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Short Communication

Higher accuracy analytical approximations to the Duffing-harmonic oscillator

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Abstract

A new method has been presented for analytically solving the Duffing-harmonic oscillator. The method is obtained by combining Newton's method with the harmonic balance method. By using the method, one obtains linear algebraic equations instead of nonlinear algebraic equations. The complexity of the HB method is greatly simplified. Iteration of procedure yields rapid convergence with respect to exact solution. The results are valid for the complete range of oscillation amplitude, including the limiting cases of amplitude approaching zero and infinity.

Consider a one-dimensional, nonlinear oscillator governed by

$$\frac{d^2u}{dt^2}+\frac{u^3}{1+u^2}=0, \quad u(0)=A, \quad \frac{du}{dt}(0)=0, \tag{1}$$

which is an example of conservative nonlinear oscillatory systems having a rational form for the restoring force. For small u, the equation is that of a Duffing-type nonlinear oscillator, while for large u, the equation approximates that of a linear harmonic oscillator; hence, Eq. (1) is called the Duffing-harmonic oscillator [1]. The system will oscillate between symmetric bounds $[-A, A]$, and the frequency and corresponding periodic solution of the nonlinear oscillator are dependent on the amplitude A. Note that for Eq. (1), the usual perturbation procedures, i.e. expansion with reference to a centre and in a small parameter, do not apply [1–3].

By rewriting Eq. (1) and applying the lowest order harmonic balance (HB) method [4], Mickens [1] obtained the first approximate angular frequency:

$$\omega_M(A)=\sqrt{3A^2/(4+3A^2)}. \tag{2}$$

Let $\phi=\pi/\lfloor 2F(1/\sqrt{2},\pi/2)\rfloor \approx 0.8472$ where $F(1/\sqrt{2},\pi/2)$ is the complete elliptic integral of the first kind. Then ϕA is the exact angular frequency for equation $d^2u/dt^2+u^3=0$ and the conjectured exact angular

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frequency of Mickens [1] for Eq. (1) is

$$\tilde{\omega}_M(A) = \phi A \Big/ \sqrt{1+\phi^2 A^2}. \tag{3}$$

By applying the lowest order HB method directly to Eq. (1), Lim and Wu [5] obtained two analytical approximate angular frequencies. The first one is

$$\omega_{L-W}(A) = \sqrt{1 + \frac{2}{A^2}\left(\frac{1}{\sqrt{1+A^2}} - 1\right)}. \tag{4}$$

The second one is more accurate but a little complicated, hence it is omitted here. Using a single-term approximate solution $u(t) = A\cos(\omega t)$ to Eq. (1) and the Ritz procedure [6], Tiwari et al. [7] obtained the same approximation as that in Eq. (4). The frequency–amplitude relations (2)–(4) are approximate relations. In particular, relations (2) and (4) have the largest error for A close to 0, while formula (3) is accurate for A close to 0 only.

The HB method [4] is very difficult to construct higher-order analytical approximations because it requires analytical solutions of sets of complicated nonlinear algebraic equations. To improve the HB method, Lim and Wu [5] presented an approach by combining the HB method and linearization of nonlinear oscillation equation with respect to displacement increment. For higher-order approximation, however, this method results in a complex nonlinear algebraic equation in terms of unknown frequency and its analytical solution is again difficult. To overcome the problem, analytical approximations to the solution of the Duffing-harmonic oscillator based on a new approach are presented in this Communication. This new method is obtained by combining Newton's method with the harmonic balance method. In such a way, one obtains linear algebraic equations instead of nonlinear algebraic equations at each iteration. The complexity of the HB method has been greatly simplified.

A new independent variable $\tau = \omega t$ is introduced. Then, Eq. (1) can be rewritten as

$$\Omega u''\left(1+u^2\right) + u^3 = 0, \quad u(0) = A, \quad u'(0) = 0, \tag{5}$$

where ($'$) denotes differentiation with respect to τ and $\Omega = \omega^2$. The new independent variable is chosen in such a way that the solution of Eq. (5) is a periodic function of τ of period 2π. The corresponding frequency of the nonlinear oscillator is given by $\omega = \sqrt{\Omega}$.

