(\bar{v} + \sigma_v t)$. This is an even function, with the following properties:

$$\int_{-\infty}^{+\infty} t f_\xi(t)\,\mathrm{d}t = 0 \tag{12}$$

and

$$\int_{-\infty}^{+\infty} t^2 f_\xi(t)\,\mathrm{d}t = 1. \tag{13}$$

2.3. Choice of the density function

In the following, most derivatives will be carried out with an unspecified PDF for the random variable ξ. The choice of the PDF must take into account the physical conditions of the problem.

It is important to note that in practice, the probability law of damping is usually believed to be log-normal. This type of law allows physical conditions to be satisfied, because it prevents damping from becoming zero or negative. Nevertheless, even if it is a "physical" law, it is not compactly supported.

For the present paper, the analytical derivation of the output PDF in problem (4) is reachable if a uniform probability law for damping is assumed. This is one originality of the paper.

3. Random viscous damping

3.1. Introduction of randomness

The response density of interest is considered to be the modulus of the oscillator's transfer function. The aim is to give analytical expressions of output PDF in terms of the probability density of the standardized random variable ξ. The case of viscous damping is first considered.

According to Section 2.3, Eq. (2) may be written as

$$[-\omega^2 m + 2(\bar{c} + \sigma_c \xi) i\omega + k]\hat{x} = \hat{f}, \tag{14}$$

where $\bar{c}$ and σ_c, respectively, stand for the viscous damping coefficient mean value and the standard deviation.

Writing the left-hand side of Eq. (14) as $\mathbf{L} + \mathbf{\Pi}$ where:

$$\mathbf{L} = -\omega^2 m + 2\bar{c} i\omega + k, \tag{15}$$

$$\mathbf{\Pi} = 2\sigma_c \bar{c} i\omega \tag{16}$$

gives:

$$(\mathbf{L} + \xi\mathbf{\Pi})H = 1. \tag{17}$$

$\mathbf{L}$ is referred as the deterministic resolvent operator. It is the operator that solves the deterministic problem (if $\xi = 0$) and $\mathbf{\Pi}$ stands for the stochastic resolvent operator.

To this point it is important to recall that $|\hat{H}|$ depends both on the random parameter ξ and on the driving frequency ω. Hence $|\hat{H}|^2$ may be written as

$$|\hat{H}(\omega, \xi)|^2 = \frac{1}{\beta_0 + \beta_1(\xi)\omega^2 + \beta_2\omega^4}, \tag{18}$$

where

$$\begin{cases} \beta_0 = k^2, \\ \beta_1(\xi) = -2mk + 4(\bar{c} + \sigma_c\xi)^2, \\ \beta_2 = m^2. \end{cases}$$

Defining $P_1(X) = \beta_0 + \beta_1(\xi)X + \beta_2 X^2$ where $X = \omega^2$, the discriminant Δ reads $\Delta = 16(\bar{c} + \sigma_c\xi)^2((\bar{c} + \sigma_c\xi)^2 - mk)$. A condition is obtained by enforcing Δ to be always negative which is expressed by

$$\left|\xi + \frac{\bar{c}}{\sigma_c}\right| < \frac{\sqrt{mk}}{\sigma_c}. \tag{19}$$

Condition (19) gives a bound on the choice of $\bar{c}$ and σ_c.

The resonance frequency ω_c of $|H|^2$ is found by writing:

$$\frac{\partial|\hat{H}|^2(\omega,\xi)}{\partial\omega}=0, \tag{20}$$

i.e.

$$\omega_c=\frac{\sqrt{mk-2(\bar{c}+\sigma_c\xi)^2}}{m}. \tag{21}$$

It can be noted that the resonance frequency also depends on the random variable ξ.

Furthermore, if one considers $|\hat{H}|^2$ as a function of ξ and ω, a divergent point occurs for

$$(\xi,\omega)=\left(-\frac{\bar{c}}{\sigma_c},\sqrt{\frac{k}{m}}\right), \tag{22}$$

which is physically related to the cancellation of viscous damping c at resonance frequency ω_c. Hence, divergence is only permitted when viscous damping c vanishes. Such random damping does not appear judicious since it may become negative. In order to correct this drawback, ω_c is assumed to be real in Eq. (21) thus extending the domain of definition of ω_c. This condition may be expressed by

$$\left|\xi+\frac{\bar{c}}{\sigma_c}\right|<\frac{1}{\sigma_c}\sqrt{\frac{mk}{2}}. \tag{23}$$

Condition (23) can always be satisfied for any general law of probability and replaces Eq. (19) in what follows.

3.2. Expression for the probability density function of $|\hat{H}|$

Consider a fixed value of ω. The transfer function $|\hat{H}|^2$ becomes a mapping of ξ and is renamed $|\hat{H}|^2=Y=\phi(\xi)$ for notation purposes. ϕ is strictly monotonically increasing over the interval $]-\infty,-\bar{c}/\sigma_c[$ and decreasing over the interval $]-\bar{c}/\sigma_c,\infty[$.

On the other hand, $\phi(t-(\bar{c}/\sigma_c))$ appears to be even with respect to t. Then:

$$\phi\left(\left]-\frac{\bar{c}}{\sigma_c},\infty\right[\right)=\phi\left(\left]-\infty,-\frac{\bar{c}}{\sigma_c}\right[\right)=\left]0,\frac{1}{(\omega^2m-k)^2}\right[=I. \tag{24}$$

I is then the image of $\mathbb{R}$ by ϕ. Hence, each real number $y\in I$ is the image under ϕ of two real numbers ξ_+ and ξ_- belonging, respectively, to intervals $]-\infty,-\bar{c}/\sigma_c[$ and $]-\bar{c}/\sigma_c,\infty[$ and defined by

$$\xi_+=-\frac{\bar{c}}{\sigma_c}+\frac{\sqrt{y-y^2(\omega^2m-k)^2}}{2y\sigma_c\omega} \tag{25}$$

and

$$\xi_-=-\frac{\bar{c}}{\sigma_c}-\frac{\sqrt{y-y^2(\omega^2m-k)^2}}{2y\sigma_c\omega}. \tag{26}$$

Let f_ξ be the PDF associated to ξ and f_Y the PDF associated with $Y=\phi(\xi) = |\hat{H}|^2$. f_Y is given by

$$f_Y(y)=\frac{f_\xi(\phi^{-1}(y))}{|\phi'(\phi^{-1}(y))|}, \tag{27}$$

where $'$ denotes the derivative with respect to ξ.

