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Short Communication

A decrement method for quantifying nonlinear and linear damping parameters

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Abstract

A method is presented which can estimate the linear and nonlinear damping parameters in a lightly damped system. Only a single response measurement from a free decay test is required as input. This ensures that the magnitude of the damping parameters is not compromised by phase distortion between measurements. The method uses the instantaneous energy to describe the long-term evolution of the system. Practically this is achieved by using only the peak amplitudes in each period. In this way the stiffness is effectively ignored, and only the damping forces are considered. For this reason, the method is not unlike the familiar decrement method, which can be used to estimate the apparent linear damping. The method is developed in the context of a weakly nonlinear, lightly damped system, with both linear and cubic damping. Simulated response data is used to demonstrate the accuracy of the technique. The nonlinear damping parameter is extracted from the response data to within 5% of the exact value, even though the nonlinear term contributes less than 1% to the total force in the system.

1. Introduction

Nonlinear damping can play a crucial role in the long-term behaviour of weakly nonlinear systems. For example, in self-excited systems, such as fluidelastic systems where the structural and fluid mechanics are strongly coupled, the onset of instability (flutter) is governed largely by the total linear damping in the system, but the amplitude of the resulting self-excited limit cycle oscillations is determined by the nonlinear damping in the system. This has been shown experimentally for tube arrays by Meskell and Fitzpatrick [1], where the nonlinear damping has been found to be cubic in nature. Identifying the functional form of nonlinear damping and quantifying the relevant coefficient in lightly damped systems, such as the those discussed above, can be problematic because of the relatively small contribution of the terms of interest to the overall force acting on the structure. Sophisticated system identification procedures for nonlinear system do exist. For sample in the frequency domain, nonlinear spectral estimation has been described by Rice and Fitzpatrick [2] and widely applied to various nonlinear systems. Granger [3] developed an estimation technique essentially based on Kalman filtering specifically for fluidelastic systems. In the time domain the force surface mapping technique [4,5] has been successfully applied to lightly damped fluidelastic systems [1]. However, these approaches, and

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many not listed, are not robust when the term of interest has a low contribution. Furthermore, if more than one measurement instrument is required (e.g. in a forced response test or in the case of force surface mapping), then the value of damping coefficients obtained is extremely sensitive to small relative phase distortion between the signals [6]. Therefore, it would be desirable to have a method applicable to weakly nonlinear systems that is similar to the decrement method used to determine the damping ratio for linear systems from a free response test. Such a method would require only a single input signal. As this signal is a deterministic response, ensemble averaging in the time domain can be used to reduce the effect of noise and extraneous excitation. While it is true that a decrement method for systems with viscous and Coulomb damping has been developed by Feeny and Liang [7], this technique is limited to Coulomb damping only. A general identification procedure for nth power damping was developed by Mottershead and Stanway [8], but as the stiffness is identified as part of the procedure, weak nonlinear damping terms cannot be reliably obtained. This paper presents a new method for quantifying other forms of nonlinear damping and viscous damping simultaneously based on consideration of the decrements of successive peaks in the transient response of the system. The method is demonstrated with a numerical simulation of a system with cubic as well as linear damping.

2. General formulation

Consider a single degree of freedom system which has a linear spring and a general nonlinear element. The system is in free response, and so no external force is acting on it. The method proposed here makes use of successive peaks in the free response of the system to indicate the rate of energy dissipation and hence the relative strength of the various damping components.

The mass normalized equation of motion is

$$\ddot{x} + \omega^2 x + f_D(x, \dot{x}) = 0, \tag{1}$$

where $\omega = \sqrt{k/m}$ and f_D is the mass normalized damping force, including both linear and nonlinear damping. Assuming that nonlinear forces are relatively small, the free response can be assumed to be a simple sinusoid with a time varying amplitude:

$$x(t) \approx A(t) \sin(\omega_d t + \phi), \tag{2}$$

$$\Rightarrow \dot{x} = A\omega_d \cos(\omega_d t + \phi) + \dot{A} \sin(\omega_d t + \phi). \tag{3}$$

In other words, the sub- and super-harmonics which would be expected from nonlinear components are neglected.

