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Reduction of the stochastic finite element models using a robust dynamic condensation method

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Abstract

One of the ways in which models of structural dynamics can be improved is by taking the various uncertainties that exist into consideration. This would also increase the reliability of predicted calculation trends in these models. Here, an original, robust, multi-level dynamic condensation method of stochastic models is proposed. The first-level condensation is based on a strategy that combines the stochastic finite element method (SFEM) with the robust condensation model. It is based on a discretization technique of random fields that was established using the Karhunen–Loeve procedure. In addition, the use of dynamic condensation was aided by random residual static responses. The consequent loads are representative of local modifications per zone (or component) of the mechanical structure. For the second-level condensation use of the polynomial chaos (PC) approach allows the presence of uncertainties in the design parameters to be taken into account and, also, the variability of the response can be analysed in a less costly manner than by using the Monte Carlo method. Alternatively, a modal perturbation (MP) approach allows rapid synthesis of the random response. We show how either of these can be used to give an accurate prediction of the condensed model and a considerable reduction of the calculation costs. Two numerical examples are presented to illustrate the performance of the proposed method.

1. Introduction

During the design and development cycle of dynamically loaded structures or mechanical systems, one commonly uses models of varying complexity in an iterative manner, to predict the response and guide the design process. In particular, one can be effective to combine detailed models (such as thin meshing) that are used for the optimization of components with simplified models that are used for the development of specifications for these components. When these components are not defined in detail, and when some uncertainties exist regarding the predicted mechanical characteristics of these components, it is necessary to be

Abbrevations: CB, Craig and Bampton method; CBE, enriched Craig and Bampton method; CPU, central processing unit; Dof(s), degree(s) of freedom; FEM, finite elements model; MC, Monte Carlo simulation; MP, modal perturbation; PC, polynomial chaos; REF, reference complete model; SFEM, stochastic finite elements method; SVD, singular values decomposition

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Nomenclature

A	area
$\mathbf{c}$	deterministic generalized coordinates vector
$\mathbf{c}(\omega,\theta)$	stochastic generalized coordinates vector
$C(x_1, x_2)$	covariance function
E	Young modulus
EA	membrane effect
EI	flexion rigidity
$\mathbf{f}_e(\omega)$	applied harmonics forces vector
$\mathbf{f}_\Delta(\omega, \theta)$	localized stochastic forces vector
$\mathbf{F}_\Delta$	static loadings basis
$H(x, \theta)$	random field
i	number of internal dof
I	quadratic moment
j	number of junctions dof
m	number of master dof (to be conserved)
$\mathbf{M}_0$, $\mathbf{K}_0$	mean mass and stiffness matrices
$\mathbf{M}(\theta)$, $\mathbf{K}(\theta)$	stochastic mass and stiffness matrices
$\mathbf{R}$	static residual basis
$\mathbf{R}_\Delta$	random static residual basis
s	number of slave dof (to be eliminated)
$\mathbf{T}_0$	standard condensation basis
$\mathbf{T}$	robust condensation basis of the model
$\Delta\mathbf{T}$	correction basis composed by the random residuals vectors
$\mathbf{U}_0(\omega)$	deterministic response vector
$\mathbf{U}_0(\omega, \theta)$	random response vector
$\mathbf{Z}_0$, $\Delta\mathbf{Z}$	dynamic stiffness matrix and perturbation matrices

Greek letters

μ	mean value of the random field
$\xi_r(\theta)$	standard random variables
λ_r, f_r	eigenvalues and Eigenvectors of $C(x_1, x_2)$
$\omega_j(\theta)$, $Y_j(\theta)$	random eigenmodes
$\mathbf{\Psi}$	static sub-basis
$\mathbf{\Theta}$	dynamic sub-basis
$\Psi_n\left(\{\xi_r\}_{r=1}^q\right)$	multidimensional hermite polynomials
$\partial\mathbf{U}(\omega)/\partial\xi_r$	first-order sensitivity of the response vector
$\partial\omega_j/\partial\xi_r, \partial Y_j/\partial\xi_r$,	first-order sensitivity of the random eigenmodes
$\mathbf{I}$	identity matrix
$\bar{\mathbf{Y}}, \bar{\mathbf{\Lambda}}$	deterministic modal and spectral matrix

