Available online at www.sciencedirect.com
SCIENCE DIRECT®

Journal of Sound and Vibration 296 (2006) 1059–1067

JOURNAL OF SOUND AND VIBRATION

www.elsevier.com/locate/jsvi

Short Communication

Several numerical solution techniques for nonlinear eardrum-type oscillations

Y.Z. Chen*, X.Y. Lin

Division of Engineering Mechanics, Jiangsu University, Zhenjinag, Jiangsu, 212013, PR China

Received 9 August 2005; received in revised form 1 February 2006; accepted 16 March 2006
Available online 11 May 2006

Abstract

Three numerical solution techniques for nonlinear eardrum-type oscillations with an even function in the restoring force are studied. The first suggested technique is named the target function technique. In the technique, the second zeros of the target function is the circular frequency of motion. The second suggested technique is named the multiple-parameter technique, and the involved parameters are evaluated from the governing equation, the initial conditions, and properties of motion. The third suggested technique is named the direct integration technique. All techniques depend on the computer computation significantly and provide very accurate numerical results. Finally, numerical examples are given.
© 2006 Elsevier Ltd. All rights reserved.

1. Introduction

Problems of nonlinear vibration in conservative systems have a long history. The well known nonlinear vibration of the Duffing equation is an example in this field. The governing equation for the problem was formulated in Refs. [1–3]. In the case of ε being a small parameter, the equation is solved by using the Lindstedt-Poincare technique, the method of multiple scales, and the method of averaging [1–3]. Almost all perturbation methods are based on small parameter ε so that the approximate solutions can be expressed in a series of small parameter. There are some disadvantages in the perturbation method. Generally, in the perturbation method one can only get, for example, several-terms solution for the small parameter. In the case of ε being a larger value, the perturbation method is no longer valid. Also, in this method it is not easy to judge how the approximation is achieved. Some typical numerical examples show that the error is increasing if the parameter ε becomes larger [4]. The limitation of the perturbation method was also pointed out in Ref. [5].

On the other hand, many nonlinear vibration problems were solved by using the harmonic balance method [6–11]. The merit of the harmonic balance method is to balance the coefficients of Fourier series in the governing equation of the nonlinear equation, once the assumed motion is substituted in the equation.

In this paper, the nonlinear eardrum-type oscillations are studied. The nonlinear eardrum-type oscillation is defined such that an even function for the displacement may contain the nonlinear restoring force [6,12]. For

*Corresponding author. Fax: +86 511 879 1739.
E-mail address: chens@ujs.edu.cn (Y.Z. Chen).

0022-460X/$ - see front matter © 2006 Elsevier Ltd. All rights reserved.
doi:10.1016/j.jsv.2006.03.019

the nonlinear eardrum oscillation, an iteration procedure for determining the motion and period of the oscillation was suggested [6]. In the study, the results up to second round iteration were presented.

In this study, several numerical solution techniques for nonlinear eardrum-type oscillations are suggested. Earlier, it was suggested that one could evaluate the eigenvalues of the ordinary differential equation (ODE) by the iteration of solving the ODE [13–15]. Recently, a similar idea was developed, and the target function method for evaluating the vibration frequency in the nonlinear vibration was suggested [4]. It is found that the idea of target function method is a general one, which can also be used to the present analysis.

The idea of target function method can be described as follows. Assume that the ODE for the nonlinear eardrum vibration with an initial condition ($u = A$, $du/dt = 0$ at the time $t = 0$) is integrated in the interval $(0, t_p)$. It is found that the function $v(t_p)$ is the mentioned target function, where the dependent variable v ($= du/dt$) is the velocity of motion. The second zero of the target function, denoted by $t_p = T_p$, will be the period of motion.

The multiple-parameter technique is also suggested in this paper. The mentioned multiple parameters are those undetermined values in the assumed solution. In the five-parameters technique, the motion is assumed as $u(t) = c_0 + c_1 \cos(\omega_p t) + c_2 \cos(2\omega_p t) + c_3 \cos(3\omega_p t)$, where ω_p, c_0, c_1, c_2, c_3 are the circular frequency of motion and Fourier coefficients, respectively. The five undetermined parameters ω_p, c_0, c_1, c_2, c_3 are determined by using the following conditions, for example, (a) the motion must have definite displacement and acceleration at the starting time $t = 0$, (b) any displacement–velocity pair (u, v) ($v = du/dt$) must be located on the motion trajectory of phase plane, etc. In the harmonic balance method, some equations are obtained from the substitution of the assumed motion in the governing equation. However, in the present approach, one equation is obtained from a substitution of the assumed motion in the trajectory equation on the phase plane (see Eq. (3) below). Meantime, two equations are obtained from the acceleration condition at the times $\omega_p t = 0$ and $\omega_p t = \pi$. In this sense, the multiple-parameters technique is not very the same as the harmonic balance method.

