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# Acoustic waves in a medium bounded by curved surfaces

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**Abstract**

A method for obtaining quasi-analytic solutions to the three-dimensional (3D) Helmholtz equation for the case of an acoustic medium bounded by two identical curved surfaces is presented. The method can be extended to a semi-infinite medium with a curved boundary for the study of Rayleigh waves on a non-planar surface, albeit the solution procedure entails the numerical matrix method. The formulation of the method is based on the differential-geometry argument employing the curved coordinates $(u^1, u^2, u^3)$ where $u^1$ and $u^2$ are along the local tangent plane of one of the bounding surfaces $z = z(x)$ and $u^3$ is perpendicular to the local tangent plane. This choice of coordinates allows the 3D Helmholtz equation, subject to boundary conditions specified on non-planar surfaces, to be solved with relative ease. Normal-mode solutions are shown for the case of a fluid layer with two pressure-release boundaries, where the bounding surfaces are given by the ramp, the Gaussian, and the sinusoidal functions, respectively.


## 1. Introduction

The problem of finding acoustic-wave solutions in fluid-layer structures with simple geometry can be found in several classical textbooks [1–3]. We present quasi-analytical solutions of eigenmodes in curved fluid-layer structures in cases where the layer surface can be described mathematically as $(x, y, z(x))$. The analysis can be extended to the study of surface waves in curved solid-layer structures (or Rayleigh-wave solutions in semi-infinite solid structures subject to curved surfaces), however, the stress-free boundary conditions imposed at curved surfaces require, e.g., a numerical matrix method based on expanding the solution in terms of eigenmodes obeying three ordinary differential equations (ODEs) (which—the present work shows—can be found quasi-analytically). Surface waves [4–6] and their propagation characteristics are important for the understanding of, e.g. earthquake spectra [7–10], surface and subsurface defect detection [11], and vibrations near crystal substrate–film interfaces being of interest in the application of surface acoustic wave spectroscopy to thin-film characterization [12].

Surface waves are characterized by their confinement to a small-region near a certain (two-dimensional, 2D) surface. Hence, such waves are basically 2D in character. This fact allows considerable simplification in their mathematical description in the general surface geometry case by use of differential-geometry methods. In the present paper, a study of solutions to the three-dimensional (3D) Helmholtz equation is given for waves

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confined near a surface $(x, y, z(x))$ where $z(x)$ denotes any function of $x$. The 3D Laplacian operator is expressed in terms of curved coordinates $(u^1, u^2, u^3)$ with $u^1$ and $u^2$ along the local tangent plane of the surface $z = z(x)$ and $u^3$ perpendicular to the local tangent plane. Discarding terms of order one and higher in $u^3$ in the expansion of the Laplacian allows the 3D Helmholtz equation to be separated in three ODEs. Previous studies indicate that this approximation ($u^3 = 0$) works surprisingly well in several concrete examples [13]. Two of the three ODEs can be solved analytically while the third ODE in the general case can be easily solved using, e.g., the finite-difference method.

Three case studies (of relevance in the study of waves in fluid-layer structures) are finally presented and solved in terms of eigenstates and eigenvalues (wavenumbers) for pressure-release boundary conditions corresponding to the plane-sloped surface, the Gaussian surface, and the sinusoidal-shaped surface, respectively. We also present the general stress-free boundary conditions in curved coordinates relevant for determining Rayleigh-wave states propagating near a solid surface $z = z(x)$. The solution to this problem, however, involves numerical methods in the general case and will not be pursued in the present work.

## 2. Theory

In papers by Jensen and Koppe and da Costa [14–16] studying a quantum mechanical particle confined to a surface $\Sigma$, a crucial ingredient was the approximation of the Laplace operator in $\mathbb{R}^3$ near the surface $\Sigma$:

$$\Delta_{\mathbb{R}^3} \approx \Delta_\Sigma + (M^2 - K) + \partial_3^2, \tag{1}$$

where $\Delta_\Sigma$ is the Laplace–Beltrami operator on the surface $\Sigma$, $M$ and $K$ are the mean and Gaussian curvatures, respectively, and $\partial_3$ is the derivative in the direction normal to the surface. Recently, it was found that the approximation is surprisingly good for several concrete examples [13].

In this paper we will use the same technique [13] to study waves confined near a surface $z = z(x)$.

### 2.1. Laplacian in the curvilinear coordinates

The first fundamental form of a surface $z = z(x, y)$ is given by

$$\begin{bmatrix} g_{11} & g_{12} \\ g_{12} & g_{22} \end{bmatrix} = \begin{bmatrix} 1 + z_1^2 & z_1 z_2 \\ z_1 z_2 & 1 + z_2^2 \end{bmatrix}, \tag{2}$$

where

$$z_1 = \partial_1 z = \frac{\partial z}{\partial x} \quad \text{and} \quad z_2 = \partial_2 z = \frac{\partial z}{\partial y}. \tag{3}$$

We shall also make use of the second-order derivatives:

$$z_{11} = \frac{\partial^2 z}{\partial x^2}, \; z_{22} = \frac{\partial^2 z}{\partial y^2}, \; z_{12} = \frac{\partial^2 z}{\partial x \partial y}. \tag{4}$$

The determinant of the first fundamental form is

$$g = 1 + z_1^2 + z_2^2, \tag{5}$$

and the inverse is

$$\begin{bmatrix} g^{11} & g^{12} \\ g^{12} & g^{22} \end{bmatrix} = \frac{1}{1 + z_1^2 + z_2^2} \begin{bmatrix} 1 + z_2^2 & -z_1 z_2 \\ -z_1 z_2 & 1 + z_1^2 \end{bmatrix}. \tag{6}$$