Superscripts

-1	inverse
T	transpose
c	index of the condensed model

Operators

$\lvert\bullet\rvert$	absolute value
$\lVert\bullet\rVert$	Euclidean norm

able to manipulate the calculations so as not to limit the number of explored solutions. The uncertainties are often linked to the geometric properties, material characteristics and boundary conditions of the model. These variables are taken into account in models according to two approaches:

The non-parametric approach [1], which introduces the uncertainties directly into the global matrices of the model or into the reduced matrices of the nominal condensed model [2,3]. The advantage of this approach lies in its numerical efficiency in terms of correctly predicting the responses and introducing the uncertainties of the model, without having recourse to the parametric reactualization of the stochastic model. Furthermore, this approach is placed upstream of an optimization procedure to give predicted stochastic solutions. Its main drawback is the difficulty in making a direct link with the uncertain physical parameters of the model.

Parametric approaches using the stochastic finite element method (SFEM), which combines classic analysis by finite elements and statistical analysis. Generally, the stochastic characteristics of the random responses can be determined from the knowledge of design parameter uncertainties. These uncertain parameters are often modelled by random variables. Various techniques exist to solve such problems [4–6]. These methods are generally classified into three categories:

- The Monte Carlo (MC) simulation method is often considered to be the reference method [7–9]. The MC simulation provides accurate solutions for the robust condensed model (first-level condensation); however, the disadvantage of this method is its high numerical cost and the quantities of calculations required.

- Perturbation methods have been widely used in the field of stochastic mechanics to estimate the response statistics. These methods are based on a Taylor series expansion [10,11] or a Neumann series expansion [12–14], both of which analyse responses according to the means of random variables.
- Spectral methods that use basic functions of the Hilbert space are associated with random problems. Generally, these functions can be considered as orthogonal polynomials and, in particular, as Polynomial Chaos (PC) [15–17]. In this case, random variables can be used; the continuous random fields are discretized.

These three categories of methods take into account the random variables from which the probability density is known. Some Gaussian variables are used in most cases [15].

If all these methods allow the calculation of some static solutions, the calculation of dynamic responses proves to be more delicate, especially for neighbouring resonances. The convergence of these methods is not assured; the matrix systems become either badly conditioned or even singular, especially for the Taylor series expansion or the Neumann series expansion methods. The Padé approximations [18], which are part of the convergence accelerations methods [19], can be used to increase the validity domain of the approximation based on the Taylor series expansion. The method of projection on the orthogonal polynomials is an interesting alternative, which makes it possible to avoid these problems. Recently, a modal perturbation (MP) approach [20] using some exact methods [21–23] was used to allow efficient calculation of the random eigenmodes and rapid synthesis of the random frequency response, thereby avoiding the bad conditioning of matrices around resonances. Mean values, standard deviations and extreme statistics can be obtained, which can give a good estimation of the solutions envelope.

Otherwise, using estimated calculations, finite elements models that are both stochastic and of large size need to be manipulated. With the aim of reducing the calculation costs and obtaining good predictivity of the model, an original, robust, multi-level dynamic condensation method of stochastic models is proposed. In the first-level condensation, the stochastic model is condensed by a robust reduction basis. The obtained condensed stochastic model will then be exploited in the calculation of the stochastic responses by using second-level condensation, which is based on two spectral strategies:

- the first is based on the MP approach;
- the second is based on the PC approach.

Note that this method allows the coupling of the SFEM method and a robust dynamic condensation method in the presence of perturbations or structural modifications [24], in view of constructing a robust, reduced random model. This method makes it possible to assign specific uncertainty levels to zones or components. Furthermore, with respect to parametric modifications, the robust condensation method allows the treatment of complex structures in a dynamic substructuring context.