It is easy to derive the following property:

$$|\phi'(\phi^{-1}(y))|=|\phi'(\xi_+)|=|\phi'(\xi_-)|=4\sigma_c y\omega\sqrt{y-y^2(\omega^2m-k)^2}. \tag{28}$$

Then f_Y vanishes over $\mathbb{R}\backslash I$ and may be simplified as

$$\forall y \in I, \quad f_Y(y) = \frac{f_\xi(\xi_+) + f_\xi(\xi_-)}{4\sigma_c y\omega\sqrt{y - y^2(\omega^2 m - k)^2}}. \tag{29}$$

In order to derive the expression of PDF of the transfer function modulus $|\hat{H}|$ noted f_H, one proceeds in the same way.

Considering a new random variable ψ such that $|\hat{H}| = \psi(Y)$, then:

$$\psi(y) = \sqrt{y}. \tag{30}$$

ψ is obviously injective over subset I since $\psi^{-1}(h) = h^2$. Letting $J =]0, 1/|\omega^2 m - k|[$ and applying Eq. (27) one finally obtains that f_H is zero on $\mathbb{R}\backslash J$ and that $\forall h \in J$:

$$f_H(h) = \frac{g(\psi^{-1}(h))}{|\psi'(\psi^{-1})(h)|} = \frac{f_\xi(y_+) + f_\xi(y_-)}{2\sigma_c h^2 \omega\sqrt{1 - h^2(\omega^2 m - k)^2}}, \tag{31}$$

with

$$y_+ = -\frac{\bar{c}}{\sigma_c} + \frac{\sqrt{1 - h^2(\omega^2 m - k)^2}}{2h\sigma_c\omega}, \tag{32}$$

$$y_- = -\frac{\bar{c}}{\sigma_c} - \frac{\sqrt{1 - h^2(\omega^2 m - k)^2}}{2h\sigma_c\omega}. \tag{33}$$

3.3. Application with the uniform law

Since resolving $y_+(h) = \pm\sqrt{3}$ or $y_-(h) = \pm\sqrt{3}$ leads to essentially the same calculations, applying Eq. (27) with the uniform law:

$$f_\xi(\theta) = \frac{\mathbb{1}_\xi}{2\sqrt{3}} \quad \text{with } \mathbb{1}_\xi = \begin{cases} 1 & \text{if } \xi \in [-\sqrt{3}, \sqrt{3}], \\ 0 & \text{otherwise.} \end{cases} \tag{34}$$

This leads to two values h_1 and h_2 given by

$$h_1 = ((\omega^2 m - k)^2 + 4\omega^2(\bar{c} + \sqrt{3}\sigma_c)^2)^{-1/2} \tag{35}$$

and

$$h_2 = ((\omega^2 m - k)^2 + 4\omega^2(\bar{c} - \sqrt{3}\sigma_c)^2)^{-1/2}. \tag{36}$$

Here two cases shall be distinguished. If condition $(\bar{c}/\sigma_c) > \sqrt{3}$ holds, then

$$f_H(h) = \frac{\mathbb{1}_h}{4\sqrt{3}\sigma_c h^2\omega\sqrt{1 - h^2(\omega^2 m - k)^2}} \quad \text{with } \mathbb{1}_h = \begin{cases} 1 & \text{if } h \in [h_1, h_2], \\ 0 & \text{otherwise.} \end{cases} \tag{37}$$

Moreover, if a lower bound provided by $(\bar{c}/\sigma_c) \leqslant \sqrt{3}$ prevails, then:

$$f_H(h) = \frac{\mathbb{1}_h}{4\sqrt{3}\sigma_c h^2\omega\sqrt{1 - h^2(\omega^2 m - k)^2}} + \frac{\mathbb{1}_{h_{\lim}}}{2\sqrt{3}\sigma_c h^2\omega\sqrt{1 - h^2(\omega^2 m - k)^2}}$$

$$\text{with } \mathbb{1}_{h_{\lim}} = \begin{cases} 1 & \text{if } h \in \left]h_2, \frac{1}{|\omega^2 m - k|}\right[, \\ 0 & \text{otherwise.} \end{cases} \tag{38}$$

Previous conditions related to Eqs. (37) and (38) are essentially correlated to the choice of viscous damping parameter c. Indeed, condition (38) means that damping can become zero or negative which results in the

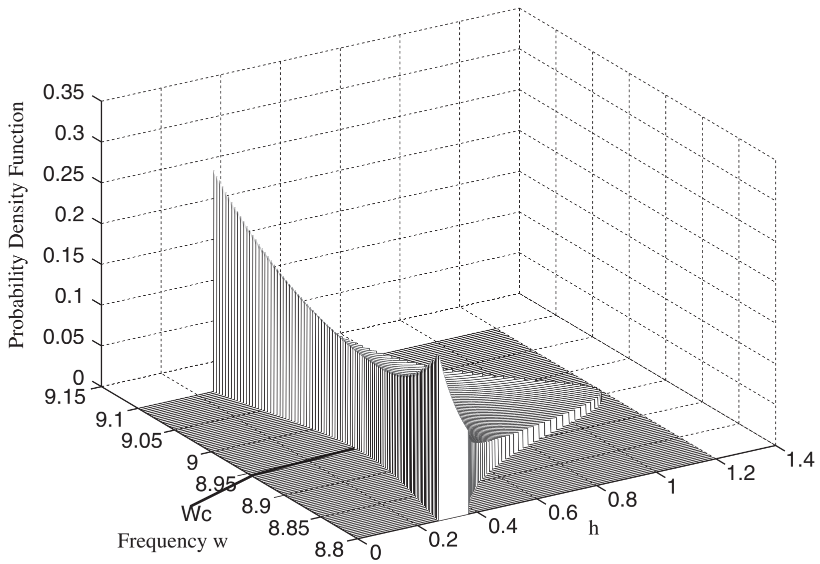

Fig. 1. PDF of frequency response function in the neighborhood of $\omega_c \simeq 8.9443$ for parameters: $m = 1$, $k = 80 \times m$, $\bar{c} = 0.1$, $\sigma_c = 0.3 \times \bar{c}$. The variable h is the variable of $f_H(h)$.

divergence of the PDF at point $(\xi, \omega) = (-\bar{c}/\sigma_c, \sqrt{k/m})$. Here it must be pointed out that it is hopeless to think that moments will converge. On the contrary, if condition (37) is met, viscous damping realizations always remain in the physically acceptable domain. As an example, density f_H is plotted in Fig. 1 for ω close to the resonance frequency in the case of a uniform input probability law.