The second fundamental form is

$$\begin{bmatrix} b_{11} & b_{12} \\ b_{12} & b_{22} \end{bmatrix} = \frac{1}{\sqrt{1 + z_1^2 + z_2^2}} \begin{bmatrix} z_{11} & z_{12} \\ z_{12} & z_{22} \end{bmatrix} \tag{7}$$

so the mean and Gaussian curvature is

$$M = \frac{1}{2}\frac{z_{11}(1+z_2^2) - 2z_{12}z_1z_2 + z_{22}(1+z_1^2)}{(1+z_1^2+z_2^2)^{3/2}}, \tag{8}$$

$$K = \frac{z_{11}z_{22} - z_{12}^2}{(1+z_1^2+z_2^2)^2}. \tag{9}$$

Using tensor notation the Laplace–Beltrami operator is

$$\Delta_\Sigma = g^{-1/2}\partial_\ell g^{k\ell} g^{1/2}\partial_k = g^{k\ell}\partial_k\partial_\ell + \left(\frac{g^{k\ell}}{2}\frac{\partial_\ell g}{g} + \partial_\ell g^{k\ell}\right)\partial_k \tag{10}$$

$$= \frac{1+z_2^2}{1+z_1^2+z_2^2}\partial_1^2 - \frac{2z_1z_2}{1+z_1^2+z_2^2}\partial_1\partial_2 + \frac{1+z_1^2}{1+z_1^2+z_2^2}\partial_2^2 - \frac{2M}{\sqrt{1+z_1^2+z_2^2}}(z_1\partial_1 + z_2\partial_2). \tag{11}$$

If $z$ is a function of $x$ only, $z = z(x)$, then $z_2 = z_{12} = z_{22} = 0$ and we have

$$g = 1 + z_1^2, \tag{12}$$

$$K = 0, \tag{13}$$

$$M = \frac{1}{2}\frac{z_{11}}{(1+z_1^2)^{3/2}}, \tag{14}$$

$$\Delta_\Sigma = \frac{1}{1+z_1^2}\partial_1^2 - \frac{z_1z_{11}}{(1+z_1^2)^2}\partial_1 + \partial_2^2, \tag{15}$$

$$\Delta_{\mathbb{R}^3} \approx \Delta_\Sigma + (M^2 - K) + \partial_3^2 \tag{16}$$

$$= \frac{1}{1+z_1^2}\partial_1^2 - \frac{z_1z_{11}}{(1+z_1^2)^2}\partial_1 + \partial_2^2 + \frac{1}{4}\frac{z_{11}^2}{(1+z_1^2)^3} + \partial_3^2. \tag{17}$$

### 2.2. *Rayleigh-wave solutions in the curvilinear coordinates*

In the previous Section, an expression [Eq. (17)] for the Laplacian in the relevant curved coordinates was derived. Substitution into the Helmholtz equation leads to

$$\Delta_{\mathbb{R}^3}v + k^2v = \left(\frac{1}{1+z_1^2}\partial_1^2 - \frac{z_1z_{11}}{(1+z_1^2)^2}\partial_1 + \partial_2^2 + \frac{1}{4}\frac{z_{11}^2}{(1+z_1^2)^3} + \partial_3^2\right)v + k^2v = 0, \tag{18}$$

where $v$ is the acoustic pressure in a fluid or any component of the longitudinal or transverse particle–displacement vector in a solid due to acoustic vibrations, $k = \omega/c$ is the wavenumber, $c$ is the wave speed, and $\omega$ is the angular frequency of the acoustic disturbance. Proceeding in the usual fashion (stimulated by the simple form of the Laplacian), we introduce a general separable solution to Eq. (18)

$$v(u^1, u^2, u^3) = \chi_1(u^1)\chi_2(u^2)\chi_3(u^3), \tag{19}$$

and simple manipulations allow us to obtain

$$\partial_1^2\chi_1 - \frac{z_1z_{11}}{1+z_1^2}\partial_1\chi_1 + \left[\frac{1}{4}\frac{z_{11}^2}{(1+z_1^2)^2} - (c_1+c_2)(1+z_1^2)\right]\chi_1 = 0, \tag{20}$$

$$\partial_2^2\chi_2 + c_2\chi_2 = 0, \tag{21}$$

$$\partial_3^2\chi_3 + \left(k^2 + c_1\right)\chi_3 = 0, \tag{22}$$

where $c_1, c_2$ are separation constants to be determined by imposing appropriate boundary conditions.

Consider now a finite surface corresponding to the parameter values: $0 \leqslant u^1 \leqslant L$, $0 \leqslant u^2 \leqslant H$ and assume that the surface boundary is free, i.e., $v = 0$ in the case of a fluid layer (pressure-release boundary conditions). Application of the boundary conditions immediately leads to

$$\chi_2 = \sin\left(\frac{n\pi}{H}u^2\right), \quad c_2 = \left(\frac{n\pi}{H}\right)^2, \tag{23}$$

where $n = 1, 2, 3, \ldots$ . Next, the possible set of wavenumber values $k$ as a function of $c_1$ are determined by the boundary conditions along the curved surfaces specified by the two $u^3$ values

$$\chi_3(u^3 = 0) = 0, \quad \chi_3(u^3 = -T) = 0, \tag{24}$$

where $T$ is the layer thickness. These boundary conditions imposed on Eq. (22) immediately give

$$\chi_3 = \sin\left(\frac{l\pi}{T}u^3\right), \quad k = \sqrt{-c_1 + \left(\frac{l\pi}{T}\right)^2}, \tag{25}$$

where $l = 1, 2, 3, \ldots$ .