If the damping is now assumed to be small in comparison to the stiffness and inertial forces, the rate of energy dissipation must be small and so the rate of change of the amplitude of vibration is also small (i.e. $\dot{A} \approx 0$). In addition the frequency of vibration $\omega_d \approx \omega$. Using these assumptions, the response of the system can now be written as

$$x = A(t) \sin(\omega t + \phi), \quad \dot{x} = A(t)\omega \cos(\omega t + \phi), \quad \ddot{x} = -A(t)\omega^2 \sin(\omega t + \phi). \tag{4}$$

The mass normalized (specific) instantaneous mechanical energy in the system

$$e(t) = \tfrac{1}{2}\omega^2 x^2(t) + \tfrac{1}{2}\dot{x}^2(t). \tag{5}$$

Substituting Eq. (4) into this relates the instantaneous specific energy to the instantaneous amplitude of vibration:

$$e(t) = \tfrac{1}{2}\omega^2 A^2(t). \tag{6}$$

Alternatively, the specific energy can be obtained by considering the energy dissipated by the non-conservative (i.e. damping) force f_D. The instantaneous energy is simply

$$e(t) = e_0(t) - w(t), \tag{7}$$

where e_0 is the initial energy at $t = 0$. The specific work done by the damping force is

$$w(t) = \int_0^t f_D(\tau)\dot{x}(\tau)\,\mathrm{d}\tau. \tag{8}$$

Combining Eqs. (6)–(8) yields

$$A^2(t) = A_0^2 - \frac{2}{\omega^2}\int_0^t f_D(\tau)\dot{x}(\tau)\,\mathrm{d}\tau, \tag{9}$$

where A_0 is the amplitude at $t = 0$. Substituting the system velocity Eq. (4), changing the variable in the integral to $\theta = \omega\tau + \phi$ and evaluating this equation over one period of vibration yields the basic equation for the decrement method proposed:

$$A_1^2 = A_0^2 - \frac{2}{\omega^3}\int_0^{2\pi} f_D(\theta)\dot{x}(\theta)\,\mathrm{d}\theta. \tag{10}$$

The values A_0 and A_1 are successive peak amplitudes in the response signal for the system undergoing a free decay from an initial perturbation. The damping force $f_D(\theta)$ and response $\dot{x}(\theta)$ depends on the instantaneous amplitude $A(\theta)$, but as the system is lightly damped, this will change slowly, and so can be assumed to be a constant value over one cycle. The presence of $f_D(\theta)$ within the integral means that the application of this general formulation to a particular system depends on the functional form of nonlinearity present. In order to demonstrate how Eq. (10) can be used to obtain parameter estimates for the damping coefficients, a specific example will be considered.

3. Example—a system with linear and cubic damping

3.1. System specific formulation

Consider a system with both viscous (linear) and cubic damping. The damping force is given by

$$f_D = c\dot{x} + \beta\dot{x}^3. \tag{11}$$

Note that the damping coefficients are already mass normalized. Assuming a constant response amplitude of A_c, the work done over one period of vibration is

$$\int_0^{2\pi} f_D(\theta)\dot{x}(\theta)\,\mathrm{d}\theta = cA_c^2\int_0^{2\pi}\omega^2\cos^2(\theta)\,\mathrm{d}\theta + \beta A_c^4\int_0^{2\pi}\omega^4\cos^4(\theta)\,\mathrm{d}\theta \tag{12}$$

$$= cA_c^2\omega^2\pi + \beta A_c^4\omega^4\frac{3\pi}{4}. \tag{13}$$

Combining this with Eq. (10) yields

$$A_1^2 = A_0^2 - \lambda_1 A_c^2 - \lambda_2 A_c^4, \tag{14}$$

where

$$\lambda_1 = \frac{2\pi}{\omega}c, \tag{15}$$

$$\lambda_2 = \frac{3\pi\omega}{2}\beta. \tag{16}$$

There are two obvious assumptions for the amplitude A_c, either A_0 or A_1. If it is assumed that $A_c = A_0$ then Eq. (14) becomes

$$p_1 = (1 - \lambda_1) + \lambda_2 p_2, \tag{17}$$

where the quantities p_1 and p_2 are defined as

$$p_1 = \left(\frac{A_1}{A_0}\right)^2, \tag{18}$$

$$p_2 = -A_0^2. \tag{19}$$

The quantities p_1 and p_2 are calculated directly from the peaks of the decay response. Data pairs can be produced for each pair of successive peaks in the free decay response. Fitting a straight line to this data will readily yield an estimate of the parameters λ_1 and λ_2 and hence the damping parameters. The regression should employ a total least squares approach as errors are comparably likely in p_1 as in p_2. The response amplitude in one cycle is always over-estimated with this assumed value for A_c. Since the energy drop in one cycle is known (from the test data), this over-estimation of the response amplitude in Eq. (10) will lead to an under-estimation of the damping parameters.

Now assuming an under-estimate for the response amplitude, $A_c = A_1$, and substituting into Eq. (14) an alternative to Eq. (17) is obtained:

$$q_1 = (1 + \lambda_1) + \lambda_2 q_2, \tag{20}$$

where the quantities q_1 and q_2 are defined as

$$q_1 = \left(\frac{A_0}{A_1}\right)^2, \tag{21}$$

$$q_2 = A_1^2. \tag{22}$$

As before, the damping parameters of the system can be estimated by fitting a straight line to the transformed response data. However, as the response amplitude has now been systematically under-estimated, the damping parameters will be systematically over-estimated.