3.4. Calculation of the three first moments with the uniform law

The first three moments of the transfer function modulus are calculated in case (37), i.e. when $\bar{c}/\sigma_c > \sqrt{3}$. The general moment of order k is given by

$$M_k = \int_{\mathbb{R}} t^k f_H(t)\,\mathrm{d}t = \int_{h_1}^{h_2} \frac{t^k}{4\sqrt{3}\sigma_c t^2 \omega \sqrt{1 - t^2(\omega^2 m - k)^2}}\,\mathrm{d}t, \tag{39}$$

with M_0 being equal to 1.

Analytical expression of moments for orders $k = 1, 2, 3$ are developed in the following equations:

$$\begin{aligned} M_1 &= \frac{\sqrt{3}}{12\omega\sigma_c} \int_{h_1|\omega^2 m - k|}^{h_2|\omega^2 m - k|} \frac{\mathrm{d}u}{u\sqrt{1-u^2}} \\ &= \frac{\sqrt{3}}{12\omega\sigma_c} \ln\left(\frac{\sqrt{(\omega^2 m - k)^2 + 4\omega^2(\bar{c} - \sqrt{3}\sigma_c)^2} - 2\omega(\bar{c} - \sqrt{3}\sigma_c)}{\sqrt{(\omega^2 m - k)^2 + 4\omega^2(\bar{c} + \sqrt{3}\sigma_c)^2} - 2\omega(\bar{c} + \sqrt{3}\sigma_c)} \right), \end{aligned} \tag{40}$$

$$\begin{aligned} M_2 = \frac{\sqrt{3}}{12\omega\sigma_c|\omega^2 m - k|} &\left[\arcsin\left(\frac{|\omega^2 m - k|}{\sqrt{(\omega^2 m - k)^2 + 4\omega^2(\bar{c} - \sqrt{3}\sigma_c)^2}} \right) \right. \\ &\left. - \arcsin\left(\frac{|\omega^2 m - k|}{\sqrt{(\omega^2 m - k)^2 + 4\omega^2(\bar{c} + \sqrt{3}\sigma_c)^2}} \right) \right], \end{aligned} \tag{41}$$

$$M_3 = \frac{\sqrt{3}}{6\sigma_c(\omega^2 m - k)^2}\left(\frac{\bar{c} + \sqrt{3}\sigma_c}{\sqrt{(\omega^2 m - k)^2 + 4\omega^2(\bar{c} + \sqrt{3}\sigma_c)^2}} - \frac{\bar{c} - \sqrt{3}\sigma_c}{\sqrt{(\omega^2 m - k)^2 + 4\omega^2(\bar{c} - \sqrt{3}\sigma_c)^2}}\right). \quad (42)$$

3.5. Numerical validation

Results obtained by using formulas (40) and (41) are compared with numerical Monte Carlo simulations involving 10 000 and 6 millions outcomes as depicted in Fig. 2. In addition to the fact that the analytical expressions require less computational time compared to the Monte Carlo method, Fig. 2 shows that for a high standard deviation level (50% of the selected average initial value), Monte Carlo simulations requires 6 millions outcomes to be in good agreement with the exact ones provided by analytical solutions. Moreover, a statistical approach as Monte Carlo demonstrates that it is difficult to evaluate convergence speed with this method. It converges with a speed of $\sigma_H/\sqrt{n}$, where σ_H is the standard deviation of the response and n the number of outcomes. The problem is that σ_H is not known a priori.

Analytical results constitute a reference solution. The present simulations, of course, illustrate an academic problem in which calculations have been led up to an order 2 with a large initial standard deviation parameter. Such analysis may however be justified in the case of small viscous damping, for which accurate experimental values are difficult to measure.

4. Random hysteretic damping

4.1. Introduction of randomness

Proceeding in the same way as for the viscous damping, Eq. (5) becomes:

$$[-\omega^2 m + k(1 + i(\bar{\eta} + \sigma_\eta \xi))]\hat{x} = \hat{f}. \quad (43)$$

Using the same decomposition $\mathbf{L} + \mathbf{\Pi}$ as in Section 3:

$$\mathbf{L} = -\omega^2 m + k(1 + i\bar{\eta}), \quad (44)$$

$$\mathbf{\Pi} = ki\sigma_\eta, \quad (45)$$

with

$$(\mathbf{L} + \xi\mathbf{\Pi})\hat{H} = 1. \quad (46)$$

Then

$$|\hat{H}(\omega, \xi)|^2 = \frac{1}{\beta_0 + \beta_1\omega^2 + \beta_2\omega^4}, \quad (47)$$

where

$$\begin{cases} \beta_0 = k^2(1 + (\bar{\eta} + \sigma_\eta\xi)^2), \\ \beta_1 = -2mk, \\ \beta_2 = m^2. \end{cases}$$

Considering $P(X) = \beta_0 + \beta_1 X + \beta_2 X^2$, (with $X = \omega^2$), the discriminant Δ of P may be written as