The direct integration technique is studied finally. Since the period of motion can be evaluated in a closed form, the direct integration technique can be used to obtain the motion of the problem. Numerical examples are provided to prove the efficiency of the three suggested techniques. It is found that all three techniques can give accurate results.

2. General analyses and the target function technique

In following analysis, the nonlinear eardrum oscillation is taken as an example. The oscillation is defined by [6,12]

$$\frac{d^2u}{dt^2} + \omega_o^2 u(1 + \varepsilon u) = 0, \tag{1}$$

where u is the displacement, ω_o is the circular frequency given beforehand, ε is a constant which may not be a small value. The imposed boundary conditions take the form

$$u|_{t=0} = A, \quad \left.\frac{du}{dt}\right|_{t=0} = 0, \tag{2}$$

where A is a positive value.

Multiplying both sides of Eq. (1) by $2du$, and making integration will yield

$$v^2 + G(u) = H, \tag{3}$$

where

$$v = \frac{du}{dt}, \tag{4}$$

$$G(u) = \omega_o^2\left(u^2 + \frac{2\varepsilon}{3}u^3\right), \quad H = G(u)|_{u=A} = \omega_o^2\left(A^2 + \frac{2\varepsilon}{3}A^3\right). \tag{5}$$

The motion trajectory in phase plane defined by Eq. (3) is plotted in Fig. 1. On the motion trajectory in phase plane, the point (u, v) may move to the left side, and one more solution with the form $(u, v) = (B, 0)$ may exist. This point (u, v) is indicated with notation Q_2 in Fig. 1. With this condition, from Eqs. (3) and (5) we have

$$B^2 + \frac{2\varepsilon}{3}B^3 = A^2 + \frac{2\varepsilon}{3}A^3. \tag{6}$$

From Eq. (6), we can obtain a solution for B as follows:

$$B = \frac{-(3 + 2\varepsilon A) + \sqrt{3(3 + 2\varepsilon A)(1 - 2\varepsilon A)}}{4\varepsilon}. \tag{7}$$

It is proved that, B is a negative value ($B<0$), and $-B>A$ is valid in general. From Eq. (7) we see that, since B is a real value, $1-2\varepsilon A$ must be positive. Thus, the following condition should be satisfied:

$$1 - 2\varepsilon A > 0 \quad \text{or} \quad 2\varepsilon A < 1. \tag{8}$$

If the circular frequency of motion is denoted by ω_p, from Eqs. (3) and (4) and the trajectory of motion (Fig. 1) we will find $\mathrm{d}t/\mathrm{d}u = -1/\sqrt{H - G(u)}$ (for $0 \leqslant \omega_p t \leqslant \pi$) and $\mathrm{d}t/\mathrm{d}u = 1/\sqrt{H - G(u)}$ (for $\pi \leqslant \omega_p t \leqslant 2\pi$). From this relation, we can obtain the period of the motion [1,16]

$$T_p = 2\int_B^A \frac{\mathrm{d}u}{\sqrt{H - G(u)}}. \tag{9}$$

The period of motion T_p and the circular frequency of motion ω_p can be expressed by the relation

$$T_p = \frac{2\pi}{\omega_p}, \quad \omega_p = \frac{2\pi}{T_p}. \tag{10}$$

For the harmonic motion case ($\varepsilon = 0$ in Eq. (1)), we have the period of motion T_o and the circular frequency of motion ω_o as follows:

$$T_o = \frac{2\pi}{\omega_o}, \quad \omega_o = \frac{2\pi}{T_o}. \tag{11}$$

Furthermore, a reduced (or magnified factor) for the circular frequency is defined by

