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A new approach for free vibration analysis of thin circular cylindrical shell

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Abstract

A new method for calculating the free vibration frequencies of a thin circular cylindrical shell is presented, based on Flügge's shell theory equations for orthotropic materials. A general displacement representation is introduced, and a type of coupled polynomial eigenvalue problem is developed in present study. Numerical examples are given for isotropic and orthotropic shells. Comparison with that from classical dynamic approach is studied for frequency of an isotropic shell.

1. Introduction

The free vibration problems of thin circular cylindrical shells have been of great interest to many structural engineers in recent years. Many investigations following the pioneering work of Arnold and Warburton [1,2] have been summarized by Leissa [3]. Methods of solution for shell vibrations have ranged from the approximate energy methods used by Arnold and Warburton [1,2] and by Sharma and Johns [4,5] to the exact solutions as studied by Forsberg [6–8], Warburton [9], Warburton and Higgs [10] and Goldman [11]. The circular cylindrical shell supported at both ends by shear diaphragms (SD–SD) has received the most attention in the literatures. This is due to the fact that one simple form of the solutions to the eighth-order differential equations of motion is also capable of satisfying the SD–SD boundary conditions exactly. The frequency equation is a polynomial of order six in shell frequency parameter.

However, only a few papers are devoted to the study of the vibration characteristics of cylindrical shells with different boundary conditions at the two ends. Authors of Refs. [6–11] have chosen exponential functions for the modal displacements along the axial direction, substituted them into the equations of motion and then enforced the eight specified boundary conditions. This leads to an eighth-order algebraic equation and an eighth-order frequency determinant which are coupled together. The simultaneous solution of these two systems of equations involves extremely laborious computation as pointed out by Leissa [3].

In this paper, a general type of displacement of shell is used for free vibration analysis. A different eighth-order characteristic equation is acquired. The equation have 8 roots, which the imaginary branch represent the

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Nomenclature

E_x, E_θ	Young's moduli, axial and circumferential directions, respectively
G	shear modulus
i	$\mathrm{i}^2 = -1$
m	number of axial half-waves
n	number of circumferential waves
P, p, T	xial, circumferential and torsional prestresses, respectively
R, h, L	radius, thickness, length of shell
t	time
u, v, w	axial, circumferential, radial displacements
x	axial coordinate
z	radial coordinate, positive inward
v_x, v_θ	Poisson's ratios, axial and circumferential directions, respectively
θ	circumferential coordinate
ρ	mass density of shell wall material
ω	(= 2πf) circular frequency (rad/s)

free vibration frequency with shear diaphragm boundary conditions. Some numerical examples are given in this paper. The results obtained by this method are compared with those from classic method.

2. Formulation of problem

Consider the orthotropic circular cylindrical shell as shown in Fig. 1. Its geometry is described by R, the radius of the shell middle surface, and h, the thickness of the shell. The material is linearly elastic. Suppose that the axial and circumferential directions are principal axes of the orthotropic material. There exists the relation

$$v_x E_\theta = v_\theta E_x, \tag{1}$$

where E_x, E_θ are Young's moduli and v_x ,v_θ are Poisson's ratios in the axial and circumferential directions.

Following Flügge's [12] exact derivation for the buckling of cylindrical shells, the differential equations of motions for free vibration become:

$$L_{11}u + L_{12}v + L_{13}w = \rho h \frac{\partial^2 u}{\partial t^2}, \quad L_{21}u + L_{22}v + L_{23}w = \rho h \frac{\partial^2 v}{\partial t^2},$$

$$L_{31}u + L_{32}v + L_{33}w = \rho h \frac{\partial^2 w}{\partial t^2}, \tag{2}$$