$$\Delta = -4m^2k^2(\bar{\eta} + \sigma_\eta\xi)^2. \quad (48)$$

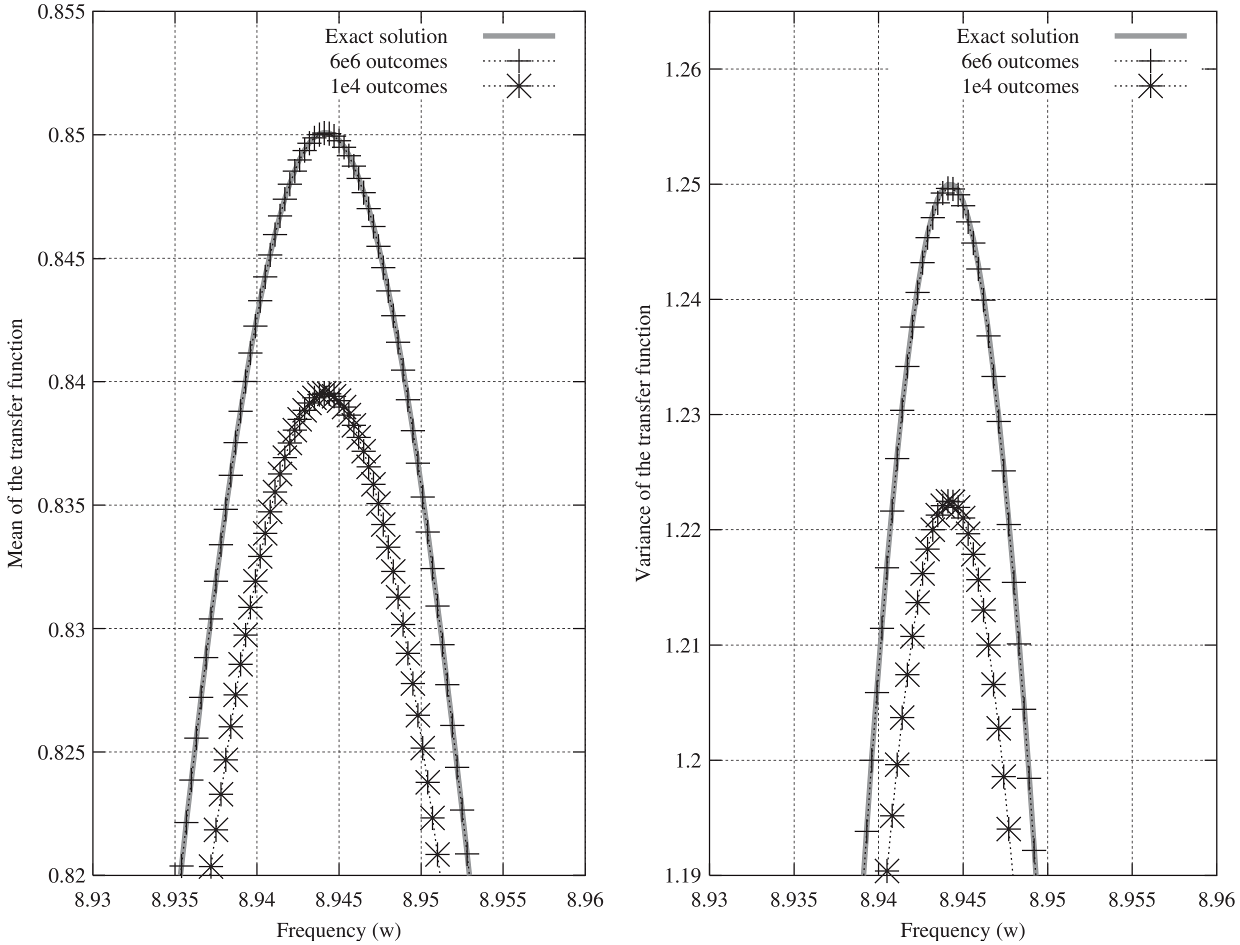

Fig. 2. Effect of random damping around $\omega_c \simeq 8.9443$. Comparison between analytical approach (–) and Monte Carlo method with, respectively, 10 000 (⁎) and 6 million (+) outcomes for moments of order 1 and 2 of H: $m = 1$, $k = 80 \times m$, $\bar{c} = 0.1$, $\sigma_c = 0.5 \times \bar{c}$.

Except for the critical value $\xi = -\bar{\eta}/\sigma_\eta$, which has no physical meaning, $|\hat{H}|^2$ is not singular in ω. A condition identical to Eq. (23) holds for ξ.

4.2. Expression of the probability density function $|\hat{H}|$

As in Section 2, PDF f_H is zero over subset $\mathbb{R}\backslash J$ and for all $h \in J$:

$$f_H(h) = \frac{f_\xi(y_+) + f_\xi(y_-)}{\sigma_\eta h^2 k \sqrt{1 - h^2(\omega^2 m - k)^2}}, \tag{49}$$

with

$$y_+ = -\frac{\bar{\eta}}{\sigma_\eta} + \frac{\sqrt{1 - h^2(\omega^2 m - k)^2}}{h \sigma_\eta k}, \tag{50}$$

$$y_- = -\frac{\bar{\eta}}{\sigma_\eta} - \frac{\sqrt{1 - h^2(\omega^2 m - k)^2}}{h \sigma_\eta k}. \tag{51}$$

4.3. Application with the uniform law

Similarly to Section 2, h_1 and h_2 are given by

$$h_1 = ((\omega^2 m - k)^2 + k^2(\bar{\eta} + \sqrt{3}\sigma_\eta)^2)^{-1/2}, \tag{52}$$

$$h_2 = ((\omega^2 m - k)^2 + k^2(\bar{\eta} - \sqrt{3}\sigma_\eta)^2)^{-1/2}. \tag{53}$$

Two cases shall be distinguished: If $\bar{\eta}/\sigma_\eta > \sqrt{3}$, then:

$$f_H(h) = \frac{\mathbb{1}_h}{2\sqrt{3}\sigma_\eta h^2 k\sqrt{1 - h^2(\omega^2 m - k)^2}} \quad \text{with } \mathbb{1}_h = \begin{cases} 1 & \text{if } h \in [h_1, h_2], \\ 0 & \text{otherwise.} \end{cases} \tag{54}$$

If $\bar{\eta}/\sigma_\eta \leqslant \sqrt{3}$, then:

$$f_H(h) = \frac{\mathbb{1}_h}{2\sqrt{3}\sigma_\eta h^2 k\sqrt{1 - h^2(\omega^2 m - k)^2}} + \frac{\mathbb{1}_{h_{\text{lim}}}}{\sqrt{3}\sigma_\eta h^2 k\sqrt{1 - h^2(\omega^2 m - k)^2}}$$

$$\text{with } \mathbb{1}_{h_{\text{lim}}} = \begin{cases} 1 & \text{if } h \in \left]h_2, \dfrac{1}{|\omega^2 m - k|}\right[, \\ 0 & \text{otherwise.} \end{cases} \tag{55}$$

Eqs. (54) and (55) are similar to criteria (37) and (38). As expected, they naturally depend on the choice of hysteretic damping η. Since η must be positive, only Eq. (54) is considered.