The possible discrete set of $c_1$ values are found by solving Eq. (20) [in the present work by use of the finite-difference method] subject to the pressure-release boundary conditions:

$$\chi_1(u^1 = 0) = \chi_1(u^1 = L) = 0. \tag{26}$$

Once possible $c_1$ values are determined the eigennumber values $k$ are fixed by Eq. (25). Specification of $c_1$ values will be given for three different cases of curved surfaces in the "Results and Discussions" section.

In the case of surface waves in solid structures the above boundary conditions [Eq. (24)] are replaced by

$$\lim_{u^3 \to -\infty} \chi_3 = 0. \tag{27}$$

In the latter case, we assume that wave propagation is restricted to the semi-infinite medium $u^3 \leqslant 0$ and that waves are damped towards the interior of the medium in agreement with the boundary conditions in Eq. (24). This yields

$$\chi_3(u^3) = \exp\left(\sqrt{-k^2 - c_1}u^3\right), \tag{28}$$

where

$$k^2 < -c_1. \tag{29}$$

Note that wave speed $c$ has the value $c_l$ ($c_t$) in the case of longitudinal (transverse) wave components. In the case of Rayleigh waves, the appropriate boundary conditions are stress-free boundary conditions at $u^3 = 0$ [derived in curved coordinates in the following subsection] and the full displacement vector will be a linear combination of longitudinal and transverse eigenstate solutions. This solution, due to the complexity of the stress-free boundary conditions in the general case of curved solid surfaces, must be found employing a numerical procedure.

### *2.3. Stress-free boundary conditions in curved coordinates*

In this section, we derive the general stress-free boundary conditions in curved coordinates relevant for solid structures and, e.g., Rayleigh-wave problems. We assume first that the surface varies with both $x$ and $y$, hence allowing for more general surfaces than considered elsewhere in this work.

Consider a surface given by the function:

$$z = z(x, y). \tag{30}$$

The normal to the surface **N** is then

$$\mathbf{N} = \frac{1}{\sqrt{1+z_1^2+z_2^2}}\begin{bmatrix} -z_1 \\ -z_2 \\ 1 \end{bmatrix}, \tag{31}$$

and the parametrization of a neighborhood to the surface (with $u^3$ small) reads

$$\mathbf{X} = \begin{bmatrix} X^1 \\ X^2 \\ X^3 \end{bmatrix} = \begin{bmatrix} u^1 - u^3 z_1 \Big/ \sqrt{1+z_1^2+z_2^2} \\ u^2 - u^3 z_2 \Big/ \sqrt{1+z_1^2+z_2^2} \\ z(u^1,u^2) + u^3 \Big/ \sqrt{1+z_1^2+z_2^2} \end{bmatrix}. \tag{32}$$

We now see that

$$\left[\frac{\partial X^i}{\partial u^j}\right]\bigg|_{u^3=0} = \begin{bmatrix} 1 & 0 & \dfrac{-z_1}{\sqrt{1+z_1^2+z_2^2}} \\ 0 & 1 & \dfrac{-z_2}{\sqrt{1+z_1^2+z_2^2}} \\ z_1 & z_2 & \dfrac{1}{\sqrt{1+z_1^2+z_2^2}} \end{bmatrix}, \tag{33}$$

and the inverse is

$$\left[\frac{\partial u^i}{\partial X^j}\right]\bigg|_{u^3=0} = \begin{bmatrix} \dfrac{1+z_2^2}{1+z_1^2+z_2^2} & \dfrac{-z_1 z_2}{1+z_1^2+z_2^2} & \dfrac{z_1}{1+z_1^2+z_2^2} \\ \dfrac{-z_1 z_2}{1+z_1^2+z_2^2} & \dfrac{1+z_1^2}{1+z_1^2+z_2^2} & \dfrac{z_2}{1+z_1^2+z_2^2} \\ \dfrac{-z_1}{\sqrt{1+z_1^2+z_2^2}} & \dfrac{-z_2}{\sqrt{1+z_1^2+z_2^2}} & \dfrac{1}{\sqrt{1+z_1^2+z_2^2}} \end{bmatrix}. \tag{34}$$

If the displacement vector has components $v^i$ then the components $\varepsilon_{k\ell}$ of the strain are given by

$$\varepsilon_{11} = \frac{\partial v^1}{\partial X^1} = \frac{\partial v^1}{\partial u^i}\frac{\partial u^i}{\partial X^1} = \frac{\dfrac{\partial v^1}{\partial u^1}(1+z_2^2) - \dfrac{\partial v^1}{\partial u^2} z_1 z_2 - \dfrac{\partial v^1}{\partial u^3} z_1 \sqrt{1+z_1^2+z_2^2}}{1+z_1^2+z_2^2}, \tag{35}$$

$$\varepsilon_{22} = \frac{\partial v^2}{\partial X^2} = \frac{\partial v^2}{\partial u^i}\frac{\partial u^i}{\partial X^2} = \frac{-\dfrac{\partial v^2}{\partial u^1} z_1 z_2 + \dfrac{\partial v^2}{\partial u^2}(1+z_1^2) - \dfrac{\partial v^2}{\partial u^3} z_2 \sqrt{1+z_1^2+z_2^2}}{1+z_1^2+z_2^2}, \tag{36}$$

$$\varepsilon_{33} = \frac{\partial v^3}{\partial X^3} = \frac{\partial v^3}{\partial u^i}\frac{\partial u^i}{\partial X^3} = \frac{\dfrac{\partial v^3}{\partial u^1} z_1 + \dfrac{\partial v^3}{\partial u^2} z_2 + \dfrac{\partial v^3}{\partial u^3}\sqrt{1+z_1^2+z_2^2}}{1+z_1^2+z_2^2}, \tag{37}$$