The efficiency of the proposed methods will be illustrated through two numerical examples: the first simulation investigates the random responses variability of a rotor dynamic model with dispersions of three design parameters, and compares the MC and PC (of order 4) simulation methods for the reference and the reduction methods, respectively. The second simulation calculates the random response of planar frame structure with two uncertain parameters, and compares the following simulation methods to the reference model: MC, PC (of order 4) and perturbation methods.

2. General formulation of the stochastic dynamic model

2.1. Discretization of the random fields

A random field $H(x, \theta)$ is a collection of random variables that are indexed by a continuous parameter $x \in \Omega$, where Ω is a bounded subset of $\mathbb{R}^d$, thereby describing the geometry of the system (d is the dimension of x, that is $d = 1$ or $d > 1$). One procedure of discretization is based on the approximation of $H(\bullet)$ by $\hat{H}(\bullet)$, which is defined by means of a finite set of variables, $\{\chi_r, r = 1, \ldots, n\}$, which are grouped into a random vector: $\chi : \hat{H}(x, \theta) = F[x, \chi(\theta)]$.

This method allows the best approximation compared with some error estimators, one of which exploits a minimal number of random variables. The most efficient methods—called *Series Expansion Methods*—consist of coupling a series development of the random field and a spectral analysis, which aims to select the most important terms [15,25,26]. Therefore, in the case of the homogeneous Gaussian field:

$$\hat{H}(x,\theta) = \mu + \sum_{r=1}^{q} H_r(x)\xi_r(\theta), \tag{1}$$

where $\{\xi_r(\theta), r = 1,\ldots,q\}$ are the independent Gaussian standard normal variables and $\{H_r(x), r = 1,\ldots,q\}$ are the deterministic functions.

C(x_1, x_2) is the known covariance function that is associated with $H(\bullet)$, which is assumed to be bound, symmetrical and positive definite. The spectral decomposition of $C(x_1, x_2)$ can be written as:

$$C(x_1,x_2) = \sum_{r=1}^{\infty} \lambda_r f_r(x_1) f_r(x_2), \tag{2}$$

where (λ_r, f_r) represent, respectively, the eigenvalues and eigenvectors of $C(x_1, x_2)$. The decomposition of the Karhunen–Loeve of $H(\bullet)$ on the basis of the eigenfonctions $f_r(x)$ is given by:

$$\begin{aligned} &H(x,\theta) = \mu + \textstyle\sum_{r=1}^{\infty} \sqrt{\lambda_r} f_r(x)\xi_r(\theta), \\ \text{the trucated form}: \quad &\hat{H}(x,\theta) = \mu + \textstyle\sum_{r=1}^{q} \sqrt{\lambda_r} f_r(x)\xi_r(\theta). \end{aligned} \tag{3}$$

2.2. Stochastic finite elements model

Generally, the selected uncertain parameters are linked to the geometric properties (such as A, I), the material characteristics (E) and, therefore, the products (EA, EI) of the model. The stochastic mass matrix is obtained using assembly of the elementary matrices. The dispersions on the random variables (mass density, for example) are introduced so that the mass matrix will be positive definite:

$$\mathbf{M}(\theta) = \bigcup_e \left[\int_{\Omega_e} H(x,\theta) N^{\mathrm{T}} N \, \mathrm{d}\Omega_e \right]. \tag{4}$$

By replacing $H(x, \theta)$ by its expression (see Eq. (1)), this matrix becomes:

$$\mathbf{M}(\theta) \approx \mathbf{M}_0 + \sum_{r=1}^{q} \mathbf{M}_r \xi_r(\theta), \tag{5}$$

with

$$\mathbf{M}_0 = \bigcup_e \left[\int_{\Omega_e} \mu N^{\mathrm{T}} N \, \mathrm{d}\Omega_e \right], \quad \mathbf{M}_r = \bigcup_e \left[\int_{\Omega_e} H_r(x) N^{\mathrm{T}} N \, \mathrm{d}\Omega_e \right], \tag{6}$$