$$\alpha = \frac{\omega_p}{\omega_o}. \tag{12}$$

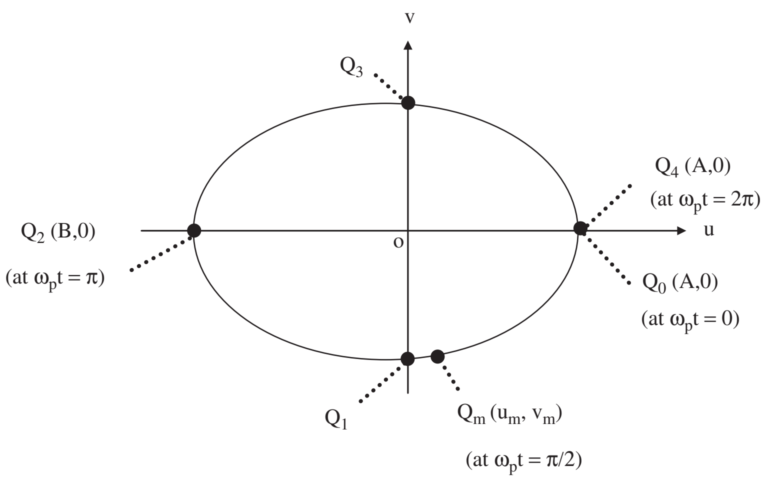

Fig. 1. The v versus u trajectory for solution of the eardrum oscillation on the phase plane.

From above-mentioned analysis we see that it is important to find the motion of the nonlinear eardrum oscillation.

In the small value of ε, the Lindstedt-Ponicare perturbation technique is used [1,2]. After some manipulation for Eq. (1) under condition (2), we have

$$\omega_p = \alpha\omega_o, \quad \alpha = 1 - \frac{5}{12}(\varepsilon A)^2, \tag{13}$$

$$u(t) = c_0 + c_1 \cos(\omega_p t) + c_2 \cos(2\omega_p t) + c_3 \cos(3\omega_p t), \tag{14}$$

where

$$c_0 = -\frac{\varepsilon A}{2}\left(1 + \frac{2\varepsilon A}{3}\right)A, \quad c_1 = \left(1 + \frac{1}{3}\varepsilon A + \frac{29}{144}(\varepsilon A)^2\right)A,$$
$$c_2 = \frac{\varepsilon A}{6}\left(1 + \frac{2\varepsilon A}{3}\right)A, \quad c_3 = \frac{(\varepsilon A)^2}{48}A. \tag{15}$$

The target function method is introduced as follows. For a given value t_p, we perform the integration for Eq. (1) with the initial boundary value condition (2) on the interval (0, t_p), and get the function $u(t)$ and $v(t)$ $(0 \leqslant t \leqslant t_p)$ and $v(t_p)$, where $v(t) = \mathrm{d}u/\mathrm{d}t$. The obtained $v(t_p)$ is called the target function in this paper. Obviously, the value of $v(t_p)$ is a function of the given time t_p, and it is not equal to zero in general. We can define the target function by $f(t_p) = v(t_p)$. The governing equation of the target function technique takes the form

$$f(t_p) = v(t_p) = 0. \tag{16}$$

Assume that $t_p = T_{p1}$ and $t_p = T_p$ are two successful zeros of the target function $v(t_p)$. Clearly, at the time $t_p = T_{p1}$ the point $(u(t_p)\ v(t_p))$ is just at the point Q_2 on the trajectory of the phase plane (Fig. 1). Meantime, at the time $t_p = T_p$ the point $(u(t_p)\ v(t_p))$ is just at the point Q_4, which in turn is the starting point of the motion Q_0 (Q_4 and Q_0 have the same position). The half-division technique is used to evaluate the second zero of the function $v(t_p)$, which is denoted by $t_p = T_p$ Note that $v(t_p)$ is the value of $\mathrm{d}u/\mathrm{d}t$ at the time $t = t_p$ from the solution of Eqs. (1) and (2). Therefore, it is necessary to perform the numerical integration for Eqs. (1) and (2). In this study, Runge–Kutta rule is used for the numerical integration [17]. After the value of T_p is obtained numerically, ω_p can be obtained immediately using Eq. (10).

A particular advantage of the suggested method is that one can get the motion of the oscillation in addition to the circular frequency ω_p. The obtained displacement may be expressed in the form:

$$u(t) = c_0 + \sum_{k=1}^{M} c_k \cos(k\omega_p t) \quad (0 \leqslant t \leqslant T_p,\ M\text{-integer}). \tag{17}$$

Clearly, the involved Fourier coefficients can be evaluated easily from the obtained displacement $u(t)$.

