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Short Communication

Modes for the coupled Timoshenko model with a restrained end

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Abstract

The modes of the second-order Timoshenko system for the displacement and rotation of a fixed beam with a restrained end at the left are formulated in terms of a fundamental spatial response. This is done without decoupling the system into fourth-order scalar equations. The restrained end leads to time–space boundary conditions which introduce the frequency as a parameter into the system of equations for determining the modes. These equations involve first-order derivatives and, consequently, the modes are determined by solving a non-conservative differential system. This modal differential equation is discussed in terms of a fundamental matrix response. It is determined by applying a closed formula that was obtained by the first author and involves the characteristic polynomial of the modal differential equation.

1. Introduction

Vibrations of beams according to the Timoshenko theory has been considered by several authors [1–4]. Here, we characterize the modes of the Timoshenko model for the displacement and rotation of a fixed beam with a mass welded to the other end and attached to a translational spring [5].

In the literature, the Timoshenko model is usually decoupled into two fourth-order equations of the same type [6,7]. The frequency equation and mode shapes are then formulated in terms of the classical Euler basis. This later is constructed in terms of the roots of the associated characteristic polynomial of a fourth-order differential equation. The involved constants in the solution modes are then determined by substitution into the original coupled system and boundary conditions.

Another approach has been given in Refs. [8–11]. It consists in using a basis generated by a fundamental matrix response which is characterized by given initial conditions. This avoids the need of decoupling the second-order Timoshenko system. It is kept as a system formed by two coupled equations of second order. The eigenanalysis is studied in terms of such fundamental response. It turns out that the amplitude vector of a mode satisfies a non-conservative second-order differential equation. As a basis of solutions, we choose the one that is generated by a fundamental matrix response and its derivative. This response can be computed in closed-form through a formula derived in Ref. [8]. See also Ref. [9] or Ref. [10] for mth-order systems in continuous and discrete time.

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After determining the modes for the coupled Timoshenko equations, we observe that the components of the modes satisfy the fourth-order equation that is obtained by decoupling the system.

2. Formulation of the problem

In Clark [1] or Ginsberg [5], the unforced coupled equations of the Timoshenko model f or the total deflection u and the bending slope ψ of a uniform beam are derived and given by

$$\rho A \frac{\partial^2 u(t,x)}{\partial t^2} - \kappa G A \frac{\partial}{\partial x}\left(\frac{\partial u(t,x)}{\partial x} - \psi(t,x)\right) = 0,$$

$$\rho I \frac{\partial^2 \psi(t,x)}{\partial t^2} - EI \frac{\partial^2 \psi(t,x)}{\partial x^2} - \kappa G A\left(\frac{\partial u(t,x)}{\partial x} - \psi(t,x)\right) = 0. \tag{1}$$

Here ρ is the mass density, A the cross-sectional area, ρA the mass per unit length, ρI the mass moment of inertia per unit length about the neutral axis which passes through the center, E the modulus of elasticity, G the shear modulus, and κ a numerical factor depending on the shape of the cross-section.

When a cube of mass m and side dimensions b is welded to the end of a left end clamped beam of length L, and a spring of stiffness k is attached to the cube, the boundary conditions, that include translational and rotatory inertia of the cube as shown in (Fig. 1), are given by

$$u(t,0) = 0,$$

$$\psi(t,0) = 0,$$

$$\kappa G A\left(\frac{\partial u}{\partial x} - \psi\right)(t,L) + k\left(u + \frac{b}{2}\psi\right) + m\left(\ddot{u} + \frac{b}{2}\ddot{\psi}\right)(t,L) = 0,$$

$$EI\frac{\partial \psi}{\partial x} - \kappa G A \frac{b}{2}\left(\frac{\partial u}{\partial x} - \psi\right) + \frac{1}{6} m b^2 \ddot{\psi}(t,L) = 0.$$

3. Eigenanalysis

In order to determine the eigenfunctions of above system, we substitute the functions $u = \mathrm{e}^{\mathrm{i}\omega t}U(x)$, $\psi = \mathrm{e}^{\mathrm{i}\omega t}\Psi(x)$ into the equations given in Eq. (1).