In the same way, the stochastic stiffness matrix is given by

$$\mathbf{K}(\theta) \approx \mathbf{K}_0 + \sum_{r=1}^{q} \mathbf{K}_r \xi_r(\theta), \tag{7}$$

with

$$\mathbf{K}_0 = \bigcup_e \left[\int_{\Omega_e} \mu B^T D B \, \mathrm{d}\Omega_e \right], \quad \mathbf{K}_r = \bigcup_e \left[\int_{\Omega_e} H_r(x) B^T D B \, \mathrm{d}\Omega_e \right], \tag{8}$$

where D is the matrix of elastic coefficients.

3. Proposed robust dynamic condensation method

3.1. Stochastic dynamic model per zone or per component

When a level of uncertainty can be attributed to zones or components of a structure, the stochastic equilibrium equation of the structure is submitted to a harmonic deterministic excitation, which is written in the following form:

$$\left(-\omega^2\mathbf{M}(\theta)+\mathrm{j}\omega\mathbf{B}(\theta)+\mathbf{K}(\theta)\right)\mathbf{U}(\omega,\theta)=f_e(\omega):j^2=1, \tag{9}$$

with $\mathbf{M}(\theta)$, $\mathbf{B}(\theta)$ and $\mathbf{K}(\theta)$ which are the stochastic mass, damping and stiffness matrices, respectively; $\mathbf{U}(\omega, \theta)$, which is the stochastic response vector of the model; and $\mathbf{f}_e(\omega)$, which is the applied harmonics forces vector. The stochastic damping $\mathbf{B}(\theta)$ is generally a linear combination of the stochastic mass and stiffness matrices. The stochastic damping that is used in this study is that of structural damping:

$$\mathbf{B}(\theta)=\left(\eta/\omega\right)\mathbf{K}(\theta). \tag{10}$$

Eq. (9) can be rewritten in the form:

$$\left(-\omega^2\mathbf{M}(\theta)+(1+\mathrm{j}\eta)\mathbf{K}(\theta)\right)\mathbf{U}(\omega,\theta)=\mathbf{f}_e(\omega). \tag{11}$$

Or as

$$[\mathbf{Z}_0(\omega)+\Delta\mathbf{Z}(\omega,\theta)]\mathbf{U}(\omega,\theta)=\mathbf{f}_e(\omega), \tag{12}$$

where $\mathbf{Z}_0(\omega)=\left(-\omega^2\mathbf{M}_0+(1+\mathrm{j}\eta)\mathbf{K}_0\right)$ is the mean dynamic stiffness matrix and $\Delta\mathbf{Z}(\omega,\theta)=\sum_{r=1}^{q}\left(-\omega^2\mathbf{M}_r+(1+\mathrm{j}\eta)\mathbf{K}_r\right)\xi_r$ is the stochastic dynamic stiffness matrix.

Eq. (12) can be rewritten in the following form:

$$\mathbf{Z}_0(\omega)\mathbf{U}(\omega,\theta)=\mathbf{f}_\Delta(\omega,\theta)+\mathbf{f}_e(\omega). \tag{13}$$

$\mathbf{f}_\Delta(\omega,\theta)=-\Delta\mathbf{Z}(\omega,\theta)\mathbf{U}(\omega,\theta)$ is the random forces vector that is associated with unknown modifications of the initial structure.

To calculate the stochastic response of the structure in presence of uncertain parameters, the proposed method in this paper exploits a concept which is similar to that introduced by Adhikari and Manohar [27] in view to construct a robust basis against the uncertainties on the random parameters. Nevertheless, the proposed method presents some advantages:

- The exploitation of a random static residual reduced basis. Consequently, the enriched condensation basis allows the reduction in a drastic manner of the stochastic finite elements model with a great size.
- It is known that in the case of important structural modifications, the perturbation methods or the methods based on Taylor series expansion must be exploited at high order, which can lead to high calculation cost. The proposed method which is based on the robust extended basis is able to treat, in addition of the uncertainties, these important structural modifications without significant additional calculations costs. This property is particularly interesting in the optimization procedure where the predictivity of the reduction basis is essential. In fact, such robust basis allows a fast approached and economic reanalysis in terms of calculations, without updating of the nominal model basis. As same, this robust extended basis concept can be exploited in the frequency or time domains of the dynamic analysis of finite elements models with a great size in presence of localized nonlinearities.