Following the lowest order HB approximation, we set

$$u_1(\tau) = A\cos\tau, \tag{6}$$

which satisfies the initial conditions in Eq. (5). Substituting Eq. (6) into Eq. (5) and setting the coefficient of resulting $\cos\tau$ to zero yield the first analytical approximation to the frequency in terms of A:

$$\omega_1(A) = \sqrt{\Omega_1(A)} = \sqrt{\frac{3A^2}{4+3A^2}}. \tag{7}$$

Therefore, the first approximate periodic solution is

$$u_1(t) = A\cos\tau, \quad \tau = \omega_1(A)t. \tag{8}$$

Note that results in Eqs. (7) and (8) have been reported by Mickens [1].

Using $u_1(\tau)$ and $\Omega_1(A)$ as initial approximations to the solution of Eq. (5), we apply the combination of Newton's method and the HB method to solve Eq. (5). The first step is the Newton procedure. The periodic solution and the square of frequency of Eq. (5) can be expressed as

$$u = u_1 + \Delta u_1, \quad \Omega = \Omega_1 + \Delta\Omega_1. \tag{9}$$

Substituting Eq. (9) into Eq. (5) and linearizing with respect to the correction terms Δu_1 and $\Delta\Omega_1$ lead to

$$\begin{aligned}&\Omega_1(1+u_1^2)u_1''+u_1^3+\Delta\Omega_1(1+u_1^2)u_1''+\Omega_1(1+u_1^2)\Delta u_1''+2\Omega_1 u_1 u_1''\Delta u_1+3u_1^2\Delta u_1=0,\\ &\Delta u_1(0)=0,\quad \Delta u_1'(0)=0,\end{aligned} \tag{10}$$

where Δu_1 is a periodic function of τ of period 2π, and both Δu_1 and $\Delta\Omega_1$ are undetermined.

The resulting linear equation in Δu_1 and $\Delta\Omega_1$ in Eq. (10) will be solved by the HB method. The second approximate solution to Eq. (10) can be developed by setting $\Delta u_1(\tau)$ as

$$\Delta u_1(\tau)=x_1(\cos\tau-\cos 3\tau), \tag{11}$$

which satisfies the initial condition in Eq. (10) at the outset. Substituting Eqs. (6), (7) and (11) into Eq. (10), expanding the resulting expression in a trigonometric series and setting the coefficients of $\cos\tau$ and $\cos 3\tau$ to zeros, respectively, yield

$$\begin{aligned}&(12A^2+24A^4)x_1-(16A+24A^3+9A^5)\Delta\Omega_1=0,\\ &(96A^2+48A^4)x_1-(4A^3+3A^5)\Delta\Omega_1+4A^3=0.\end{aligned} \tag{12}$$

Solving Eq. (12) for x_1 and $\Delta\Omega_1$ yield

$$x_1=-\frac{A(4+3A^2)}{96+117A^2+30A^4},\quad \Delta\Omega_1=-\frac{4A^2(1+2A^2)}{128+252A^2+157A^4+30A^6}. \tag{13}$$

Therefore, the second approximations to frequency and periodic solution of the nonlinear oscillator are

$$\begin{aligned}&\omega_2(A)=\sqrt{\Omega_2(A)},\quad u_2(t)=u_1(\tau)+\Delta u_1(\tau)=X(A)\cos\tau+Y(A)\cos 3\tau,\\ &\tau=\omega_2(A)t,\end{aligned} \tag{14}$$

where

$$\begin{aligned}\Omega_2(A)&=\frac{A^2(23+10A^2)}{32+39A^2+10A^4},\quad X=\frac{2A(46+57A^2+15A^4)}{96+117A^2+30A^4},\\ Y&=\frac{A(4+3A^2)}{96+117A^2+30A^4}.\end{aligned} \tag{15}$$

Based on the second approximations, the periodic solution and the frequency of Eq. (5) can be further expressed as

$$u=u_2+\Delta u_2,\quad \Omega=\Omega_2+\Delta\Omega_2. \tag{16}$$

Substituting Eq. (16) into Eq. (5) and linearizing with respect to the correction terms Δu_2 and $\Delta\Omega_2$ yield

$$\begin{aligned}&\Omega_2(1+u_2^2)u_2''+u_2^3+\Delta\Omega_2(1+u_2^2)u_2''+\Omega_2(1+u_2^2)\Delta u_2''+2\Omega_2 u_2 u_2''\Delta u_2+3u_2^2\Delta u_2=0,\\ &\Delta u_2(0)=0,\quad \Delta u_2'(0)=0,\end{aligned} \tag{17}$$

where Δu_2 is a periodic function of τ of period 2π, and both Δu_2 and $\Delta\Omega_2$ are undetermined unknowns.