The reduced factor α and the Fourier coefficients c_0, c_1, c_2, c_3, c_4 depend on A and ε, and for the case $A = 0.45$ the computed results are listed in Table 1. In computation the $M = 100$ divisions is used in the Runge–Kutta method for the numerical solution of ODE [17]. From Table 1 we see that the target function technique gives very accurate results, which coincide with the exact solution. Meantime, a set of motions $u(t)$ is plotted in Fig. 2. From Fig. 2 we see that the influence of ε to the motion is significant. For example, for case of $A = 0.45$, $\varepsilon = 0.1$ we have $B = -0.464$ and $-B/A = -1.031$, and for case of $A = 0.45$, $\varepsilon = 1.0$ we have $B = -0.705$ and $-B/A = -1.567$.

In addition, if the perturbation technique is used, under same condition ($A = 0.45$) the results for the reduced factor α and the Fourier coefficients c_0, c_1, c_2, c_3 are listed in Table 2, which are obtained by using Eqs. (13)–(15). If the error tolerance for the reduced factor α is assumed to be $\delta < 1\%$, the perturbation technique is valid for $\varepsilon < 0.5$ in the case of $A = 0.45$.

Table 1
Computed results from the target function technique, α value and the calculated Fourier coefficients for the solution of the eardrum oscillation $d^2u/dt^2 + \omega_o^2 u(1 + \varepsilon u) = 0$ with the condition $u(0) = A = 0.45$ and $u'(0) = 0$ (see Eqs. (1), (2), (9) and (10))

ε	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
α_{ex}	0.9991	0.9964	0.9915	0.9842	0.9739	0.9598	0.9408	0.9146	0.8769	0.8153
α_{target}	0.9991	0.9964	0.9915	0.9842	0.9739	0.9598	0.9408	0.9146	0.8769	0.8153
β	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
c_0	−0.0105	−0.0217	−0.0338	−0.0472	−0.0622	−0.0793	−0.0995	−0.1244	−0.1574	−0.2084
c_1	0.4569	0.4643	0.4723	0.4809	0.4903	0.5009	0.5130	0.5273	0.5449	0.5687
c_2	0.0035	0.0072	0.0113	0.0159	0.0211	0.0272	0.0346	0.0442	0.0576	0.0803
c_3	0.0000	0.0001	0.0002	0.0004	0.0007	0.0011	0.0018	0.0028	0.0046	0.0085
c_4	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0001	0.0002	0.0003	0.0008

α_{ex}—reduced factor from the exact solution using Eqs. (9), (10) and (12). α_{target}—reduced factor from the target function technique. β—percentage error defined by $\beta = 100 \times (\alpha_{target} - \alpha_{ex})/\alpha_{ex}$, c_0, c_1, c_2, c_3, c_4—The Fourier coefficients in Eq. (17).

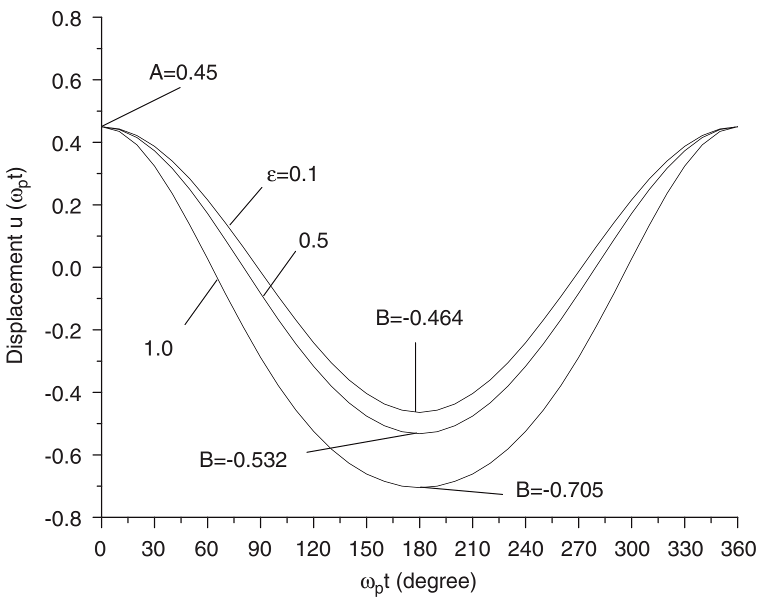

Fig. 2. Motion of the eardrum oscillation ($u(\omega_p t)$, $0 \leqslant \omega_p t \leqslant 2\pi$) in case of $A = 0.45$ and $\varepsilon = 0.1$, 0.5 and 1.0.