4.4. Calculation of the three first moments with the uniform law

In the case where $\bar{\eta}/\sigma_\eta > \sqrt{3}$, the first moment M_0 is equal to 1 and the first three moments of the output transfer function modulus are given by

$$M_1 = \frac{\sqrt{3}}{6k\sigma_\eta} \int_{h_1|\omega^2 m - k|}^{h_2|\omega^2 m - k|} \frac{\mathrm{d}u}{u\sqrt{1 - u^2}} \tag{56}$$

$$= \frac{\sqrt{3}}{6k\sigma_\eta} \ln\left(\frac{\sqrt{(\omega^2 m - k)^2 + k^2(\bar{\eta} - \sqrt{3}\sigma_\eta)^2} - k(\bar{\eta} - \sqrt{3}\sigma_\eta)}{\sqrt{(\omega^2 m - k)^2 + k^2(\bar{\eta} + \sqrt{3}\sigma_\eta)^2} - k(\bar{\eta} + \sqrt{3}\sigma_\eta)}\right), \tag{57}$$

$$M_2 = \frac{\sqrt{3}}{6k\sigma_\eta|\omega^2 m - k|}\left[\arcsin\left(\frac{|\omega^2 m - k|}{\sqrt{(\omega^2 m - k)^2 + k^2(\bar{\eta} - \sqrt{3}\sigma_\eta)^2}}\right) - \arcsin\left(\frac{|\omega^2 m - k|}{\sqrt{(\omega^2 m - k)^2 + k^2(\bar{\eta} + \sqrt{3}\sigma_\eta)^2}}\right)\right], \tag{58}$$

$$M_3 = \frac{\sqrt{3}}{6\sigma_\eta(\omega^2 m - k)^2}\left(\frac{\bar{\eta} + \sqrt{3}\sigma_\eta}{\sqrt{(\omega^2 m - k)^2 + k^2(\bar{\eta} + \sqrt{3}\sigma_\eta)^2}} - \frac{\bar{\eta} - \sqrt{3}\sigma_\eta}{\sqrt{(\omega^2 m - k)^2 + k^2(\bar{\eta} - \sqrt{3}\sigma_\eta)^2}}\right). \tag{59}$$

4.5. Numerical validation

Same comments and conclusions as in Section 2 may be drawn for hysteretic damping. Results are not shown, since they are very similar to Fig. 2. Practically speaking, estimating hysteric damping is not easy to perform experimentally and predictions are highly variable. This acknowledges the relevance of considering hysteretic damping as a random parameter that greatly influences the output transfer function behavior. A new concept of envelope is therefore introduced hereafter to carefully take into account these aspects.

5. Concept of envelope

From an engineering point of view, the ability to assess the probability of making errors due to intrinsic parameters randomness, is of great interest. This leads to the definition of randomness factors which is closely related to the concept of confidence interval.

5.1. Definition of confidence interval

Given the probability density function f_X of a random variable X, the distribution function which characterizes the law of probability may be written as

$$F(\beta) = P(X < \beta) = \int_{-\infty}^{\beta} f_X(t)\,\mathrm{d}t. \tag{60}$$

The confidence interval $[\beta_1, \beta_2]$ may also be defined by

$$P(\beta_1 < X < \beta_2) = F(\beta_2) - F(\beta_1) = \int_{\beta_1}^{\beta_2} f_X(t)\,\mathrm{d}t = \alpha, \tag{61}$$

where α is a target probability satisfying $0 < \alpha \leqslant 1$.

Confidence intervals are always subject to arbitrariness because it should merely surround the mean value of X. It is also important to answer the following question: "How to choose interval bounds β_1 and β_2 such that for a given probability α, random outcomes of X remain within interval $[\bar{X} - \beta_1, \bar{X} + \beta_2]$?"

Such a probability may be depicted by

$$P(\bar{X} - \beta_1 < X < \bar{X} + \beta_2) = \int_{\bar{X}-\beta_1}^{\bar{X}+\beta_2} f_X(t)\,\mathrm{d}t. \tag{62}$$

Hence, the definition of confidence interval around $\bar{X}$ is

$$\{\beta_1, \beta_2\} \Big/ \int_{\bar{X}-\beta_1}^{\bar{X}+\beta_2} f_H(t)\,\mathrm{d}t = \alpha. \tag{63}$$

5.2. Definition of the envelope

5.2.1. A first definition

Sections 2 and 3 provide analytical expressions for the PDF of the transfer function modulus f_H which depends on ω. From this point it is then possible to exactly compute the distribution function for a given ω using Eq. (63) with PDF given either by Eq. (27) or (49). For the problem of interest, Eq. (63) rewrites:

$$\{\beta_1, \beta_2\} \Big/ \int_{\bar{H}-\beta_1}^{\bar{H}+\beta_2} f_H(t)\,\mathrm{d}t = \alpha. \tag{64}$$

From a practical point, it can be useful to predict possible outcomes of H around its average value for a given probability α. The definition for an α-envelope may then be written as:

$$P(\beta_1^\omega < H - \bar{H} < 0) = \alpha_1 = \int_{\beta_1^\omega}^{0} f_H(t - \bar{H})\,\mathrm{d}t, \tag{65}$$

$$P(0 \leqslant H - \bar{H} < \beta_2^\omega) = \alpha_2 = \int_{0}^{\beta_2^\omega} f_H(t - \bar{H})\,\mathrm{d}t, \tag{66}$$

$$\alpha_1 + \alpha_2 = \alpha. \tag{67}$$

The α-envelope of H is written Υ_H^α where:

$$\Upsilon_H^\alpha = \bigcup_\omega [\beta_1^\omega, \beta_2^\omega]. \tag{68}$$

This is however incorrectly defined.

There is no reason indeed for the sum of probabilities $\mathbf{P}(\beta_1^\omega < H - \bar{H} < 0) + \mathbf{P}(0 \leqslant H - \bar{H} < \beta_2^\omega)$ to be equal to the probability of the sum $\alpha_1 + \alpha_2$. Both $\beta_1^\omega(\alpha_1)$ and $\beta_2^\omega(\alpha_2)$ quantities may exhibit singular points when α_1 and/or α_2 are lower than 1. This definition is ill-posed since it uses the complicated expression of mean value $\bar{H}$ given by Eqs. (40) or (57) which makes the resolution of Eqs. (65) or (66) quite difficult. The output PDF f_H is not symmetric and as a consequence, the output mean value is not centered on interval $[t_1, t_2]$ which makes it difficult to define symmetric envelopes with respect to the mean value. Considering the current definition, the choice of a pair $\{\alpha_1, \alpha_2\}$ appears to be arbitrary in the definition of α-envelopes. It is also virtually possible to define the envelope position according to the probability below the mean value (α_1) or above the mean value (α_2). This last point is not too restrictive since equiprobable envelopes are usually considered.

5.2.2. *Envelope of H for compactly supported input PDF*

Random parameters governed by non-compact densities lead to output transfer function outcomes that may not always be physically admissible. Focusing on the case of random damping for instance, the forced response of the oscillator with a vanishing or negative damping is simply diverging towards infinity. Thus, considering a compactly supported input PDF enables one to derive the expression of PDF of H given by Eqs. (27) or (49) which in turn also appears to be compactly supported. Another advantage is that simple integrations can be carried out to calculate the output law of probability.