The HB method will again be applied to solve Eq. (17) for Δu_2 and $\Delta\Omega_2$. Here $\Delta u_2(\tau)$ is taken as

$$\Delta u_2(\tau)=y_1(\cos\tau-\cos 3\tau)+y_2(\cos 3\tau-\cos 5\tau), \tag{18}$$

which satisfies the initial conditions in Eq. (17). Substituting Eqs. (14), (15) and (18) into Eq. (17), expanding the resulting expression in a trigonometric series and setting the coefficients of $\cos\tau$, $\cos 3\tau$ and $\cos 5\tau$ to zeros, respectively, yield three relations for y_1, y_2 and $\Delta\Omega_2$ as follows:

$$\begin{aligned}&-(4X+3X^3+11X^2Y+38XY^2)\Delta\Omega_2\\ &+(-4\Omega_2+6X^2+2\Omega_2X^2-6XY+54\Omega_2XY+6Y^2-38\Omega_2Y^2)y_1\\ &+(3X^2-11\Omega_2X^2+6XY-6\Omega_2XY-3Y^2+43\Omega_2Y^2)y_2\\ &+(-4\Omega_2X+3X^3-3\Omega_2X^3+3X^2Y-11\Omega_2X^2Y+6XY^2-38\Omega_2XY^2)=0,\end{aligned} \tag{19a}$$

$$
\begin{aligned}
&-(X^3+36Y+22X^2Y+27Y^3)\Delta\Omega_2\\
&\quad+(36\Omega_2-3X^2+19\Omega_2X^2+12XY-44\Omega_2XY-9Y^2+81\Omega_2Y^2)y_1\\
&\quad+(-36\Omega_2+3X^2+5\Omega_2X^2-6XY+70\Omega_2XY+9Y^2-81\Omega_2Y^2)y_2\\
&\quad+(X^3-\Omega_2X^3-36\Omega_2Y+6X^2Y-22\Omega_2X^2Y+3Y^3-27\Omega_2Y^3)=0,
\end{aligned} \tag{19b}
$$

$$
\begin{aligned}
&-(11X^2Y+19XY^2)\Delta\Omega_2\\
&\quad+(-3X^2+11\Omega_2X^2+16\Omega_2XY+3Y^2-19\Omega_2Y^2)y_1\\
&\quad+(100\Omega_2-3X^2+43\Omega_2X^2+6XY-38\Omega_2XY-6Y^2+86\Omega_2Y^2)y_2\\
&\quad+(3X^2Y-11\Omega_2X^2Y+3XY^2-19\Omega_2XY^2)=0.
\end{aligned} \tag{19c}
$$

The linear equations. (19a–c) in unknowns y_1, y_2 and $\Delta\Omega_2$ can be solved, and these solutions can be expressed as functions of A. Consequently, one may obtain the third approximations to the frequency and the corresponding periodic solution of the nonlinear oscillator as

$$
\begin{aligned}
&\omega_3(A)=\sqrt{\Omega_3(A)},\\
&u_3(t)=\big(X(A)+y_1(A)\big)\cos\tau+\big(Y(A)-y_1(A)+y_2(A)\big)\cos 3\tau-y_2(A)\cos 5\tau,\quad \tau=\omega_3(A)t,
\end{aligned} \tag{20}
$$

where

$$\Omega_3(A)=C(A)/D(A),$$

$$
\begin{aligned}
C(A)=A^2(&6\,629\,381\,323\,200\,921\,600+67\,776\,657\,351\,647\,428\,608A^2\\
&+324\,181\,140\,708\,366\,434\,304A^4+963\,763\,044\,959\,455\,887\,360A^6\\
&+1\,995\,414\,900\,881\,199\,438\,208A^8+3\,055\,182\,354\,907\,875\,823\,120A^{10}\\
&+3\,584\,852\,660\,636\,245\,925\,296A^{12}+3\,296\,040\,031\,717\,526\,802\,633A^{14}\\
&+2\,407\,402\,837\,392\,073\,420\,890A^{16}+1\,407\,400\,421\,302\,961\,257\,356A^{18}\\
&+660\,232\,177\,524\,469\,119\,960A^{20}+247\,999\,449\,491\,216\,103\,600A^{22}\\
&+74\,057\,760\,578\,406\,852\,000A^{24}+17\,349\,552\,015\,153\,480\,000A^{26}\\
&+3\,119\,930\,639\,690\,400\,000A^{28}+415\,788\,841\,656\,000\,000A^{30}\\
&+38\,695\,631\,040\,000\,000A^{32}+2\,245\,903\,200\,000\,000A^{34}+61\,236\,000\,000\,000A^{36}),
\end{aligned}
$$