Table 2
Results from the perturbation technique, α value and the calculated Fourier coefficients for the solution of the eardrum oscillation $d^2u/dt^2 + \omega_o^2 u(1 + \varepsilon u) = 0$ with the condition $u(0) = A = 0.45$ and $u'(0) = 0$ (see Eqs. (13)–(15))

ε	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
α_{ex}	0.9991	0.9964	0.9915	0.9842	0.9739	0.9598	0.9408	0.9146	0.8769	0.8153
α_{pert}	0.9992	0.9966	0.9924	0.9865	0.9789	0.9696	0.9587	0.9460	0.9317	0.9156
β	0.0027	0.0240	0.0893	0.2354	0.5175	1.0228	1.8999	3.4302	6.2456	12.3036
c_0	−0.0104	−0.0215	−0.0331	−0.0454	−0.0582	−0.0717	−0.0858	−0.1004	−0.1157	−0.1316
c_1	0.4569	0.4642	0.4719	0.4799	0.4883	0.4971	0.5062	0.5157	0.5256	0.5359
c_2	0.0035	0.0072	0.0110	0.0151	0.0194	0.0239	0.0286	0.0335	0.0386	0.0439
c_3	0.0000	0.0001	0.0002	0.0003	0.0005	0.0007	0.0009	0.0012	0.0015	0.0019

α_{ex}—reduced factor from the exact solution using Eqs. (9), (10) and (12). α_{pert}—reduced factor from the perturbation technique shown by Eq. (13). β—percentage error defined by $\beta = 100 \times (\alpha_{pert} - \alpha_{ex})/\alpha_{ex}$, c_0, c_1, c_2, c_3—The Fourier coefficients in Eqs. (14) and (15).

3. The multiple-parameter technique and the direct integration technique

Below, the multiple-parameter technique is suggested to study the nonlinear eardrum oscillation. The multiple parameters are those undetermined values in the assumed solution. In the five-parameters technique, the motion is assumed as

$$u(t) = c_0 + c_1 \cos(\omega_p t) + c_2 \cos(2\omega_p t) + c_3 \cos(3\omega_p t), \tag{18}$$

where ω_p, c_0, c_1, c_2, c_3 stand for five undetermined parameters. These parameters can be evaluated from the general properties of motion. We can derive five conditions for the five parameters step by step.

From Eq. (18), we can obtain the velocity and the acceleration for the motion

$$\frac{\mathrm{d}u}{\mathrm{d}t} = -\omega_p(c_1 \sin(\omega_p t) + 2c_2 \sin(2\omega_p t) + 3c_3 \sin(3\omega_p t)), \tag{19}$$

$$\frac{\mathrm{d}^2u}{\mathrm{d}t^2} = -\omega_p^2(c_1 \cos(\omega_p t) + 4c_2 \cos(2\omega_p t) + 9c_3 \cos(3\omega_p t)). \tag{20}$$

In the first step, we consider the conditions at time $\omega_p t = 0$. From the first condition in Eqs. (2) and (18), we have

$$c_0 + c_1 + c_2 + c_3 = A. \tag{21}$$

Meantime, from Eqs. (1) and (2), we can obtain the acceleration at the time $\omega_p t = 0$

$$\frac{\mathrm{d}^2u}{\mathrm{d}t^2}|_{\omega_p t=0} = -\omega_o^2 A(1 + \varepsilon A). \tag{22}$$

Substituting Eq. (20) into the left-hand side of Eq. (22) yields

$$\omega_p^2(c_1 + 4c_2 + 9c_3) = \omega_o^2 A(1 + \varepsilon A). \tag{23}$$

Similarly, at the time $\omega_p t = \pi$, the (u, v) pair is just on the point Q_2 (Fig. 1). From above-mentioned analysis and Eqs. (1) and (2) we have

$$u|_{\omega_p t=\pi} = B, \tag{24}$$

$$\frac{\mathrm{d}^2u}{\mathrm{d}t^2}|_{\omega_p t=\pi} = -\omega_o^2 B(1 + \varepsilon B). \tag{25}$$