$f_H(t)$ is defined on the compact set $[t_1^\omega, t_2^\omega]$. Since f_H is a density probability function:

$$\int_{t_1^\omega}^{t_2^\omega} f_H(t)\,\mathrm{d}t = 1 \quad \forall \omega > 0. \tag{69}$$

Moreover H necessarily belongs to $[t_1^\omega, t_2^\omega]$. Thus an envelope containing all outcomes of H (i.e. with a probability of 1) is given by

$$\bigcup_\omega [t_1^\omega, t_2^\omega]. \tag{70}$$

An α-envelope Υ_H^α may be similarly defined as follows. α_1 and α_2 are two probabilities and one is interested in finding the set $\{\beta_1^\omega, \beta_2^\omega\}$ such that:

$$\int_{t_1^\omega}^{t_1^\omega + \beta_1^\omega} f_H(t)\,\mathrm{d}t = \alpha_1, \tag{71}$$

$$\int_{t_2^\omega - \beta_2^\omega}^{t_2^\omega} f_H(t)\,\mathrm{d}t = \alpha_2, \tag{72}$$

$$\alpha_1 + \alpha_2 = \alpha, \tag{73}$$

involving positive values β_1^ω and β_2^ω.

The α-envelope of H namely Υ_H^α may finally be defined as the complementary of the union of intervals:

$$\Upsilon_H^\alpha = \overline{\bigcup_\omega [t_1^\omega + \beta_1^\omega, t_2^\omega - \beta_2^\omega]}. \tag{74}$$

At first sight, this definition preserves an arbitrary choice for probability levels α_1 and α_2. The meaning of these latter values is now directly connected to the probabilities that an outcome of H does not belong to the envelope defined by the total probability α. Hence, an α-envelope can also be called '$(1-\alpha)$%-envelope'. For instance, a 0.20-envelope may be understood as an 80%-envelope.

5.2.3. Equiprobable envelope

An α-envelope with $\alpha_1 = \alpha_2 = \alpha/2$ is called an equiprobable α-envelope. It should be noticed that an equiprobable envelope is not necessarily centered, i.e. that following property holds:

$$\bar{H} - \beta_1^\omega(\alpha/2) \neq \bar{H} + \beta_2^\omega(\alpha/2). \tag{75}$$

Following applications concentrate on equiprobable envelopes and one will investigate outcomes of H around its average within these envelopes.

5.3. Resolution and application

5.3.1. Expression of Υ_H^α for a compactly supported PDF

The problem consists in solving:

$$\int_{t_1^\omega}^{t_1^\omega+\beta_1^\omega} f_H(t)\,\mathrm{d}t = \alpha_1, \tag{76}$$

$$\int_{t_2^\omega-\beta_2^\omega}^{t_2^\omega} f_H(t)\,\mathrm{d}t = \alpha_2, \tag{77}$$

where β_1^ω and β_2^ω are unknown variables.

Only the case of viscous damping is developed here. Hysteretic damping can be treated in a similar way.

5.3.2. Application to viscous damping

The resolution of Eqs. (76) and (77) leads to

$$\beta_2^\omega(\alpha_2) = \frac{t_2\left(\sqrt{1 + 48t_2^2\alpha_2^2\omega^2\sigma^2 + 8t_2\sqrt{3}\alpha_2\omega\sigma\sqrt{1 - t_2^2(\omega^2 m - k)^2}} - 1\right)}{\sqrt{1 + 48t_2^2\alpha_2^2\omega^2\sigma^2 + 8t_2\sqrt{3}\alpha_2\omega\sigma\sqrt{1 - t_2^2(\omega^2 m - k)^2}}}, \tag{78}$$

$$\beta_1^\omega(\alpha_1) = \frac{t_1\left(1 - \sqrt{1 + 48t_1^2\alpha_1^2\omega^2\sigma^2 - 8t_1\sqrt{3}\alpha_1\omega\sigma\sqrt{1 - t_1^2(\omega^2 m - k)^2}}\right)}{\sqrt{1 + 48t_1^2\alpha_1^2\omega^2\sigma^2 - 8t_1\sqrt{3}\alpha_1\omega\sigma\sqrt{1 - t_1^2(\omega^2 m - k)^2}}}. \tag{79}$$

Note that $\beta_1^\omega(0) = \beta_2^\omega(0) = 0$, which corresponds to an envelope with a maximum range, i.e. it contains all outcomes of H. This shows that a 0-envelope brings us back to Eq. (70).

Moreover, it can be verified that $t_2^\omega - \beta_2^\omega(1) = t_1^\omega$ and $t_1^\omega + \beta_1^\omega(1) = t_2^\omega$ which means that upper and lower envelopes are reversed when $\alpha_1 = 1 (\alpha_2 = 0)$ or $\alpha_2 = 1 (\alpha_1 = 0)$. Note $t_2^\omega - \beta_2^\omega(\alpha) = t_1^\omega + \beta_1^\omega(1-\alpha)$, i.e. a 1-envelope (or 0%-envelope) has a null range.

Eqs. (78) and (79) display the 80%-envelope for the output transfer function modulus for the specific case of random viscous damping. Fig. 3 displays the 80%-envelope together with the mean value curve and the 100%-envelope. Generally speaking, numerical simulations are in reasonable agreement with expected results. The 80%-envelope is located within the maximum envelope and the mean value curve remains well inside the boundaries of these two envelopes. Simulations have been also carried out to check whether computed envelopes correspond to probabilities that can be numerically obtained with a significant number of drawings. The 80%-envelope is superposed to 100 outcomes of H related to random drawings of the input standardized random variable ξ. All numerical simulations are in reasonable agreement with exact ones, the outcomes of H remaining within the 100%-envelope. From a statistical point of view, outcomes of H that are not within the 80%-envelope have been counted and thus the percentage of missed drawings has been estimated. For example, for 50 000 drawings a figure of 20.03% of missed outcomes is obtained.