where

$$L_{11} = (D_x - P)\frac{\partial^2}{\partial x^2} + \left(\frac{D_{x\theta}}{R^2} + \frac{K_{x\theta}}{R^4} - \frac{p}{R}\right)\frac{\partial^2}{\partial \theta^2} - 2\frac{T}{R}\frac{\partial^2}{\partial x \partial \theta}, \quad L_{12} = \frac{v_\theta D_x + D_{x\theta}}{R}\frac{\partial^2}{\partial_x \partial_\theta},$$

$$L_{13} = \left(-\frac{v_\theta D_x}{R} - p\right)\frac{\partial}{\partial x} + \frac{K_x}{R}\frac{\partial^3}{\partial x^3} - \frac{K_{x\theta}}{R^3}\frac{\partial^3}{\partial x \partial \theta^2}, \quad L_{21} = L_{12},$$

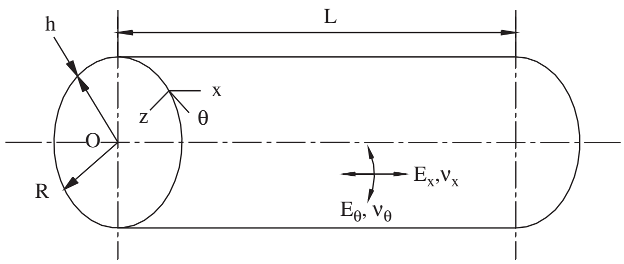

Fig. 1. An orthotropic circular cylindrical shell.

$$L_{22} = \left(D_{x\theta} + \frac{3K_{x\theta}}{R^2} - P\right)\frac{\partial^2}{\partial x^2} + \left(\frac{D_\theta}{R^2} - \frac{p}{R}\right)\frac{\partial^2}{\partial \theta^2} - 2\frac{T}{R}\frac{\partial^2}{\partial x \partial \theta}, \quad L_{23} = \left(-\frac{D_\theta}{R^2} + \frac{p}{R}\right)\frac{\partial}{\partial \theta} + \frac{v_x K_\theta + 3K_{x\theta}}{R^2}\frac{\partial^3}{\partial x^2 \partial \theta} + 2\frac{T}{R}\frac{\partial}{\partial x},$$

$$L_{31} = -L_{13}, \quad L_{32} = -L_{23},$$

$$L_{33} = -K_x\frac{\partial^4}{\partial x^4} - \frac{2v_x K_\theta + 4K_{x\theta}}{R^2}\frac{\partial^4}{\partial x^2 \partial \theta^2} - \frac{K_\theta}{R^4}\frac{\partial^4}{\partial \theta^4} - \left(\frac{2K_\theta}{R^4} + \frac{p}{R}\right)\frac{\partial^2}{\partial \theta^2} - \left(\frac{D_\theta}{R^2} + \frac{K_\theta}{R^4}\right) - P\frac{\partial^2}{\partial x^2} - 2\frac{T}{R}\frac{\partial^2}{\partial x \partial \theta},$$

where P, p and T are axial, circumferential and torsional prestresses, respectively. The quantities D_x, D_θ, and $D_{x\theta}$ are extensional rigidities; K_x, K_θ, and $K_{x\theta}$ are flexural rigidities. These are defined as follows:

$$D_x = E_x h/(1 - v_x v_\theta), \quad D_\theta = E_\theta h/(1 - v_x v_\theta), \quad D_{x\theta} = D_{x\theta} = Gh,$$

$$K_x = E_x h^3/12(1 - v_x v_\theta), \quad K_\theta = E_\theta h^3/(1 - v_x v_\theta), \quad K_{x\theta} = Gh^3/12, \tag{3}$$

where G is the shear modulus.