The relationship (Eq. (13)) is interpreted as a dynamic equilibrium equation of the initial deterministic model that is submitted to solicitations $\mathbf{f}_\Delta(\omega,\theta)$. Hence, the force basis that is associated with stochastic modifications is generalized by the following approximation:

$$\mathbf{f}_\Delta(\omega,\theta)=-\Delta\mathbf{Z}(\omega,\theta)\mathbf{U}(\omega,\theta)\approx-\Delta\mathbf{Z}(\omega,\theta)\mathbf{U}_0(\omega), \tag{14}$$

where the deterministic response $\mathbf{U}_0(\omega)$ is known.

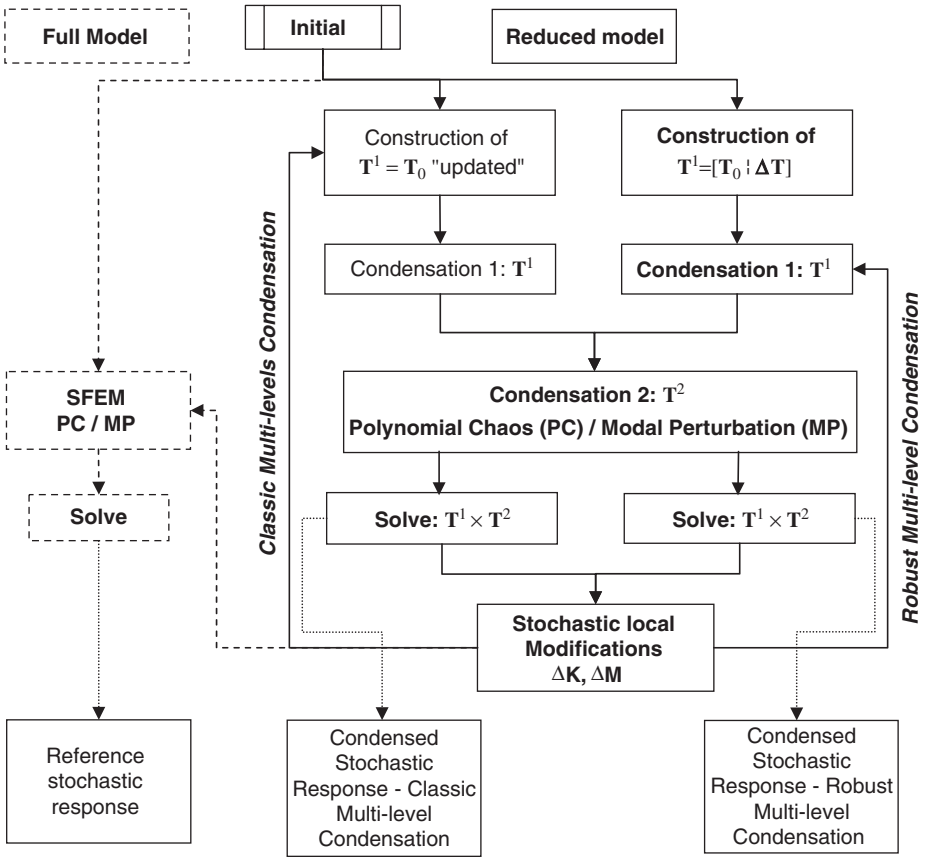

Fig. 1. Principle of the robust condensation.