6. A sensitivity study

6.1. The case of viscous damping

The modulus of frequency response function is a mapping involving variables ξ and ω:

$$|\hat{H}|(\omega, \xi) = \frac{1}{\sqrt{(\omega^2 m - k)^2 + 4\omega^2(\bar{c} + \sigma_c \xi)^2}}. \tag{80}$$

Normalized variable ξ belongs to an interval $[a, b]$, and $|\hat{H}| \in \mathscr{C}^1(\mathbb{R})$. The result that follows may straightforwardly be written by applying the mean-value theorem

$$\frac{\partial |\hat{H}|(\omega, \xi)}{\partial \xi} = \frac{-4\omega^2 \sigma_c(\bar{c} + \sigma_c \xi)}{((\omega^2 m - k)^2 + 4\omega^2(\bar{c} + \sigma_c \xi)^2)^{3/2}}. \tag{81}$$

The bound value K_v is given by

$$K_v = \frac{4\omega^2 \sigma_c(\bar{c} + \sigma_c b)}{((\omega^2 m - k)^2 + 4\omega^2(\bar{c} + \sigma_c a)^2)^{3/2}}, \tag{82}$$

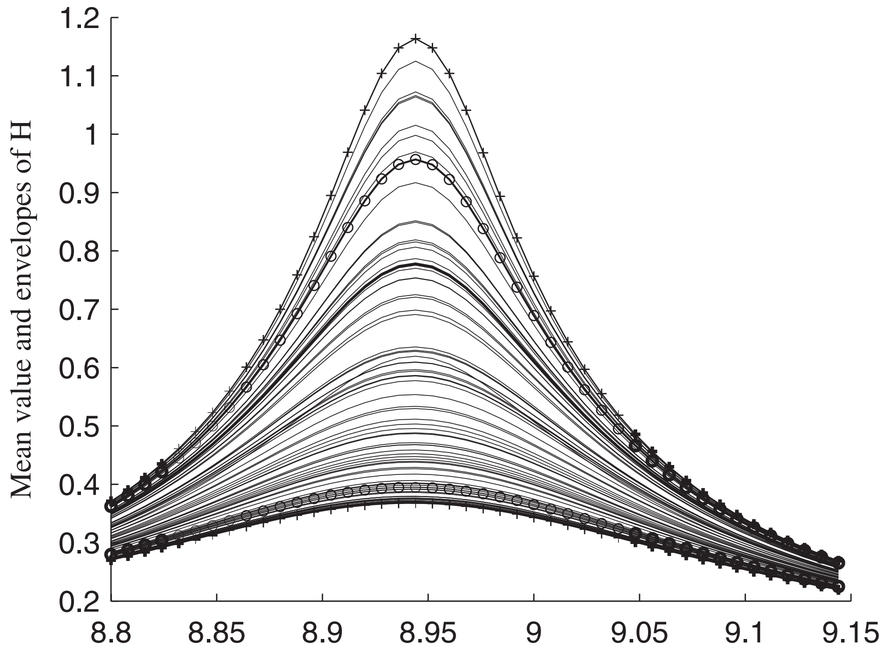

Fig. 3. Mean value (–), 80%-envelopes (○), 100%-envelopes (+) for H and 100 outcomes of H: $\bar{c} = 0.1$, $m = 1$, $k = 80 \times m$, $\sigma_c = 0.3 \times \bar{c}$.

which gives

$$\left||\hat{H}|(\omega,0)-|\hat{H}|(\omega,\xi)\right|\leqslant M_v|b-a|. \tag{83}$$

Assuming that the random parameter is governed by a uniform law ($a=-\sqrt{3}$ and $b=\sqrt{3}$), an inequality holds for $|\hat{H}|$:

$$\left||\hat{H}|(\omega,0)-|\hat{H}|(\omega,\xi)\right|\leqslant\frac{8\sqrt{3}\omega^2\sigma_c(\bar{c}+\sigma_c\sqrt{3})}{((\omega^2m-k)^2+4\omega^2(\bar{c}-\sigma_c\sqrt{3})^2)^{3/2}}. \tag{84}$$

6.2. The case of hysteretic damping

The transfer function modulus is related to ξ and ω,

$$|\hat{H}|(\omega,\xi)=\frac{1}{\sqrt{(\omega^2m-k)^2+k^2(\bar{\eta}+\sigma_\eta\xi)^2}} \tag{85}$$

and similarly:

$$\frac{\partial|\hat{H}|(\omega,\xi)}{\partial\xi}=\frac{-k^2\sigma_\eta(\bar{\eta}+\sigma_\eta\xi)}{((\omega^2m-k)^2+k^2(\bar{\eta}+\sigma_\eta\xi)^2)^{3/2}}. \tag{86}$$

The bound K_h is given by

$$K_h=\frac{k^2\sigma_\eta(\bar{\eta}+\sigma_\eta b)}{((\omega^2m-k)^2+k^2(\bar{\eta}+\sigma_\eta a)^2)^{3/2}}. \tag{87}$$

For the uniform law ($a=-\sqrt{3}$ and $b=\sqrt{3}$), one finally obtains:

$$\left||\hat{H}|(\omega,0)-|\hat{H}|(\omega,\xi)\right|\leqslant\frac{2\sqrt{3}k^2\sigma_\eta(\bar{\eta}+\sigma_\eta\sqrt{3})}{((\omega^2m-k)^2+k^2(\bar{\eta}-\sigma_\eta\sqrt{3})^2)^{3/2}}. \tag{88}$$

6.3. Comparison between envelope and sensitivity methods

The main differences between these two approaches are exhibited in Fig. 4 in the case of viscous damping. In the neighborhood of resonances, the sensitivity method largely over-estimates possible variations of transfer function. Although the selected example only presents results associated with an initial standard deviation value of 20%, variability is even more important for higher values. Similar remarks can be pointed out for hysteretic damping. Generally speaking, envelope curves displayed in Fig. 4 highlight potential benefits offered by the envelope method versus sensitivity method to reduce the randomness margins of studied parameter.

7. Identification of the uniform law

The theoretical construction of 100%-envelopes ensure that all random outcomes of transfer function H remain contained within the envelope. Extremal values obtained by following envelope curves also exhibit mean value and standard deviation of the initial random parameter. Assuming that the parameter of interest is a viscous damping governed by a uniform law, the issue is here to identify σ_c and $\bar{c}$ from 'experimental' data.