$$
\begin{aligned}
D(A)=\big(32+39A^2+10A^4\big)\big(&288\,586\,791\,747\,846\,144+2\,827\,110\,614\,223\,028\,224A^2\\
&+12\,901\,760\,886\,549\,168\,128A^4+36\,422\,203\,401\,630\,900\,224A^6\\
&+7\,122\,903\,534\,348\,970\,6624A^8+102\,396\,952\,803\,640\,001\,008A^{10}\\
&+112\,041\,973\,038\,923\,497\,908A^{12}+95\,309\,456\,481\,644\,416\,311A^{14}\\
&+63\,813\,346\,845\,672\,803\,868A^{16}+33\,822\,874\,705\,790\,864\,268A^{18}\\
&+14\,193\,355\,896\,848\,141\,280A^{20}+4\,689\,648\,931\,579\,491\,600A^{22}\\
&+1\,205\,418\,822\,799\,968\,000A^{24}+236\,059\,546\,632\,120\,000A^{26}\\
&+34\,028\,121\,546\,000\,000A^{28}+3\,404\,341\,548\,000\,000A^{30}\\
&+211\,205\,880\,000\,000A^{32}+6\,123\,600\,000\,000A^{34}\big)
\end{aligned}
$$

It should be clear how the procedure works for constructing further higher-order approximate solutions.

Table 1
Comparison of approximate frequencies with exact frequency

A	ω_e	$\omega_M(\omega_1)/\omega_e$	ω_2/ω_e	ω_3/ω_e	ω_{L-W}/ω_e	$\tilde{\omega}_M/\omega_e$
0.01	0.00847	1.02221	1.00068	1.00007	1.02220	0.99999
0.05	0.04232	1.02226	1.00069	1.00007	1.02215	1.00007
0.1	0.08439	1.02241	1.00071	1.00006	1.02199	1.00034
0.5	0.38737	1.02580	1.00130	1.00000	1.01772	1.00693
1	0.63678	1.02807	1.00236	1.00002	1.0107	1.01512
5	0.96698	1.00763	1.00290	1.00117	1.00035	1.00649
10	0.99092	1.00251	1.00121	1.00068	1.00004	1.00221
50	0.99961	1.00013	1.00007	1.00005	1.00000	1.00011
100	0.99990	1.00003	1.00002	1.00001	1.00000	1.00003

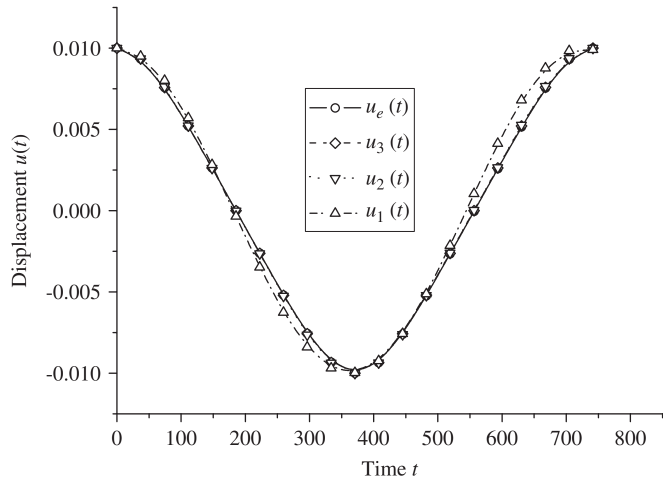

Fig. 1. Comparison of approximate periodic solutions with exact solution for $A = 0.01$.