In this way, this method brackets the true value of the damping parameters, and so improved parameter estimates can be achieved by simply averaging the under- and over-estimate values. However, at least for the nonlinear damping, the aggregate estimate can be improved compared to a simple mean.

Eqs. (17) and (20) yield two estimates for the derived nonlinear damping parameter, λ_{2p} and λ_{2q}, respectively. Now examining the last term in Eq. (14), and remembering that λ_{2p} has been obtained from an over-estimation of A_c, which is the effective mean amplitude, the error in λ_2 can be written as

$$\lambda_2 A_c^4 = \lambda_{2p}\left(\frac{A_c}{1-\varepsilon_p}\right)^4, \tag{23}$$

$$\Rightarrow \lambda_2^{1/4}(1-\varepsilon_p) = \lambda_{2p}^{1/4}. \tag{24}$$

Similarly for λ_{2q}:

$$\lambda_2 A_c^4 = \lambda_{2q}\left(\frac{A_c}{1+\varepsilon_q}\right)^4, \tag{25}$$

$$\Rightarrow \lambda_2^{1/4}(1+\varepsilon_q) = \lambda_{2q}^{1/4}. \tag{26}$$

The errors ε_p and ε_q represent the effective positive (i.e. unsigned) errors in the amplitude estimates. The two parameter estimates λ_{2p} and λ_{2q} can be combined in a simple mean:

$$\lambda_2(1-\varepsilon_1) = \tfrac{1}{2}(\lambda_{2p} + \lambda_{2q}), \tag{27}$$

$$= \lambda_2 \tfrac{1}{2}[(1-\varepsilon_p)^4 + (1+\varepsilon_q)^4]. \tag{28}$$

Thus, the error associated with a simple mean of the under- and over-estimates of the nonlinear parameter is

$$\varepsilon_1 = 1 - \tfrac{1}{2}[(1-\varepsilon_p)^4 + (1+\varepsilon_q)^4]. \tag{29}$$

Alternatively, as ε_p and ε_q by definition are both positive, simply averaging Eqs. (24), (26), will reduce the cumulative error. The resulting parameter estimate of λ_2 will be

$$\lambda_2(1+\varepsilon_4) = [\tfrac{1}{2}(\lambda_{2p}^{1/4} + \lambda_{2q}^{1/4})]^4, \tag{30}$$

$$= \lambda_2\left[1 + \frac{\varepsilon_q - \varepsilon_p}{2}\right]^4, \tag{31}$$

$$\Rightarrow \varepsilon_4 = 1 - \left[1 + \frac{\varepsilon_q - \varepsilon_p}{2}\right]^4. \tag{32}$$

It is straightforward to show that $\varepsilon_4 < \varepsilon_1$, indicating that the fourth power of the mean fourth root yields a better estimate of the parameter λ_2 than a simple average. In other words, a much improved physical nonlinear damping parameter estimate, β_4, can be obtained from Eq. (30), as will be seen clearly in the example below.

3.2. Results

Consider the system described by Eq. (11) with specific parameter values given in Table 1. The system was given an initial displacement of 10 mm and released from rest. The response was calculated using a fourth-order Runge–Kutta scheme with a constant time step equivalent to a sample frequency of 1024 Hz. Note that this level of response and parameter values is typical of the fluidelastic system investigated by Meskell and Fitzpatrick [1].

The successive peaks of the response can be easily selected automatically. Fig. 1 plot the data associated with Eqs. (17) and (20). The units are not physically meaningful and so have been omitted. As can be seen, both approximations lead to a straight line. The procedure can be applied to peak values obtained from the displacement, velocity or acceleration response, but the displacement amplitude must be calculated using Eq. (4). The various estimates for λ_1 and λ_2 which are related to the intercept and slope of the fitted line can now be used to quantify the damping parameters. The normalized estimate values for the damping parameters c and β, along with the mean estimates $\bar{c}$ and $\bar{\beta}$, are shown in Table 2. Note that the estimates from Eq. (17) are denoted with a subscript p, while those from Eq. (20) use a q subscript. The improved cumulative estimate for the nonlinear damping obtained with Eq. (30) is referred to as β_4.