Expressions of t_1^ω and t_2^ω in Eq. (70) are given by condition (37) as follows:

$$t_1^\omega=((\omega^2m-k)^2+4\omega^2(\bar{c}+\sqrt{3}\sigma_c)^2)^{-1/2}, \tag{89}$$

$$t_2^\omega=((\omega^2m-k)^2+4\omega^2(\bar{c}-\sqrt{3}\sigma_c)^2)^{-1/2}. \tag{90}$$

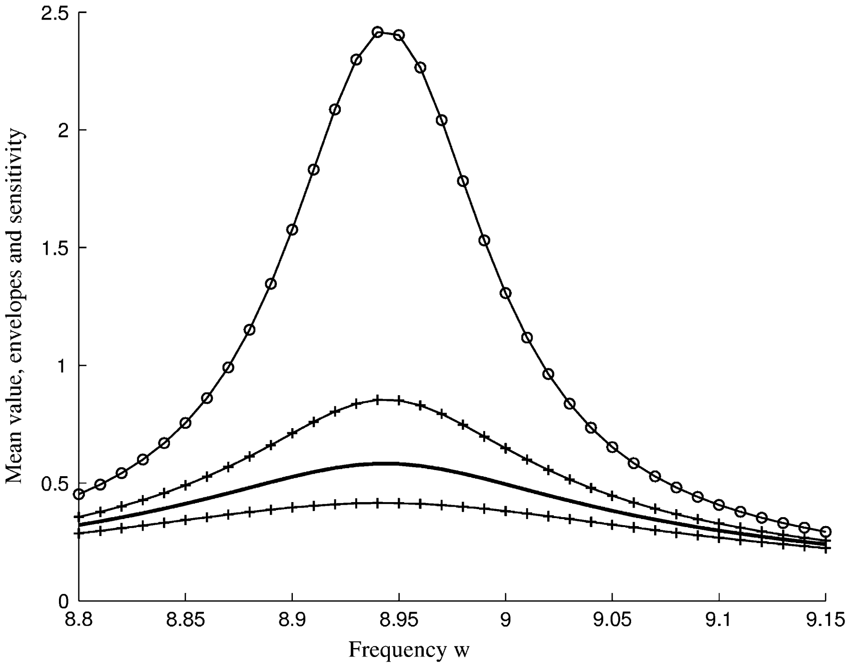

Fig. 4. Comparison between sensitivity and proposed envelope methods. Mean value (–), 100%-envelope (+) and estimates given by the sensitivity study (○) : $m = 1$, $k = 80 \times m$, $\bar{c} = 0.1m$, $\sigma_c = 0.2 \times \bar{c}$.

When resonance is encountered

$$t_1^{\omega_c} = \frac{1}{2\omega_c(\bar{c} + \sqrt{3}\sigma_c)}, \tag{91}$$

$$t_2^{\omega_c} = \frac{1}{2\omega_c(\bar{c} - \sqrt{3}\sigma_c)}. \tag{92}$$

Hence, reading values of $t_1^{\omega_c}$ and $t_2^{\omega_c}$ on the envelope curve and solving previous system of equations finally yields:

$$\bar{c} = \frac{1}{4}\sqrt{\frac{m}{k}}\frac{t_1^{\omega_c} + t_2^{\omega_c}}{t_1^{\omega_c} t_2^{\omega_c}}, \tag{93}$$

$$\sigma_c = \frac{\sqrt{3}}{12}\sqrt{\frac{m}{k}}\frac{t_2^{\omega_c} - t_1^{\omega_c}}{t_1^{\omega_c} t_2^{\omega_c}}. \tag{94}$$

Verification of analytical expressions is made with numerical simulations computed for $\bar{c} = 0.1$ and $\sigma_c = 0.03$ as displayed in Fig. 3. After plotting 10 000 outcomes of the transfer function, $t_1^{\omega_c}$ and $t_2^{\omega_c}$ are estimated numerically and correspond to the experimental 100%-envelope. Applying the proposed identification technique on 100%-envelope numerical data, one finds expected $\bar{c} = 0.999992$ instead of 0.1 and $\sigma_c = 0.0299997$ instead of 0.03 with a precision of order 10^{-6}. This means that the theoretical and experimental 100%-envelope match.

Considering a 'thinner' envelope increases prediction error committed on standard deviation estimate. For instance, if a 50%-envelope is constructed in the following: among the 10 000 outcomes of the transfer function, 25% are discarded starting from each curve delimiting the experimental 100%-envelope. This 50%-envelope gives relatively wide spread estimates $\bar{c} = 0.1052$ and $\sigma_c = 0.0149999$: standard deviation is only evaluated to half of its expected value (the envelope probability is 50%) whereas mean value error is only about 5%. For this particular case, the envelope method may be understood as an identification method for mean and standard deviation values of random parameter c that holds for an input uniform law.

8. Conclusion

The purpose of the present work has been to derive the analytical expression of the output transfer function norm PDF of a simple linear dynamic oscillator featuring a single random parameter. The PDF analytical expression obviously depends on the input density function of the random damping parameter. Regarding numerical applications, only PDFs with compact support for the random parameters have been considered to prevent the occurrence of physical aberrations. Straightforward calculation also provide analytical expressions of the first three moments of the output transfer function probability law in the case of a uniform probability law characterizing damping randomness. Comparison with numerical Monte Carlo drawings shows that analytical results indeed provide exact solution of the investigated problem.

The present analytical methodology brings a better understanding of the influence of input parameter randomness upon the output variability since the output probability density function dispersion is known exactly.

Analyzing analytical formulas (31) or (49) in the particular case of a uniform input probability law shows two important results. Firstly, an initially centered input probability law does not necessarily remain centered for the PDF of the transfer function. Secondly, the property of support compactness is preserved by the oscillator's transfer function. This compactness property of PDF support was originally used to introduce the notion of envelope and to give their analytical expression although no property of symmetry is available for probability density functions. Given the choice of mean and standard deviation values of initial random parameter, the problem of defining both an envelope and a probability that outcomes of H remain within envelope is well posed. The envelope method may be useful to find out parameters of an input law from experimental data.

In the present paper, envelopes were used to identify statistical values such as mean or standard deviation when a random parameter is governed by a uniform law. This technique was numerically tested and identified data proved to be in good agreement with the expected ones. The main disadvantage of the envelope method is in practice related to the experimental construction of the 100% confidence envelope. Important errors may occur in estimating standard deviation as demonstrated in Section 6, which highlights the difficulty in finding available statistical information.

Future prospects deal with the validation of the proposed identification technique using experimental data (for instance, does the identification method work in presence of noise) and the feasibility to draw an envelope sketch from a relatively reasonable number of experimental outcomes. Applying analytical formulas after a calibration process would allow one to characterize the dispersion of viscous or hysteretic damping for the specific case of uniform law. The general envelope procedure also needs further investigation to incorporate other classical laws of probability. Other research topics under study are related to the generalization of present works to treat multiple degree-of-freedom systems or multiple random parameters in relationship with modal synthesis.

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