For each stochastic zone (i), a force sub-basis $\mathbf{F}_{\Delta i}$ can be defined from the initial modal properties ($\bar{\mathbf{Y}}, \bar{\mathbf{\Lambda}}$: deterministic eigensolutions) and the stiffness and mass matrices of the stochastic zones $\mathbf{K}_i(\theta)$ and $\mathbf{M}_i(\theta)$:

$$\mathbf{F}_{\Delta i}(\theta) = \left[\mathbf{F}^K_{\Delta i} \,\vdots\, \mathbf{F}^M_{\Delta i}\right] = \left[\sum_{r=1}^{q} \mathbf{K}^i_r \,\xi_r(\theta)\,\overline{\mathbf{Y}} \,\vdots\, \sum_{r=1}^{q} \mathbf{M}^i_r \,\xi_r(\theta)\,\overline{\mathbf{Y}}\,\overline{\Lambda}\right] \tag{15}$$

The representative force basis of the stochastic modifications group by the concatenation of the sub-basis $\mathbf{F}_{\Delta i}(\theta)$, followed by singular values decomposition (SVD), leads to linear independence of the columns of this basis. From this, the random static vectors can be constructed:

$$\mathbf{R}_\Delta(\theta) = \mathbf{R}\mathbf{F}_\Delta(\theta), \tag{16}$$

with $\mathbf{R} = \mathbf{K}_0^{-1} - \bar{\mathbf{Y}}\bar{\mathbf{\Lambda}}^{-1}\bar{\mathbf{Y}}^{\mathrm{T}}$ being the static residual matrix of the nominal model.

For structures with rigid body modes, the matrix $\mathbf{K}_0$ is singular and the method outlined in Ref. [28] can be used.

In practice, the resolution of the problem (13) using the MC simulation approach leads to high numerical costs. The condensation of this model by a standard reduction method is proving insufficient in terms of robustness towards parametric perturbations. Therefore, it is proposed that a dynamic condensation method can be exploited by adapting it to stochastic models (Fig. 1).

The dynamic response of the perturbed system (Eq. (13)) can be expressed by the nominal condensation basis $\mathbf{T}_0$ that is obtained from the mean model enriched with a static residual $\mathbf{R}$, such that:

$$\mathbf{U}(\omega,\theta) \simeq \mathbf{T}_0\mathbf{c}(\omega,\theta) + \mathbf{R}\mathbf{f}_\Delta(\omega,\theta). \tag{17}$$

The reduction basis $\mathbf{T}$, which is common to both the initial and the perturbed systems, is constructed by the nominal condensation basis $\mathbf{T}_0$ and the static displacements $\mathbf{R}\mathbf{f}_\Delta$, which are associated with a set of static loads $\mathbf{F}_\Delta$ that are representative of the potential perturbations $\Delta\mathbf{Z}(\omega,\theta)$:

$$\mathbf{T} = \left[\mathbf{T}_0 \;\vdots\; \Delta\mathbf{T}\right];\; \Delta\mathbf{T} = \mathbf{R}\,\mathbf{f}_\Delta \tag{18}$$

where $\mathbf{T}_0$ is the nominal reduction basis, $\Delta\mathbf{T}$ is the correction basis due to the stochastic terms $\Delta\mathbf{Z}$ and $\mathbf{T}$ is the robust extended basis.

In the practice, the nominal reduction basis $\mathbf{T}_0$ can be, for example, a Ritz basis of fixed or free normal modes of component mode synthesis method [29,30] or simply a truncated modal basis in the direct dynamic condensation.