This results in the system

$$\kappa G A U'' - \kappa G A \Psi' + \rho A \omega^2 U = 0,$$

$$EI\Psi'' + \kappa G A U' + (\rho I \omega^2 - \kappa G A)\Psi = 0. \tag{2}$$

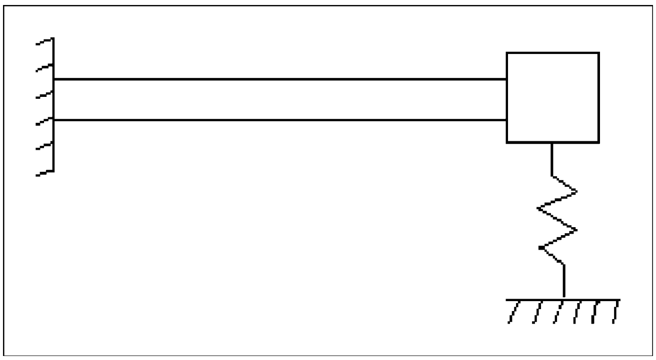

Fig. 1. A uniform cantilever beam with a restrained end.

By substituting the modal functions into the boundary conditions, we have

$$U(0) = 0,$$

$$\Psi(0) = 0, \tag{3}$$

$$\kappa GAU'(L) + (k - \omega^2 m)U(L) + \left(\frac{kb}{2} - \frac{mb\omega^2}{2} - \kappa GA\right)\Psi(L) = 0,$$

$$-\frac{\kappa GAb}{2}U'(L) + EI\Psi'(L) + \left(\frac{\kappa GAb}{2} - \frac{mb^2\omega^2}{6}\right)\Psi(L) = 0. \tag{4}$$

In matrix form, it follows that

$$\mathscr{M}\mathbf{X}''(x) + \mathscr{C}\mathbf{X}'(x) + \mathscr{K}(\omega)\mathbf{X}(x) = \mathbf{0}, \tag{5}$$

where $\mathbf{X}$ denotes the vector of the amplitudes U, Ψ for the deflection and the bending slope, respectively, and

$$\mathscr{M} = \begin{bmatrix} \kappa GA & 0 \\ 0 & EI \end{bmatrix}, \quad \mathscr{C} = \begin{bmatrix} 0 & -\kappa GA \\ \kappa GA & 0 \end{bmatrix},$$

$$\mathscr{K}(\omega) = \begin{bmatrix} \rho A\omega^2 & 0 \\ 0 & \rho I\omega^2 - \kappa GA \end{bmatrix}.$$

We observe that the effect of the shear stress introduces the matrix coefficient $\mathscr{C}$. The frequency appears in the matrix coefficient $\mathscr{K}$ as it would be with an Euler–Bernoulli beam.

In matrix form, the boundary conditions are written as

$$A_{11}\mathbf{X}(0) + B_{11}\mathbf{X}'(0) = 0,$$

$$A_{21}\mathbf{X}(L) + B_{21}\mathbf{X}'(L) = 0, \tag{6}$$

where A_{11} is the 2×2 identity matrix, B_{11} is the 2×2 zero matrix and

$$A_{21} = \begin{bmatrix} k - \omega^2 m & -\kappa GA + \frac{b}{2}(k - m\omega^2) \\ 0 & \frac{b}{2}\left(\kappa GA - \frac{\omega^2 mb}{3}\right) \end{bmatrix}, \quad B_{21} = \begin{bmatrix} \kappa GA & 0 \\ -\frac{\kappa GAb}{2} & EI \end{bmatrix}.$$

3.1. Solution basis

The general solution of Eq. (5) in terms of initial values can be written as

$$X(x) = \mathbf{h}_0(x)X(0) + \mathbf{h}_1(x)X'(0),$$

$$\mathbf{h}_0(x) = \mathbf{h}'(x)\mathscr{M} + \mathbf{h}(x)\mathscr{C},$$

$$\mathbf{h}_1(x) = \mathbf{h}(x)\mathscr{M}, \tag{7}$$

or

$$X(x) = \mathbf{h}(x)c_1 + \mathbf{h}'(x)c_2 \tag{8}$$

for arbitrary vectors c_1 and c_2. Here $\mathbf{h}(x)$ is the solution of the initial value problem

$$\mathscr{M}\mathbf{h}''(x) + \mathscr{C}\mathbf{h}'(x) + \mathscr{K}(\omega)\mathbf{h}(x) = 0, \tag{9}$$

$$\mathbf{h}(0) = 0, \quad \mathscr{M}\mathbf{h}'(0) = I, \tag{10}$$

where 0 denotes the 2×2 null matrix and I the 2×2 identity matrix.