Substituting Eqs. (18) and (20) into Eqs. (24) and (25) yields

$$c_0 - c_1 + c_2 - c_3 = B, \tag{26}$$

$$\omega_p^2(-c_1 + 4c_2 - 9c_3) = \omega_o^2 B(1 + \varepsilon B). \tag{27}$$

We consider the condition at the time $\omega_p t = \pi/2$. In this time, the (u, v) pair is assumed on some point Q_m of the trajectory (Fig. 1). From Eqs. (18) and (19), the displacement and velocity at this time can be expressed as

$$u_m = u|_{\omega_p t=\pi/2} = c_0 - c_2, \quad v_m = \frac{\mathrm{d}u}{\mathrm{d}t}|_{\omega_p t=\pi/2} = -\omega_p(c_1 - 3c_3). \tag{28}$$

Clearly, the (u_m, v_m) must satisfy Eq. (3). Substituting Eq. (28) into Eq. (3) yields

$$\omega_p^2(c_1 - 3c_3)^2 + \omega_o^2(c_0 - c_2)^2\left(1 + \frac{2\varepsilon(c_o - c_2)}{3}\right) = \omega_o^2 A^2\left(1 + \frac{2\varepsilon A}{3}\right). \tag{29}$$

Five equations (21), (23), (26), (27) and (29) are formulated to solve the five undetermined parameters ω_p, c_0, c_1, c_2, c_3. It is preferable to write the unknowns in an alternative form

$$\beta = \alpha^2 = \left(\frac{\omega_p}{\omega_o}\right)^2, \quad g_0 = \frac{c_0}{A}, \quad g_1 = \frac{c_1}{A}, \quad g_2 = \frac{c_2}{A}, \quad g_3 = \frac{c_3}{A}. \tag{30}$$

By using these notations, Eqs. (21), (26), (23), (27) and (29) are reduced to

$$g_0 + g_1 + g_2 + g_3 = 1, \tag{31}$$

$$g_0 - g_1 + g_2 - g_3 = h_1, \tag{32}$$

$$\beta(g_1 + 4g_2 + 9g_3) = 1 + h_2, \tag{33}$$

$$\beta(-g_1 + 4g_2 - 9g_3) = h_1(1 + h_1h_2), \tag{34}$$

$$\beta(g_1 - 3g_3)^2 + (g_0 - g_2)^2\left(1 + \frac{2h_2(g_0 - g_2)}{3}\right) = 1 + \frac{2h_2}{3}, \tag{35}$$

where

$$h_1 = \frac{B}{A}, \quad h_2 = \varepsilon A. \tag{36}$$

Finally, five equations (31)–(35) are formulated to solve the five undetermined parameters β, g_0, g_1, g_2, g_3. Clearly, Eqs. (31)–(35) belong to a nonlinear algebraic equation with respect to five unknowns. Therefore, it is necessary to use iteration in computation. It is found that the relevant results from the perturbation technique shown by Eqs. (13)–(15) are the suitable values used in the first round of iteration. The iteration is convergent in general.

Similarly, in case of $A = 0.45$ the computed results for the reduced factor α and the Fourier coefficients c_0, c_1, c_2, c_3, are listed in Table 3. It is found from Table 3 that the computed results are also very accurate in the present technique. For example, for case of $A = 0.45$ and $\varepsilon = 1.0$, the relative error for the reduced factor α is -0.2618%.

If less terms are assumed in the motion, for example, letting

$$u(t) = c_0 + c_1 \cos(\omega_p t). \tag{37}$$

In this case, simply let $c_2 = c_3 = 0$ in Eqs. (21), (26) and (29), we obtain

$$c_0 = (A + B)/2, \quad c_1 = (A - B)/2,$$

$$\alpha = \omega_p/\omega_o = \frac{2}{A - B}\sqrt{A^2 - c_0^2 + \frac{2\varepsilon}{3}(A^3 - c_0^3)}. \tag{38}$$

Similarly, in case of $A = 0.45$ the computed results for the reduced factor α and the Fourier coefficients c_0, c_1, are listed in Table 4. It is found from Table 4 that the computed results have a deviation from the accurate results. For example, for case of $A = 0.45$ and $\varepsilon = 1.0$, the relative error for the reduced factor α is 5.8966%.

Note that, in both the target function and multiple-parameter techniques, all the unknowns including the circular frequency ω_p (or the reduced factor $\alpha = \omega_p/\omega_o$) and the Fourier coefficients c_i are obtained from the numerical solution.