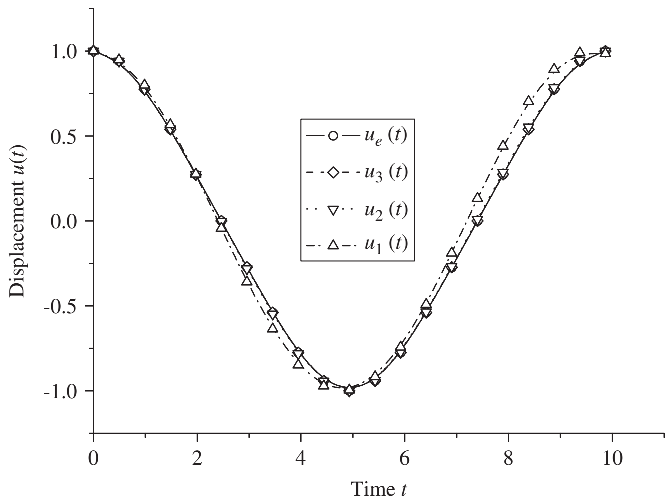

Fig. 2. Comparison of approximate periodic solutions with exact solution for $A = 1$.

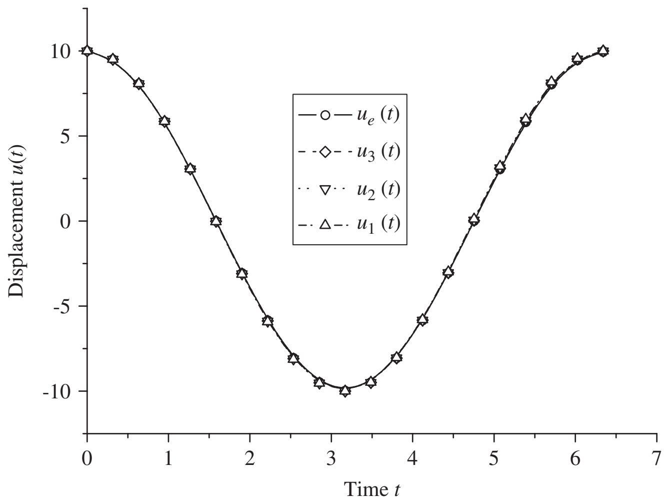

Fig. 3. Comparison of approximate periodic solutions with exact solution for $A = 10$.

For the nonlinear oscillator, the exact angular frequency is

$$\omega_e(A) = \frac{\pi}{2}\left(\int_0^{\pi/2} \frac{A\cos t\,\mathrm{d}t}{\sqrt{A^2\cos^2 t + \ln\left[1 - A^2\cos^2 t/\left(1 + A^2\right)\right]}}\right)^{-1}. \tag{21}$$

Table 1 shows the ratios of the approximate angular frequencies $\omega_1(\omega_M)$, ω_2, ω_3, ω_{L-W}, $\tilde{\omega}_M$ in Eqs. (2), (14), (20), (4) and (3), respectively, to the exact angular frequency ω_e in Eq. (21). Furthermore, we have

$$\lim_{A\to\infty} \omega_e(A) = \lim_{A\to\infty} \omega_M(A) = \lim_{A\to\infty} \omega_2(A) = \lim_{A\to\infty} \omega_3(A) = \lim_{A\to\infty} \omega_{L-W}(A) = \lim_{A\to\infty} \tilde{\omega}_M(A) = 1,$$

$$\lim_{A\to 0}\frac{\omega_M}{\omega_e} = 1.02220, \quad \lim_{A\to 0}\frac{\omega_2}{\omega_e} = 1.00068,$$
$$\lim_{A\to 0}\frac{\omega_3}{\omega_e} = 1.00007, \quad \lim_{A\to 0}\frac{\omega_{L-W}}{\omega_e} = 1.02220, \quad \lim_{A\to 0}\frac{\tilde{\omega}_M}{\omega_e} = 1. \tag{22}$$

From Table 1 and Eq. (22), it can be observed that Eqs. (14) and (20) yield excellent approximate frequencies for both small and large values of amplitude A. For amplitudes, $A = 0.01, 1$ and 10, the (numerical) exact periodic solution $u_e(t)$ obtained by numerically integrating Eq. (1) and the approximate periodic solutions $u_1(t), u_2(t)$ and $u_3(t)$ computed by Eqs. (8), (14) and (20), respectively, are plotted in Figs. 1–3. These figures show that the proposed solutions in Eqs. (14) and (20) provide excellent approximations to exact periodic solutions for both small and large amplitude.

In summary, a new method has been presented for analytically solving the Duffing-harmonic oscillator. The iteration yields rapid convergence with respect to the exact solution. The results are valid for the complete range of oscillation amplitude, including the limiting cases of amplitude approaching zero and infinity.

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