For determining $\mathbf{h}(x)$ we use the formula derived in Ref. [8]. We first consider the characteristic polynomial

$$P(s) = \det[s^2\mathscr{M} + s\mathscr{C} + \mathscr{K}] = \sum_{k=0}^{4} b_k s^{4-k}.$$

It turns out that

$$b_0 = ab, \quad b_1 = 0, \quad b_2 = ae\omega^2 + c\omega^2 b, \quad b_3 = 0, \quad b_4 = c\omega^4 e - c\omega^2 a,$$

where

$$a = \kappa GA, \quad b = EI, \quad c = \rho A, \quad e = \rho I.$$

Then we find the solution $h(x)$ of the initial value problem

$$\sum_{k=0}^{4} b_k h^{(4-k)}(x) = b_0 h^{(iv)}(x) + b_2 h''(x) + b_4 h(x) = 0,$$

$$h(0) = h'(0) = h''(0) = 0, \quad b_0 h'''(0) = 1 \tag{11}$$

and we let $\mathbf{h}_k$ be the solution of the matrix difference equation

$$\mathscr{M}\mathbf{h}_{k+2} + \mathscr{C}\mathbf{h}_{k+1} + \mathscr{K}\mathbf{h}_k = 0,$$

$$\mathbf{h}_0 = 0, \quad \mathscr{M}\mathbf{h}_1 = I. \tag{12}$$

We have

$$\mathbf{h}(x) = \sum_{j=1}^{4} \sum_{i=0}^{j-1} b_i h^{(j-1-i)}(x)\mathbf{h}_{4-j}. \tag{13}$$

The calculations run as follows. The characteristic differential Eq. (11) is

$$abh^{(iv)}(x) + (ae + cb)\omega^2 h''(x) + (e\omega^2 - a)c\omega^2 h(x) = 0,$$

$$h(0) = h'(0) = h''(0) = 0, \quad abh'''(0) = 1. \tag{14}$$

The characteristic polynomial can be conveniently written as

$$P(s) = ab(s^4 + g^2 s^2 - r^4),$$

where

$$g^2 = (e/b + c/a)\omega^2, \quad r^4 = c\omega^2(-e\omega^2 + a)/ab.$$

The roots of the characteristic polynomial $P(s)$ are then easily found to be $s = \varepsilon, -\varepsilon, \mathrm{i}\delta, -\mathrm{i}\delta$, where

$$\varepsilon = 1/2\sqrt{-2g^2 + 2\sqrt{(g^4 + 4r^4)}}, \quad \delta = \sqrt{(g^2 + \varepsilon^2)}.$$

Thus

$$g^2 = \delta^2 - \varepsilon^2, \quad r^4 = \delta^2\varepsilon^2.$$

By using the Euler basis or the Laplace transform method, we find that the solution of the initial value problem (11) is

$$h(x) = \frac{\delta\sinh(\varepsilon x) - \varepsilon\sin(\delta x)}{ab(\varepsilon^2 + \delta^2)\varepsilon\delta}. \tag{15}$$

The following matrix values are obtained by iterating the initial value problem (12):

$$\mathbf{h}_0 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad \mathbf{h}_1 = \begin{bmatrix} \frac{1}{a} & 0 \\ 0 & \frac{1}{b} \end{bmatrix},$$

$$\mathbf{h}_2 = \begin{bmatrix} 0 & \frac{1}{b} \\ -\frac{1}{b} & 0 \end{bmatrix}, \quad \mathbf{h}_3 = \begin{bmatrix} -\frac{a^2 + c\omega^2 b}{a^2 b} & 0 \\ 0 & -\frac{e\omega^2}{b^2} \end{bmatrix}.$$

Then

$$\mathbf{h}(x) = \begin{bmatrix} bh''(x) + (e\omega^2 - a)h(x) & ah'(x) \\ -ah'(x) & ah''(x) + h(x)c\omega^2 \end{bmatrix}. \tag{16}$$