$$\begin{aligned} \varepsilon_{12} &= \frac{1}{2}\left(\frac{\partial v^1}{\partial X^2} + \frac{\partial v^2}{\partial X^1}\right) = \frac{1}{2}\left(\frac{\partial v^1}{\partial u^i}\frac{\partial u^i}{\partial X^2} + \frac{\partial v^2}{\partial u^i}\frac{\partial u^i}{\partial X^1}\right) \\ &= \frac{-\dfrac{\partial v^1}{\partial u^1} z_1 z_2 + \dfrac{\partial v^1}{\partial u^2}(1+z_1^2) - \dfrac{\partial v^1}{\partial u^3} z_2\sqrt{1+z_1^2+z_2^2} + \dfrac{\partial v^2}{\partial u^1}(1+z_2^2) - \dfrac{\partial v^2}{\partial u^2} z_1 z_2 - \dfrac{\partial v^2}{\partial u^3} z_1\sqrt{1+z_1^2+z_2^2}}{2(1+z_1^2+z_2^2)}, \end{aligned} \tag{38}$$

$$\varepsilon_{13} = \frac{1}{2}\left(\frac{\partial v^1}{\partial X^3} + \frac{\partial v^3}{\partial X^1}\right) = \frac{1}{2}\left(\frac{\partial v^1}{\partial u^i}\frac{\partial u^i}{\partial X^3} + \frac{\partial v^3}{\partial u^i}\frac{\partial u^i}{\partial X^1}\right)$$
$$= \frac{\frac{\partial v^1}{\partial u^1} z_1 + \frac{\partial v^1}{\partial u^2} z_2 + \frac{\partial v^1}{\partial u^3}\sqrt{1+z_1^2+z_2^2} + \frac{\partial v^3}{\partial u^1}(1+z_2^2) - \frac{\partial v^3}{\partial u^2} z_1 z_2 - \frac{\partial v^3}{\partial u^3} z_1\sqrt{1+z_1^2+z_2^2}}{2(1+z_1^2+z_2^2)}, \quad (39)$$

$$\varepsilon_{23} = \frac{1}{2}\left(\frac{\partial v^2}{\partial X^3} + \frac{\partial v^3}{\partial X^2}\right) = \frac{1}{2}\left(\frac{\partial v^2}{\partial u^i}\frac{\partial u^i}{\partial X^3} + \frac{\partial v^3}{\partial u^i}\frac{\partial u^i}{\partial X^2}\right)$$
$$= \frac{\frac{\partial v^2}{\partial u^1} z_1 + \frac{\partial v^2}{\partial u^2} z_2 + \frac{\partial v^2}{\partial u^3}\sqrt{1+z_1^2+z_2^2} - \frac{\partial v^3}{\partial u^1} z_1 z_2 + \frac{\partial v^3}{\partial u^2}(1+z_1^2) - \frac{\partial v^3}{\partial u^3} z_2\sqrt{1+z_1^2+z_2^2}}{2(1+z_1^2+z_2^2)}. \quad (40)$$

The components $\sigma_{k\ell}$ of the stress are given by [5]

$$\sigma_{k\ell} = \frac{E}{1+\sigma}\left(\varepsilon_{k\ell} + \frac{\sigma}{1-2\sigma}\varepsilon_{ii}\delta_{k\ell}\right), \quad (41)$$

where $E$ and $\sigma$ are Young's modulus and Poisson's ratio, respectively. The components of the normal are denoted $N^\ell$, hence the stress-free condition is

$$\sigma_{k\ell}N^\ell = 0, \quad k = 1,2,3 \iff \sigma_{k,3} - z_1\sigma_{k,1} - z_2\sigma_{k,2} = 0, \quad k = 1,2,3. \quad (42)$$

In the case where $z_2 = 0$, i.e., the surface does not depend on $y$, we have

$$\left[\frac{\partial u^i}{\partial X^j}\right]\Bigg|_{u^3=0} = \begin{bmatrix} \frac{1}{1+z_1^2} & 0 & \frac{z_1}{1+z_1^2} \\ 0 & 1 & 0 \\ \frac{-z_1}{\sqrt{1+z_1^2}} & 0 & \frac{1}{\sqrt{1+z_1^2}} \end{bmatrix}. \quad (43)$$

and the stress-free condition reads

$$\sigma_{k,3} - z_1\sigma_{k,1} = 0, \quad k = 1,2,3, \quad (44)$$

or,

$$\varepsilon_{13} - \frac{z_1}{1-2\sigma}((1-\sigma)\varepsilon_{11} + \sigma(\varepsilon_{22}+\varepsilon_{33})) = 0, \quad (45)$$

$$\varepsilon_{23} - z_1\varepsilon_{12} = 0, \quad (46)$$

$$\frac{1}{1-2\sigma}((1-\sigma)\varepsilon_{33} + \sigma(\varepsilon_{11}+\varepsilon_{22})) - z_1\varepsilon_{13} = 0, \quad (47)$$

where

$$\varepsilon_{11} = \frac{1}{1+z_1^2}\frac{\partial v^1}{\partial u^1} - \frac{z_1}{\sqrt{1+z_1^2}}\frac{\partial v^1}{\partial u^3}, \quad (48)$$

$$\varepsilon_{22} = \frac{\partial v^2}{\partial u^2}, \quad (49)$$

$$\varepsilon_{33} = \frac{z_1}{1+z_1^2}\frac{\partial v^3}{\partial u^1} + \frac{1}{\sqrt{1+z_1^2}}\frac{\partial v^3}{\partial u^3}, \quad (50)$$