Table 1
System parameter values

ω^2 (rad/s)^2	c (Ns/m/kg)	β (Ns^3/m^3/kg)
1600	2	10

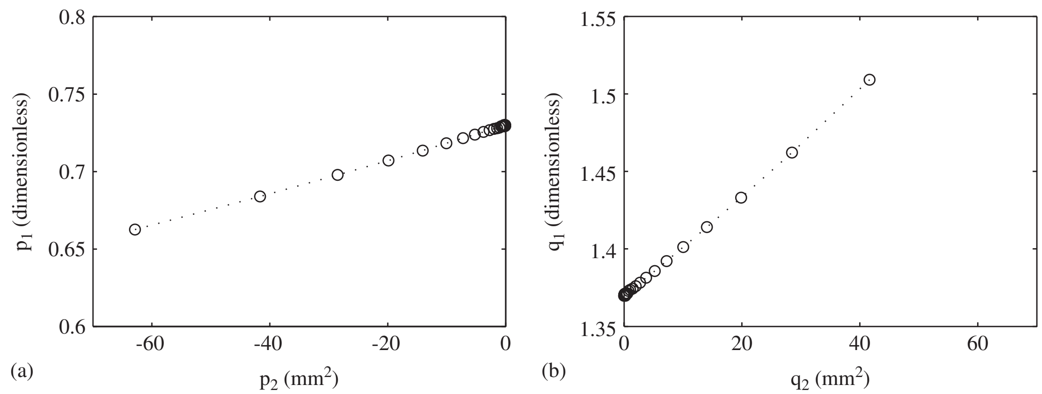

Fig. 1. Peak response data transformed into identification coordinates: (a) data associated with Eq. (17); (b) data associated with Eq. (20); ○, response data; ···, straight line fit using total least squares.

Table 2
System damping parameter estimates

	$\frac{c_p}{c}$	$\frac{c_q}{c}$	$\frac{\bar{c}}{c}$	$\frac{\beta_p}{\beta}$	$\frac{\beta_q}{\beta}$	$\frac{\bar{\beta}}{\beta}$	$\frac{\beta_4}{\beta}$
x	1.18	0.86	1.02	1.74	0.58	1.16	1.04
$\dot{x}$	1.18	0.86	1.02	1.77	0.57	1.17	1.05
$\ddot{x}$	1.18	0.86	1.02	1.77	0.56	1.17	1.04

c is the exact linear damping parameter; β is the exact cubic damping parameter; c_p and β_p are the linear and cubic damping estimated from Eq. (17); c_q and β_q are the linear and cubic damping estimated from Eq. (20); $\bar{c}$ and $\bar{\beta}$ are the averages of the estimated parameters; and β_4 is the aggregate estimate of cubic damping from Eq. (30).

It can be seen from the estimates that the linear damping c is always identified to within 2% of the actual value, even though the individual estimates c_p and c_q are out by almost 20%. The estimates for the nonlinear damping, on the other hand, have greater errors and more importantly, the over-estimate has nearly twice the error of the under-estimate. Therefore, the simple mean value is nearly 20% greater than the exact value. However, using the order specific averaging method, i.e. Eq. (30), the cumulative estimate is within 5%. It is interesting to note that the estimates obtained from the velocity are systematically worse than those from either the acceleration or displacement response. This is true also for the linear damping estimates, although much less significant.

4. Conclusions

A decrement method for identifying the linear and nonlinear damping parameters in lightly damped systems has been presented, the principle strength of which is that only a single free response signal is required. The application of the method has been demonstrated with simulated response data for a system with linear and cubic damping. The damping parameters are well estimated (within 5%) regardless of which response measurement is used (displacement, velocity or acceleration).

It is worth noting that the technique will tolerate weakly nonlinear stiffness as this will still be a conservative force and so not have a strong effect on the energy dissipation rates. However, strong nonlinearities in the stiffness will generate large superharmonics in the response, increasing the rms velocity and so enhancing the energy dissipation.

While the procedure has been illustrated for a system with just one nonlinear damping term, it is straightforward to show how it can be applied to system models with several such terms. In this case, the central equation of the method, Eq. (10), will lead to two higher order polynomial equations in the amplitude peak values, rather than the first-order relationships found for the cubic damping case, i.e. Eqs. (17) and (20). However, even in this situation, the identification process reduces to simply fitting the appropriate order polynomial to the data derived from the free response test. The physical damping parameters will be obtained directly from the fitted polynomial coefficients.

As it is currently applied, the method is restricted to single degree of freedom systems. Most system parameter identification methods balance the instantaneous forces in the system (whether in the time or frequency domain) and so require that the sub- and super-harmonics associated with any nonlinearities are also balanced. However, since the current technique is based on the long-term evolution of the energy in the system, in principle it may be possible to apply notch filtering at the fundamental frequency of each degree of freedom and hence apply the method to an equivalent set of single degree of freedom systems. Further work is needed to explore this possibility.

In conclusion, the method described here will be useful in determining from experimental data the nonlinear damping coefficients especially in fluidelastic systems, where the total damping is low.

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