3.2. Characteristic equation

We now proceed to determine the characteristic equation that will give the frequencies for which the modes $\mathbf{X}(x)$ are to be determined. Since the left end of the beam is fixed, the choice of the matrix basis $\mathbf{h}, \mathbf{h}'$ should simplify computations at such end. The substitution of the general solution

$$\mathbf{X}(x) = \mathbf{h}(x)c_1 + \mathbf{h}'(x)c_2, \quad 0 \leqslant x \leqslant L, \tag{17}$$

where c_1, c_2 are 2×1 vectors, into the boundary conditions (6) leads to the linear system

$$\mathbf{h}(0)c_1 + \mathbf{h}'(0)c_2 = 0.c_1 + \mathscr{M}^{-1}c_2 = 0,$$

$$(A_{21}\mathbf{h}(L) + B_{21}\mathbf{h}'(L))c_1 + (A_{21}\mathbf{h}'(L) + B_{21}\mathbf{h}''(L))c_2 = 0.$$

It follows that $c_2 = 0$ and the characteristic equation is

$$\det(\Delta(\omega)) = 0, \tag{18}$$

where

$$\Delta(\omega) = A_{21}\mathbf{h}(L) + B_{21}\mathbf{h}'(L). \tag{19}$$

Here the values $\mathbf{h}(L)$, $\mathbf{h}'(L)$ are to be computed from Eq. (16) and the coefficients A_{21}, B_{21} are given in Eq. (6). In particular, by using the Euler basis that leads to Eq. (15), we have

$$\mathbf{h}(x) = \begin{bmatrix} \frac{\delta A \sinh(\varepsilon x) - \varepsilon B \sin(\delta x)}{ab(\varepsilon^2 + \delta^2)\varepsilon\delta} & \frac{\cosh(\varepsilon x) - \cos(\delta x)}{b(\varepsilon^2 + \delta^2)} \\ -\frac{\cosh(\varepsilon x) - \cos(\delta x)}{b(\varepsilon^2 + \delta^2)} & \frac{\delta C \sinh(\varepsilon x) - \varepsilon D \sin(\delta x)}{ab(\varepsilon^2 + \delta^2)\varepsilon\delta} \end{bmatrix}, \tag{20}$$

where

$$A = e\omega^2 + b\varepsilon^2 - a,$$
$$B = e\omega^2 - b\delta^2 - a,$$
$$C = c\omega^2 + \varepsilon^2 a,$$
$$D = c\,\omega^2 - \delta^2 a$$

and the involved parameters and constants are as defined above.

It is clear that for each root of the characteristic Eq. (19), we have the mode

$$\mathbf{X}(x, \omega) = \mathbf{h}(x, \omega)c_1, \tag{21}$$

where c_1 is a non-zero solution of $\Delta c = 0$. The dependence of the characteristic equation as well as the modes upon frequency ω have been emphasized. When the frequency is a double one, we actually have two modes associated with such frequency. In this situation each column of the matrix **h** will be then a mode for the coupled Timoshenko model.

From Eqs. (16) and (21), we have the displacement mode

$$U(x,\omega) = \alpha(bh''(x,\omega) + (e\omega^2 - a)h(x,\omega)) + \beta(ah'(x,\omega)) \tag{22}$$

and the rotation mode

$$R(x,\omega) = \alpha(-ah'(x,\omega)) + \beta(ah''(x,\omega) + c\omega^2 h(x,\omega)) \tag{23}$$

for scalar constants α and β that both are not simultaneously zero. Here $h(x,\omega)$ is the solution (15) for each root ω of the characteristic Eq. (19).

We can that verify that $U(x,\omega)$ and $R(x,\omega)$ satisfy the characteristic fourth-order differential Eq. (14), that is

$$aby^{(iv)}(x) + (ae + cb)\omega^2 y''(x) + (e\omega^2 - a)c\omega^2 y(x) = 0 \tag{24}$$

or equations that are obtained by differentiating once or twice this later equation.

We should observe that this later equation, after dividing by the factor ab, is the fourth-order spatial equation that arises for the amplitude by substitution into the time fourth-order equation that is obtained decoupling system (2), that is

$$\alpha^2\tau^2\frac{\partial^4 w}{\partial t^4} + \left(\beta^2 - (\alpha^2 + \tau^2)\frac{\partial^2 w}{\partial x^2}\right)\frac{\partial^2 y}{\partial t^2} + \frac{\partial^4 w}{\partial x^4} = 0, \tag{25}$$

where

$$\alpha^2 = \frac{\rho}{g\kappa G}, \quad \beta^2 = \frac{\rho A}{gEI}, \quad \tau^2 = \frac{\rho}{gE}. \tag{26}$$

This equation have been also obtained by a variational argument [12] where the frequency spectrum is discussed too.

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