Assuming solutions of Eq. (2) in the form

$$u = A e^{\lambda x} e^{in\theta} e^{i\omega t}, \quad v = B e^{\lambda x} e^{in\theta} e^{i\omega t}, \quad w = C e^{\lambda x} e^{in\theta} e^{i\omega t}, \tag{4}$$

where A, B, C and λ are complex amplitudes, n is number of circumferential waves. $i^2 = -1$. ω is the frequency of shell (rad/s). Eq. (4) is the most general solutions of the problem. If one selects the real parts or imaginary parts of right hand of this equation, a conventional assumed solution for arbitrary boundary conditions is obtained (e.g. Ref. [6]). Thus, the solutions given in Eq. (4) are compatible with all possible boundary conditions. Using Eq. (4) in Eq. (2) one can obtain a system of equation of the form

$$[Q]\begin{Bmatrix} A \\ B \\ C \end{Bmatrix} = \{0\}. \tag{5}$$

The elements of matrix Q are given in Appendix A. For nontrivial solution of Eq. (5), one requires

$$\det\ ([Q]) = 0, \tag{6}$$

which is the characteristic equation of the system. Eq. (6) can be rewritten as

$$g_8\lambda^8 + g_7\lambda^7 + g_6\lambda^6 + g_5\lambda^5 + g_4\lambda^4 + g_3\lambda^3 + g_2\lambda^2 + g_1\lambda + g_0 = 0, \tag{7}$$

where

$$g = g(h, R, n, v_x, v_\theta, E_x, E_\theta, G, \omega).$$

Eq. (7) is a polynomial λ eigenvalue problem of order 8. In this equation, g_8, g_6, g_4, g_2 and g_0 are real numbers, while g_7, g_5, g_3 and g_1 are pure imaginary numbers. It should be noted that, the eighth-order characteristic equation obtained here is different from those of Refs. [6–11], where g_7, g_5, g_3 and g_1 vanished. For the usual range of shell parameter and $n \geqslant 1$, the eight roots of Eq. (7) have the form

$$\lambda_1, \ -\lambda_1, \ \lambda_2 i, \ -\lambda_2 i, \ \lambda_3 + \lambda_4 i, \ \lambda_3 - \lambda_4 i, \ -\lambda_3 + \lambda_4 i, \ -\lambda_3 - \lambda_4 i, \tag{8}$$

where λ_i ($i = 1$–4) are real, positive numbers.

Nondimensional frequency parameter is defined as follows:

$$\Omega^2 = \frac{\rho R^2(1 - v_x v_\theta)}{E_x}\omega^2. \tag{9}$$

3. Numerical examples and discussion

The algorithm used for root search procedure of determinantal equation (7) is Newton–Raphson iteration scheme in the complex domain. For an isotropic shell, the roots of Eq. (7) are computed for following cases:

i) with zero initial stress state;
ii) with hydrostatic pressure, $P = 0.5pR$.

The data of the shell are: $R/h = 100$, $v = 0.3$, $n = 4$, $\rho = 7800\,\mathrm{kg/m^3}$.

The results are shown in Figs. 2–4. An orthotropic circular cylindrical shell example is given in Fig. 5. With the increase of frequency, λ_1 vary smoothly, where λ_2 increase. The existence of hydrostatic pressure can enlarge the amplitude of λ_2, and there is a light decrease for λ_3 and λ_4.

A shell's vibration under torsional prestress is presented in Fig. 4. The characteristics of shell under torsional force are slightly different from under circumferential prestress [3,12]. Thus, the λ_2 root has irregular variation. Furthermore, there are jumps for λ_1 and λ_2 in Fig. 5, an orthotropic shell's results without any

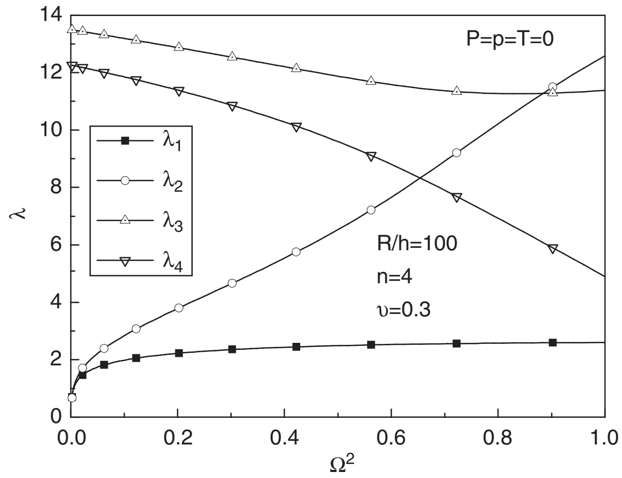

Fig. 2. $\lambda \sim \Omega^2$.