$$\varepsilon_{12}=\frac{1}{2}\frac{\partial v^1}{\partial u^2}+\frac{1}{2(1+z_1^2)}\frac{\partial v^2}{\partial u^1}-\frac{z_1}{2\sqrt{1+z_1^2}}\frac{\partial v^2}{\partial u^3}, \tag{51}$$

$$\varepsilon_{13}=\frac{z_1}{2(1+z_1^2)}\frac{\partial v^1}{\partial u^1}+\frac{1}{2\sqrt{1+z_1^2}}\frac{\partial v^1}{\partial u^3}+\frac{1}{2(1+z_1^2)}\frac{\partial v^3}{\partial u^1}-\frac{z_1}{2\sqrt{1+z_1^2}}\frac{\partial v^3}{\partial u^3}, \tag{52}$$

$$\varepsilon_{23}=\frac{z_1}{2(1+z_1^2)}\frac{\partial v^2}{\partial u^1}+\frac{1}{2\sqrt{1+z_1^2}}\frac{\partial v^2}{\partial u^3}+\frac{1}{2}\frac{\partial v^3}{\partial u^2}. \tag{53}$$

Inserting these expressions into Eqs. (45)–(47) yields Eqs. (54)–(56), respectively:

$$\begin{aligned}&-z_1\frac{\partial v^1}{\partial u^1}+\left((1-2\sigma)+2(1-\sigma)z_1^2\right)\sqrt{1+z_1^2}\frac{\partial v^1}{\partial u^3}-2\sigma z_1(1+z_1^2)\frac{\partial v^2}{\partial u^2}\\&+\left((1-2\sigma)-2\sigma z_1^2\right)\frac{\partial v^3}{\partial u^1}-z_1\sqrt{1+z_1^2}\frac{\partial v^3}{\partial u^3}=0,\end{aligned} \tag{54}$$

$$-z_1\frac{\partial v^1}{\partial u^2}+\sqrt{1+z_1^2}\frac{\partial v^2}{\partial u^3}+\frac{\partial v^3}{\partial u^2}=0, \tag{55}$$

$$\begin{aligned}&\left(2\sigma-(1-2\sigma)z_1^2\right)\frac{\partial v^1}{\partial u^1}-z_1\sqrt{1+z_1^2}\frac{\partial v^1}{\partial u^3}+2\sigma(1+z_1^2)\frac{\partial v^2}{\partial u^2}\\&+z_1\frac{\partial v^3}{\partial u^1}+\left(2(1-\sigma)+(1-2\sigma)z_1^2\right)\sqrt{1+z_1^2}\frac{\partial v^3}{\partial u^3}=0.\end{aligned} \tag{56}$$

The above equations are the appropriate boundary conditions describing surface-wave states in solids with curved surfaces: $(x, y, z(x))$. The solution to these equations must be carried out numerically in the general case since each of the (total) displacement components $v^i$ can be written as a sum of longitudinal and transverse displacement components which again are linear combinations of states satisfying the separable set: Eqs. (20)–(22). We point out that (not surprisingly) in the simple case with an infinite planar surface, analytical expressions can be obtained for Rayleigh-wave dispersion results. In actual fact, it follows immediately from Eq. (18) that the Helmholtz equation in curvilinear coordinates: $u^i, i = 1, 2, 3$ is identical to the Cartesian form of the Helmholtz equation (in $x, y, z$ coordinates) in this specific case. Hence, for an infinite planar surface, the dispersion relation for Rayleigh waves propagating along the $u^1$-axis:

$$\xi=f(u^3)\exp(\mathrm{i}(ku^1+\omega t)), \tag{57}$$

with $\xi$ the total displacement and $f$ some function of $u^3$, obtained by invoking stress-free boundary conditions at the surface $z = 0$, evidently reads:

$$\left(2k^2-\frac{\omega^2}{c_t^2}\right)^4=16k^4\left(k^2-\frac{\omega^2}{c_t^2}\right)\left(k^2-\frac{\omega^2}{c_l^2}\right), \tag{58}$$

in agreement with the classical result [5].

## 3. Results and discussions

In the following, three cases of surface functions will be analyzed: (a) the plane-sloped surface (including the flat surface), (b) the Gaussian-shaped surface, and (c) the sinusoidal-shaped surface. In all cases, the functional form of the acoustic displacement component $v$—assumed to obey the Helmholtz equation with pressure-release boundary conditions—is determined as well as allowed wavenumber values associated with eigenvalues $c_1, c_2$ determined from the $\chi_1, \chi_2$ separable differential equations.

### 3.1. Example 1. Plane-sloped surface

This case, illustrated in Fig. 1, corresponds to the surface function:

$$z(u^1, u^2) = z_0 + \delta u^1, \tag{59}$$

i.e.,

$$z_1 = \delta, \quad z_{11} = 0, \tag{60}$$

where $\delta$ is the surface slope. Inserting Eqs. (59), (60) into Eq. (20) gives

$$\partial_1^2 \chi_1 - (c_1 + c_2)(1 + \delta^2)\chi_1 = 0, \tag{61}$$

which can be easily solved so as to give

$$\chi_1 = \sin\left(\frac{m\pi}{L}u^1\right), \quad c_1 = -\frac{1}{1+\delta^2}\left(\frac{m\pi}{L}\right)^2 - c_2 = -\frac{1}{1+\delta^2}\left(\frac{m\pi}{L}\right)^2 - \left(\frac{n\pi}{H}\right)^2, \tag{62}$$

where $m = 1, 2, 3, \ldots$ and $n = 1, 2, 3, \ldots$ . Thus, the possible values for the wavenumber $k$ are

$$k = \sqrt{\frac{1}{1+\delta^2}\left(\frac{m\pi}{L}\right)^2 + \left(\frac{n\pi}{H}\right)^2 + \left(\frac{l\pi}{T}\right)^2}. \tag{63}$$