Table 3
Computed results from the multiple-parameter technique, α value and the calculated Fourier coefficients for the solution of the eardrum oscillation $\mathrm{d}^2u/\mathrm{d}t^2 + \omega_o^2u(1 + \varepsilon u) = 0$ with the condition $u(0) = A = 0.45$ and $u'(0) = 0$ (see Eqs. (1), (2), (9) and (10))

ε	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
α_{ex}	0.9991	0.9964	0.9915	0.9842	0.9739	0.9598	0.9408	0.9146	0.8769	0.8153
α_{mult}	0.9991	0.9964	0.9915	0.9842	0.9738	0.9598	0.9407	0.9144	0.8762	0.8132
β	0.0000	0.0000	−0.0002	−0.0007	−0.0019	−0.0049	−0.0118	−0.0287	−0.0756	−0.2618
c_0	−0.0105	−0.0217	−0.0338	−0.0472	−0.0622	−0.0794	−0.0997	−0.1249	−0.1585	−0.2113
c_1	0.4569	0.4643	0.4723	0.4809	0.4903	0.5009	0.5130	0.5272	0.5447	0.5681
c_2	0.0035	0.0072	0.0114	0.0159	0.0212	0.0274	0.0349	0.0448	0.0590	0.0840
c_3	0.0000	0.0001	0.0002	0.0004	0.0007	0.0011	0.0018	0.0028	0.0048	0.0092

α_{ex}—reduced factor from the exact solution using Eqs. (9), (10) and (12). α_{mult}—reduced factor from the multiple-parameter technique. β—percentage error defined by $\beta = 100 \times (\alpha_{mult} - \alpha_{ex})/\alpha_{ex}$, c_0, c_1, c_2, c_3—The Fourier coefficients in Eq. (18).

Table 4
Computed results from the multiple-parameter technique using $u(t) = c_0 + c_1\cos(\omega_p t)$, α value and the Fourier coefficients for the solution of the eardrum oscillation $\mathrm{d}^2u/\mathrm{d}t^2 + \omega_o^2 u(1+\varepsilon u) = 0$ with the condition $u(0) = A = 0.45$ and $u'(0) = 0$ (see Eqs. (1), (2), (9) and (10))

ε	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
α_{ex}	0.9991	0.9964	0.9915	0.9842	0.9739	0.9598	0.9408	0.9146	0.8769	0.8153
α_{mult}	0.9993	0.9971	0.9932	0.9874	0.9793	0.9683	0.9536	0.9338	0.9060	0.8634
β	0.0203	0.0714	0.1750	0.3262	0.5516	0.8833	1.3591	2.0961	3.3236	5.8966
c_0	−0.0070	−0.0144	−0.0225	−0.0313	−0.0410	−0.0520	−0.0648	−0.0800	−0.0995	−0.1273
c_1	0.4570	0.4644	0.4725	0.4813	0.4910	0.5020	0.5148	0.5300	0.5495	0.5773

Table 5
Computed results from the direct integration technique, α value and the calculated Fourier coefficients for the solution of the eardrum oscillation $\mathrm{d}^2u/\mathrm{d}t^2 + \omega_o^2 u(1+\varepsilon u) = 0$ with the condition $u(0) = A = 0.45$ and $u'(0) = 0$ (see Eqs. (1), (2), (9) and (10))

ε	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
α_{ex}	0.9991	0.9964	0.9915	0.9842	0.9739	0.9598	0.9408	0.9146	0.8769	0.8153
c_0	−0.0105	−0.0217	−0.0338	−0.0472	−0.0622	−0.0793	−0.0995	−0.1244	−0.1574	−0.2084
c_1	0.4569	0.4643	0.4723	0.4809	0.4903	0.5009	0.5130	0.5273	0.5449	0.5687
c_2	0.0035	0.0072	0.0113	0.0159	0.0211	0.0272	0.0346	0.0442	0.0576	0.0803
c_3	0.0000	0.0001	0.0002	0.0004	0.0007	0.0011	0.0018	0.0028	0.0046	0.0085
c_4	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0001	0.0002	0.0003	0.0008

α_{ex}—reduced factor from the exact solution using Eqs. (9), (10) and (12), c_0, c_1, c_2, c_3, c_4—The Fourier coefficients in Eq. (17).

In addition to suggested numerical techniques, the direct integration technique is introduced below. In the technique, the circular frequency ω_p is obtained from Eqs. (9) and (10), and the motion is obtained by a numerical integration.