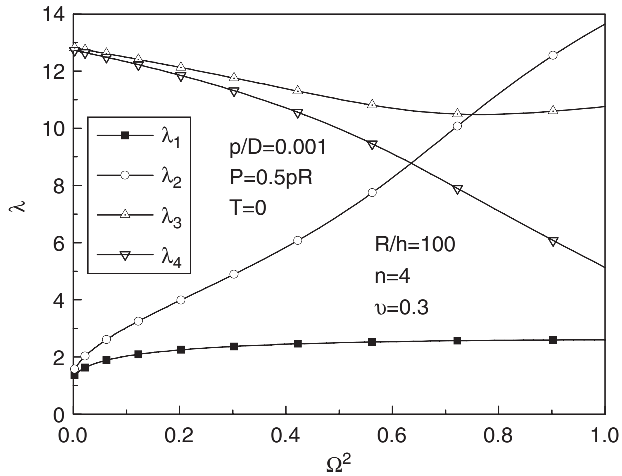

Fig. 3. $\lambda \sim \Omega^2$.

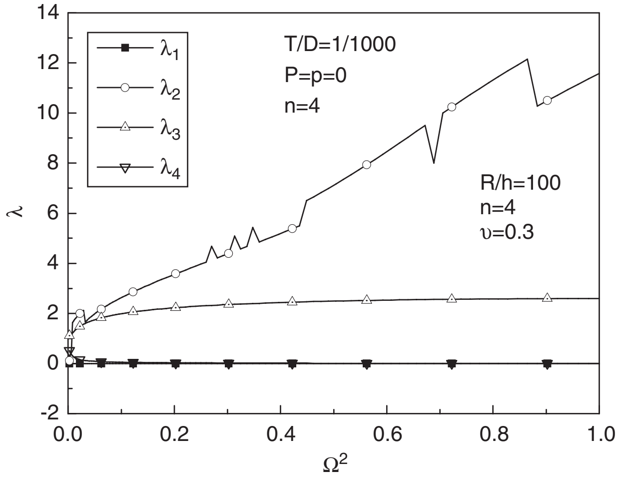

Fig. 4. $\lambda \sim \Omega^2$.

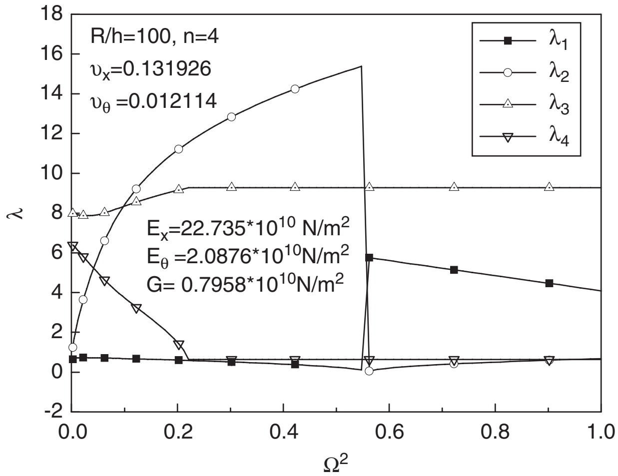

Fig. 5. $\lambda \sim \Omega^2$.

prestresses. The characteristic of an orthotropic shell can be quite different from that of an isotropic shell, which can be seen from Fig. 5. This performance is just due to shell's orthotropy, i.e. material parameters.