### 3.2. Example 2. Gaussian-shaped surface

Consider next the Gaussian-shaped surface (Fig. 2), i.e.,

$$z(u^1) = z_0 \exp\left(-\frac{\left(u^1 - \frac{L}{2}\right)^2}{\delta^2}\right), \tag{64}$$

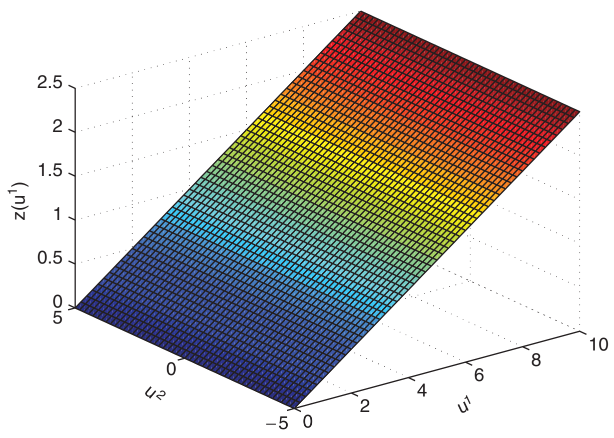


Fig. 1. Schematics of the plane-sloped surface.

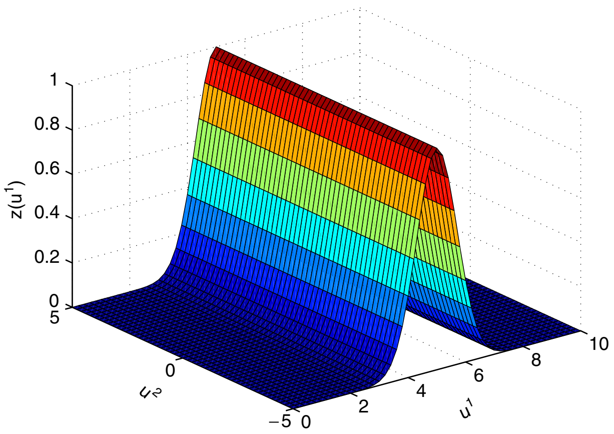


Fig. 2. Schematics of the Gaussian-shaped surface.

Table 1
The first five eigenvalues for $c_1$ [in units of $m^{-2}$] corresponding to the flat-plane surface [Case 1], the Gaussian-shaped surface with $\delta = L/2$, $z_0 = 1$ m [Case 2], the Gaussian-shaped surface with $\delta = 10L$, $z_0 = 1$ m [Case 3], the sinusoidal-shaped surface with $N = 1$ [Case 4], and the sinusoidal-shaped surface with $N = 2$ [Case 5], respectively

| $m$ | $c_1$ [Case 1] | $c_1$ [Case 2] | $c_1$ [Case 3] | $c_1$ [Case 4] | $c_1$ [Case 5] |
|---|---|---|---|---|---|
| 1 | −0.0987 | −0.3944 | −0.0987 | −0.0681 | −0.0819 |
| 2 | −0.3948 | −0.7382 | −0.3948 | −0.3070 | −0.3583 |
| 3 | −0.8883 | −1.546 | −0.8881 | −0.7208 | −0.6561 |
| 4 | −1.579 | −2.265 | −1.579 | −1.3107 | −1.2903 |
| 5 | −2.467 | −3.413 | −2.466 | −2.0521 | −1.9049 |

The eigenvalue number index is denoted $m$. Note that only negative $c_1$ values are sought for as the Rayleigh criteria $k^2 = \omega^2/c^2 < -c_1$ requires $c_1$ to be negative (for real frequencies). The computed values are for the geometry-parameter values: $L = 10$ m and $H = \infty$.

corresponding to a Gaussian-shaped surface centered at $u^1 = L/2$ of width $\delta$. The first- and second-order derivatives of $z$ with respect to $u^1$ read

$$z_1 = -\frac{2z_0}{\delta^2}\left(u^1 - \frac{L}{2}\right)\exp\left(-\frac{\left(u^1 - \frac{L}{2}\right)^2}{\delta^2}\right),$$

$$z_{11} = -\frac{2z_0}{\delta^2}\left(1 - \frac{2}{\delta^2}\left(u^1 - \frac{L}{2}\right)^2\right)\exp\left(-\frac{\left(u^1 - \frac{L}{2}\right)^2}{\delta^2}\right). \tag{65}$$

Next, insertion of these expressions in Eq. (20) yields a second-order differential equation in $\chi_1$ which can be solved for eigenvalues and eigenstates using the finite-difference method.

In Table 1, the first five eigenvalues $c_1$ are given corresponding to the parameter values: $L = 10$ m, $z_0 = 1$ m, $H = \infty$ ($H = \infty$ implies that $c_2 = 0$) for two cases with $\delta = L/2$ and $\delta = 10L$ (in Table 1 denoted Cases 2 and 3, respectively). Note in particular that for a Gaussian-shaped surface with $\delta = 10L$, approximately the same $c_1$ values are obtained as in the plane surface case (compare Case 3 and Case 1 results). This result is understandable since the Gaussian function with $\delta = 10L$ is almost constant along the $u^1$ direction.