Making a substitution $\tau = \omega_p t$ to Eqs. (1) and (2), and considering $\alpha = \omega_p/\omega_o$, one will obtain

$$\frac{\mathrm{d}^2u}{\mathrm{d}\tau^2} + \frac{1}{\alpha^2}\,u(1+\varepsilon u) = 0, \tag{39}$$

$$u|_{\tau=0} = A, \quad \left.\frac{\mathrm{d}u}{\mathrm{d}t}\right|_{\tau=0} = 0. \tag{40}$$

Since $\omega_p t$ $(\tau = \omega_p t)$ is defined in the interval $0 \leqslant \omega_p t \leqslant 2\pi$, τ must be defined in the same integral $0 \leqslant \tau \leqslant 2\pi$. Simply making integration to Eq. (39) under condition (40) in the interval $0 \leqslant \tau \leqslant 2\pi$, one will find the motion immediately. The computed results for the Fourier coefficients are listed in Table 5. The results in Table 5 exactly coincide with those in Table 1.

4. Remarks

It is known that it is a rare case that an elasticity problem or a nonlinear vibration problem can be solved in a closed form. The advanced electronic computers were not available in early years. In this case, investigators had to pay attention to some solutions, which can be performed by hand, or very elementary computation. Meantime, even though some techniques could be designed in early years, researchers could not complete the solution because of some complicated computation.

This situation was changed after the advanced electronic computers were equipped. The present study mainly depends on the successful numerical solutions and computer computation. Particular advantages for the method are as follows. Since there is no difference in the numerical solution between the linear and nonlinear differential equations, the difficulty caused from the nonlinearity disappears if target function method is used. Also, a high accurate computation scheme is used, for example, 100 division is assumed in the numerical solution of ODE, the obtained result must be very near to the exact solution. Finally, all necessary information, including the motion of vibration and the period of motion, can be obtained from solution.

Acknowledgment

The project was supported by National Natural Science Foundation of China.

References

[1] A.H. Nayfeh, D.T. Mook, *Nonlinear Oscillations*, Wiley, New York, 1978.
[2] A.H. Nayfeh, *Introduction to Perturbation Techniques*, Wiley, New York, 1981.
[3] J. Guckenheimer, P. Holmes, *Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields*, Springer, New York, 1983.
[4] Y.Z. Chen, Solution of the Duffing equation by using target function method, *Journal of Sound and Vibration* 256 (2002) 573–578.
[5] J.H. He, Approximate solution of nonlinear differential equations with convolution product nonlinearities, *Computer Methods in Applied Mechanics and Engineering* 167 (1998) 69–73.
[6] P.S. Li, B.S. Wu, An iteration approach to nonlinear oscillations of conservative single-degree-of-freedom systems, *Acta Mechanica* 170 (2004) 69–75.
[7] R.E. Mickens, Comments on the method of harmonic-balance, *Journal of Sound and Vibration* 94 (1984) 456–460.
[8] R.E. Mickens, Integration procedure for determining approximate solutions to nonlinear oscillator equations, *Journal of Sound and Vibration* 116 (1987) 185–188.
[9] A.V. Rao, B.N. Rao, Some remarks on the harmonic balance method for mixed-parity non-linear oscillation, *Journal of Sound and Vibration* 170 (1994) 571–576.
[10] M.S. Sarma, B.N. Rao, A rational harmonic balance approximation for the Duffing equation of mixed parity, *Journal of Sound and Vibration* 207 (1997) 597–599.
[11] S.B. Tiwari, B.N. Rao, N.S. Swamy, K.S. Sai, H.R. Nataraja, Analytical study on a Duffing-harmonic oscillator, *Journal of Sound and Vibration* 285 (2005) 1217–1222.
[12] R.E. Mickens, *Oscillations in Planar Dynamics Systems*, World Scientific Publishing, Singapore, 1996.
[13] Y.Z. Chen, A numerical procedure for evaluating the plastic limit load of circular plate using Mises yield criterion, *Computers and Structures* 24 (1986) 821–822.
[14] Y.Z. Chen, J.R. Xie, Evaluation of natural frequencies of non-uniform beams by numerical integration, *Computers and Structures* 29 (1988) 693–697.
[15] S.R.R. Pillai, B.N. Rao, Improved solution for the non-linear vibration of simply supported rectangular cross-ply plates, *Journal of Sound and Vibration* 150 (1991) 517–519.
[16] B.N. Rao, S.R.R. Pillai, Exact solution of the equation of motion to obtain non-linear vibration characteristics of thin plate, *Journal of Sound and Vibration* 153 (1992) 168–170.
[17] F.B. Hildebland, *Introduction to Numerical Analysis*, McGraw-Hill, New York, 1974.