There are 136 combinations of "simple" boundary conditions possible for a closed circular cylindrical shell [3]; most of the results are available for a single one of these cases—when both ends are supported by shear diaphragms. The displacement function in this paper is a general expression for any kind of these 136 boundary conditions. Usually, $g_i\ (i = 1,2,\ldots 8)$ are not zeros for simple edge conditions in static problem and λ is complex number.

As for dynamic problem, there will always be at least two roots of the form $\pm\lambda_2 i$. After study, when it occurs that any one of those initial stresses acting individually, there always will be a imaginary root such that $i\lambda_2 = im\pi R/L$. The modes associated with a pure imaginary λ_2 conform to the homogenous end conditions of a finite shell of length L given by

$$v = w = N_x = M_x = 0, \quad x = 0, L, \tag{10}$$

where m is number of axial half-wave.

This is familiarly knows as shear diaphragm (SD–SD) condition, often loosely called the simply supported condition, and is the most widely used of the shell boundary conditions. Physically, this corresponds to an end supported by means of an end plate which is highly flexible in the z-direction as well as in bending but which has a large stiffness in its own plane. The λ_2 imaginary mode being waveform-like, it obviously implies that the shell maintains an adjacent equilibrium configuration represented by this mode which is identical to the adjacent equilibrium position referred in classical vibration analysis. In this case, the vibration mode of shell along axial direction can be expressed in cosine or sine forms, which are the traditional expressions in SD–SD solutions.

Therefore, the λ_2 imaginary mode covering the entire frequency range in the $\lambda - \Omega^2$ plane represents the free vibration of all lengths (that is, $\lambda = 0$ to ∞) with shear diaphragm boundary conditions. The other three branches represent those remaining 135 cases of boundary conditions.

The shell length L can be determined for given n, λ_2 and structural parameters. In order to study the applicability of present approach, some salient comparisons are indicated in Table 1 for an isotropic shell. The results of Leissa [3] are calculated from three-dimensional theory, while Li and Chen [13] used Flügge's classical shell theory. The results from present approach should be identical to that from Ref. [13] for the same problem, theoretically. The present analysis and that of Ref. [13] used the same shell theory. As for SD–SD boundary condition, the frequency equation is a polynomial of order 6 in shell frequency parameter. A standard exact method for polynomial roots is used in Ref. [13]. The present solution for Eq. (7) is Newton–Raphson iteration, a numerical solution. Thus, there will be slight discrepancy between two methods.

Table 1
Frequency comparisons ($\nu = 0.3$)

This paper						Other literatures		
Initial stress	R/h	n	Ω^2	$i\lambda_2$	$\frac{L}{mR}\left(=\frac{\pi}{\lvert\lambda_2\rvert}\right)$	$\frac{L}{mR}$	Ω^2	Refer to Ref.
$P = p = T = 0$	100	4	4×10^{-2}	$i2.05539$	1.52846	1.52846	4.000067×10^{-2}	[13]
$p/D = 10^{-4}$ $P = 0.5pR$, $T = 0$,	100	4	9×10^{-2}	$i2.7456888$	1.1441911	1.1441911	9.010458×10^{-2}	[13]
$P = p = T = 0$	20	3	1.68691×10^{-2}	$i0.7775271$	4.0404927	4.0	1.68691×10^{-2}	[1, p. 55]
$P = p = T = 0$	500	4	1.25264×10^{-3}	$i0.7853785$	4.0000999	4.0	1.25264×10^{-3}	[1, p. 55]

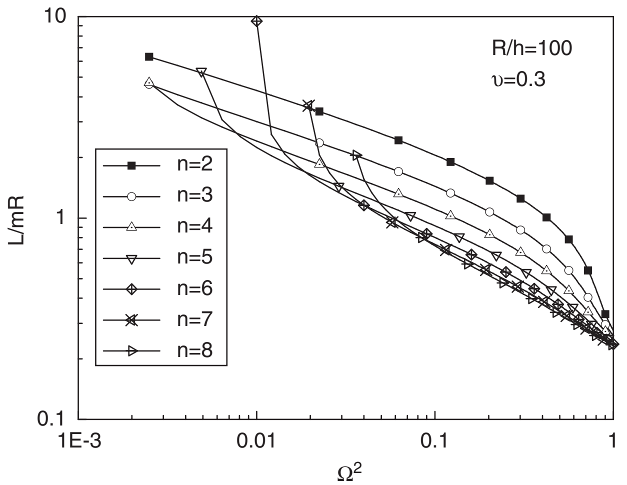

Fig. 6. Variation of frequency parameter.