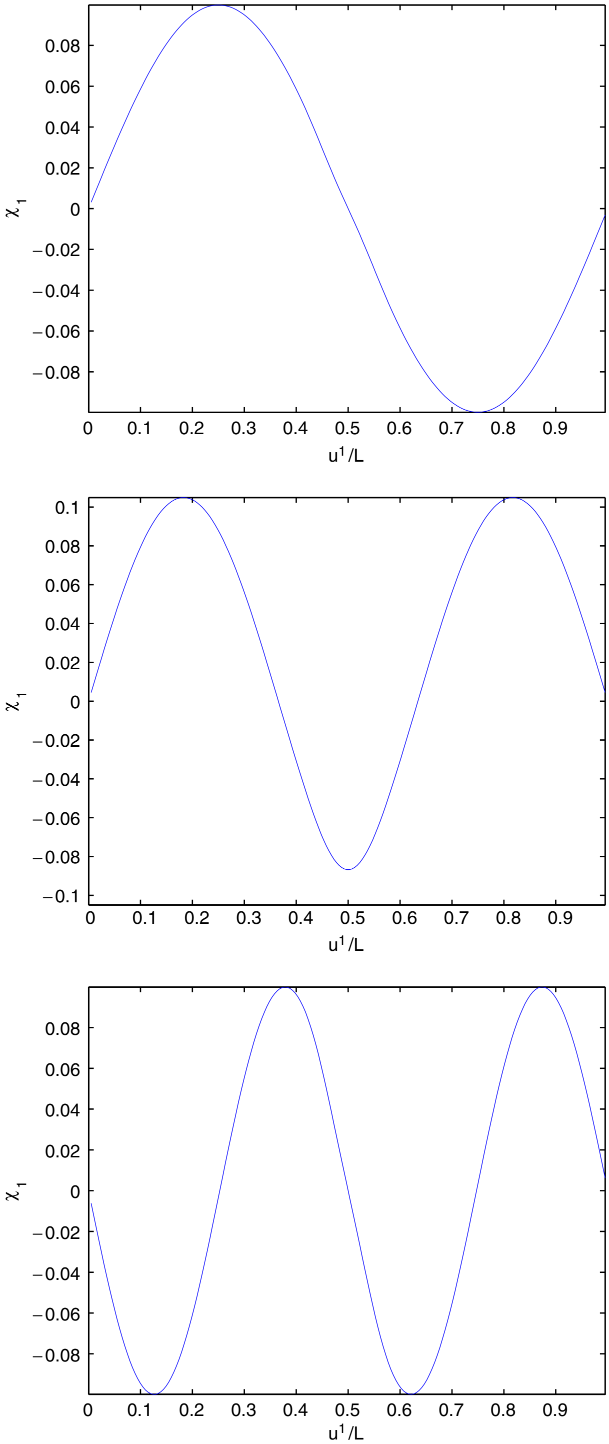


Fig. 3. Plots of the first three eigenstates along the $u^1$ direction for the Gaussian-shaped surface with $\delta = L/2$, $z_0 = 1\,\text{m}$ [Case 2]. The upper, middle, and lower plots correspond to the $c_1$ values: $-0.3944$, $-0.7382$, and $-1.546$, respectively.

In Fig. 3, the first three eigenstates are plotted corresponding to the first three $c_1$ eigenvalues in Table 1, Case 2. Note that all states are symmetrical/antisymmetrical with respect to reflection around $u^1 = L/2$ as they must be due to the symmetry of the differential equation in $\chi_1$, i.e., all terms are unchanged by the symmetry operation $u^1 = L/2 - x \to L/2 + x$ and $\partial_1 \to -\partial_1$. It is also observed that the $n$th state has $n$ nodes along the $u^1$ direction. A solution with a positive $c_1$ eigenvalue is also found (not listed in Table 1) representing Rayleigh waves with imaginary eigenfrequencies, hence, such waves are damped exponentially with time.

Similar to the case considered in Example 1, the allowed wavenumber eigenvalues become [rewriting Eq. (25)]:

$$k = \sqrt{-c_1 + \left(\frac{l\pi}{T}\right)^2}, \tag{66}$$

with $c_1$ values as mentioned above.

### 3.3. Example 3. Sinusoidal-shaped surface

Consider finally the sinusoidal-shaped surface (Fig. 4), i.e.,

$$z(u^1) = \sin\left(\frac{2\pi N}{L} u^1\right) \tag{67}$$

corresponding to a sinusoidal corrugation with $N$ periods along the $u^1$ direction. The first- and second-order derivatives of $z$ with respect to $u^1$ read

$$\begin{aligned} z_1 &= \frac{2\pi N}{L}\cos\left(\frac{2\pi N}{L} u^1\right), \\ z_{11} &= -\left(\frac{2\pi N}{L}\right)^2 \sin\left(\frac{2\pi N}{L} u^1\right). \end{aligned} \tag{68}$$

Again, insertion of these expressions in Eq. (20) yields a second-order differential equation in $\chi_1$ which can be solved using the finite-difference method.

In Table 1, the first five eigenvalues $c_1$ are given corresponding to the parameter values: $L = 10$ m, $H = \infty$ for two cases with $N = 1$ and 2 (in Table 1 denoted Cases 4 and 5, respectively).

In Fig. 5, the first three eigenstates are plotted corresponding to the first three $c_1$ eigenvalues in Table 1, Case 5. Note that all states are symmetrical/antisymmetrical with respect to reflection around $u^1 = L/2$ as they must be due to the symmetry of the differential equation in $\chi_1$, i.e., all terms are unchanged by the symmetry operation $u^1 = L/2 - x \to L/2 + x$ and $\partial_1 \to -\partial_1$. Again, it is found that the $n$th (real-frequency) state has $n$ nodes along the $u^1$ direction.

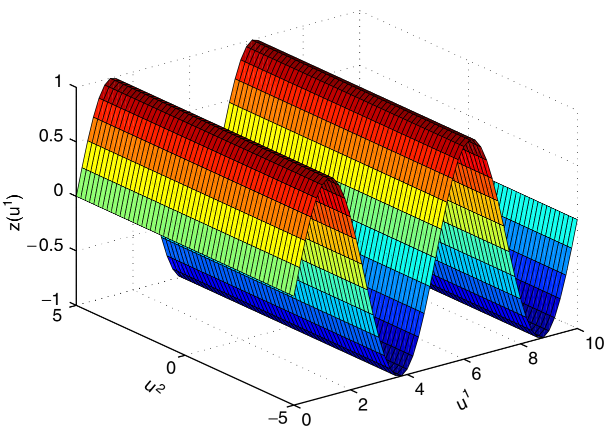


Fig. 4. Schematics of the sinusoidal-shaped surface.

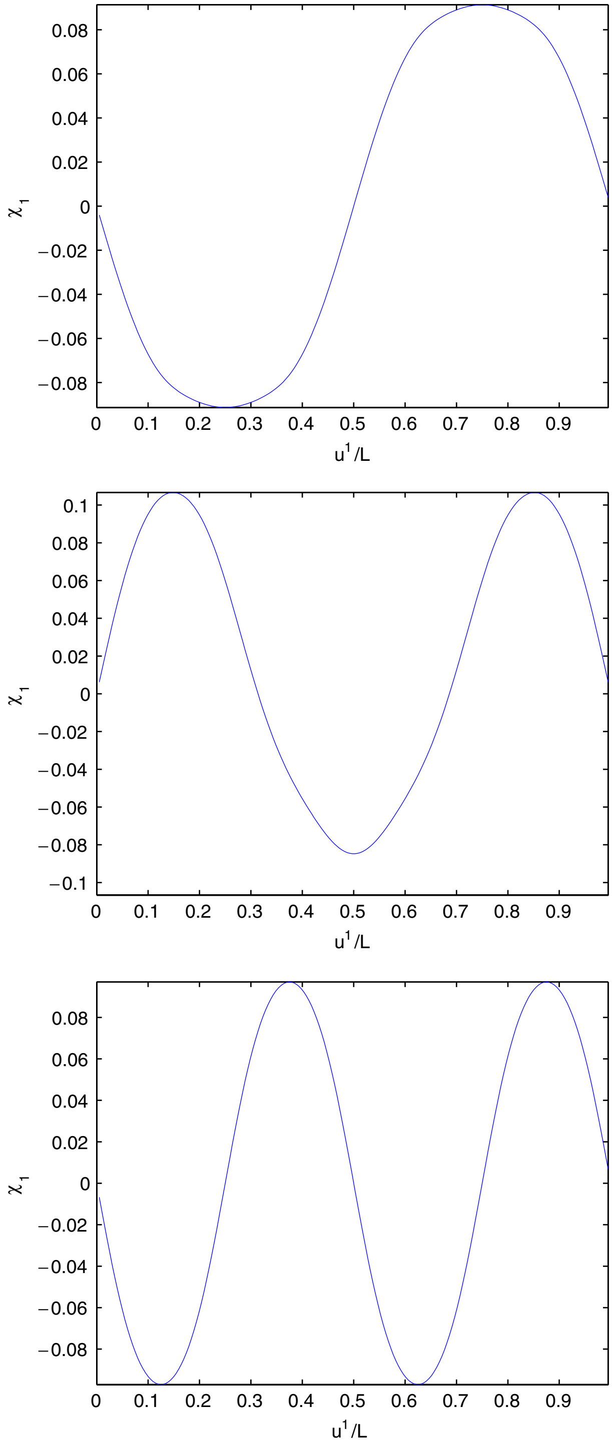


Fig. 5. Plots of the first three eigenstates along the $u^1$ direction for the sinusoidal-shaped surface with $N = 2$ [Case 5]. The upper, middle, and lower plots correspond to the $c_1$ values: $-0.0819$, $-0.3583$, and $-0.6561$, respectively.

Similar to the cases considered in Examples 1 and 2, the allowed wavenumber eigenvalues become [rewriting Eq. (25)]:

$$k = \sqrt{-c_1 + \left(\frac{l\pi}{T}\right)^2}, \tag{69}$$

with $c_1$ values as mentioned above.

## 4. Conclusions

We have presented a quasi-analytical solution method for determining fluid curved-layer eigenstates and eigenvalues corresponding to a general surface $z = z(x)$ using differential-geometry arguments. The latter method allows solution of the 3D Helmholtz equation in terms of three ODEs, hence simplifying the mathematical problem considerably, by choosing the curved coordinates $(u^1, u^2, u^3)$ with $u^1, u^2$ along the local tangent plane of the surface $z = z(x)$ and $u^3$ perpendicular to the local tangent plane. The general model is finally solved in terms of eigenstates and eigenvalues for three case studies with pressure-release boundary conditions: the flat (sloped) plane, the Gaussian-shaped surface, and the sinusoidal-shaped surface, respectively. Stress-free boundary conditions relevant for solid structures are also derived in curved coordinates. The general solution to a Rayleigh-wave problem in solids with curved surfaces involves a numerical solution method, e.g., based on expansion in eigenstates obeying three ODEs (which can be found quasi-analytically according to this work).

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