There are also slight difference between three-dimensional elastic theory and Flügge's shell theory, as shown in Table 1. From this table, it can be concluded that the present method has high accuracy.

Another applicability of present approach is illustrated in Fig. 6. In many research, (e.g. Refs. [4–6]), the characteristics of shell vibration are shown as length–radius-ratio versus frequency parameter. Variation of the frequency parameter versus axial wavelength parameter L/mR is presented in Fig. 6.

For a fixed number of circumferential waves the frequency increases monotonically with axial half-waves, for all values of shell parameters (R/h, L/R, v) and for all boundary conditions. The value of n which corresponds to the minimum frequency depends strongly upon length-to-radius ratio. This is clearly seen in Fig. 6 for SD–SD boundary condition. Frequency envelop can also be drawn from this figure easily.

4. Conclusion

A new method for calculating the free vibration frequencies of a thin circular cylindrical shell is presented, based on Flügge's shell theory equations for orthotropic materials. A general displacement representation is introduced, and a new type of coupled polynomial eigenvalue problem is developed in present study.

The merit of this approach is that, it presents another general form of separation of variables in circular cylindrical shell analysis for arbitrary boundary conditions. The general solutions of the problem can be reduced to conventional assumed solution, which has been used in many literatures. In order to study the applicability of present method, numerical examples are given for isotropic and orthotropic shells. Comparison with that from classical dynamic approach is studied for frequency of an isotropic shell. The modes associated with a pure imaginary λ_2 conform to the homogenous end conditions of a finite shell. The present method has high accuracy.

Appendix A

Elements of Q matrix:

$$Q_{11} = \lambda^2(D_x - P) - n^2\left(\frac{D_{x\theta}}{R^2} + \frac{K_{x\theta}}{R^4} - \frac{p}{R}\right) - i2\frac{T}{R}\lambda n + \rho h\omega^2, \quad Q_{12} = i\frac{v_\theta D_x + D_{x\theta}}{R}\lambda n,$$

$$Q_{13} = \left(-\frac{v_\theta D_x}{R} - p\right)\lambda + \frac{K_x}{R}\lambda^3 + \frac{K_{x\theta}}{R^3}\lambda n^2, \quad Q_{21} = Q_{12},$$

$$Q_{22} = \left(D_{x\theta} + \frac{3K_{x\theta}}{R^2} - P\right)\lambda^2 - \left(\frac{D_\theta}{R^2} - \frac{p}{R}\right)n^2 - i2\frac{T}{R}\lambda n + \rho h\omega^2,$$

$$Q_{23} = i\left(-\frac{D_\theta}{R^2} + \frac{p}{R}\right)n + i\frac{v_x K_\theta + 3K_{x\theta}}{R^2}\lambda^2 n + 2\frac{T}{R}\lambda, \quad Q_{31} = -Q_{13}, \quad Q_{32} = -Q_{23},$$

$$Q_{33} = -K_x\lambda^4 + \frac{2v_x K_\theta + 4K_{x\theta}}{R^2}\lambda^2 n^2 - \frac{K_\theta}{R^4}n^4 + \left(\frac{2K_\theta}{R^4} + \frac{p}{R}\right)n^2 - \left(\frac{D_\theta}{R^2} + \frac{K_\theta}{R^4}\right) - P\lambda^2 - i2\frac{T}{R}\lambda n + \rho h\omega^2.$$

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