The first statistical moments (mean and standard deviation) of the random response are expressed by:

$$\begin{aligned} \mathrm{E}(\mathbf{U}(\omega,\theta)) &= \frac{1}{N_{\mathrm{samp}}}\sum_{p=1}^{N_{\mathrm{samp}}}\mathbf{U}\left(\omega,\theta_p\right),\\ \sigma_{\mathbf{U}(\omega,\theta)} = [\mathrm{var}(\mathbf{U}(\omega,\theta))]^{1/2} &= \left[\frac{1}{N_{\mathrm{samp}}}\sum_{p=1}^{N_{\mathrm{samp}}}\left(\mathbf{U}\left(\omega,\theta_p\right)-\mathrm{E}(\mathbf{U}(\omega,\theta))\right)^2\right]^{1/2} \end{aligned} \tag{19}$$

The extreme statistics of the random response is introduced by

$$\begin{aligned} \mathbf{U}^{\max}(\omega,\theta) &= \max_{p=1,\ldots,N_{\mathrm{samp}}}\mathbf{U}\left(\omega,\theta_p\right),\\ \mathbf{U}^{\min}(\omega,\theta) &= \min_{p=1,\ldots,N_{\mathrm{samp}}}\mathbf{U}\left(\omega,\theta_p\right), \end{aligned} \tag{20}$$

with E(•) and var(•) which are the expectation and variance operators, respectively; N_{samp}, which is the number of samples $\left(\theta=\theta_1,\ldots,\theta_{N_{\mathrm{samp}}}\right)$.

3.2. First-level condensation of the stochastic model

The stochastic model (Eq. (13)) condensed by using the nominal reduction basis $\mathbf{T}_0$ is given, in the frequency domain, by the following form:

$$\mathbf{Z}_0^c(\omega)\mathbf{U}^c(\omega,\theta) = \mathbf{f}_\Delta^c(\omega,\theta) + \mathbf{f}_e^c(\omega), \tag{21}$$

where $\mathbf{U}^c(\omega,\theta) = \mathbf{T}_0\mathbf{U}(\omega,\theta)$ is the reduced stochastic vector; $\mathbf{Z}_0^c(\omega) = \mathbf{T}_0^{\mathrm{T}}\mathbf{Z}_0(\omega)\mathbf{T}_0 = \left(-\omega^2\mathbf{M}_0^c + (1+\mathrm{j}\eta)\mathbf{K}_0^c\right)$ is the mean condensed dynamic stiffness matrix; $\mathbf{f}_\Delta^c(\omega,\theta) = \mathbf{T}_0^{\mathrm{T}}\mathbf{f}_\Delta(\omega,\theta) = -\Delta\mathbf{Z}^c(\omega,\theta)\mathbf{U}^c(\omega,\theta)$ is the condensed stochastic forces vector that is associated with unknown reduced modifications $\Delta\mathbf{Z}^c(\omega,\theta) = \mathbf{T}_0^{\mathrm{T}}\Delta\mathbf{Z}(\omega,\theta)\mathbf{T}_0$ of the initial structure . When the approximation introduced in Eq. (14) is adopted, the reduced force basis that is associated with reduced stochastic modifications can be approximated by $\mathbf{f}_\Delta^c(\omega,\theta) \approx -\Delta\mathbf{Z}^c(\omega,\theta)\mathbf{U}^c(\omega)$; where the reduced deterministic response $\mathbf{U}_0^c(\omega) = \mathbf{T}_0\mathbf{U}(\omega)$ is known; and $\mathbf{f}_e^c(\omega) = \mathrm{T}_0^T\mathbf{f}_e(\omega)$ and is the condensed vector of the applied forces.

The transformation matrix that is proposed in Ref. [31] is used to reduce the model in the first level. It consists of an extension of the passage procedure back in physical coordinates for the current basis of model condensation. To simplify the presentation and without restricting the generality of the method, the study will be limited to fixed and free-component mode synthesis methods. For these two configurations, the nominal Ritz basis $\mathbf{T}_0$ can be expressed in the following form:

$$\mathbf{U} = \begin{Bmatrix}\mathbf{U}_j\\ \mathbf{U}_i\end{Bmatrix} = \begin{bmatrix}\mathbf{I}_j & \mathbf{0}\\ \mathbf{\Psi} & \mathbf{\Theta}\end{bmatrix}\begin{Bmatrix}\mathbf{U}_j\\ \mathbf{c}\end{Bmatrix}, \tag{22}$$

where $c\in\mathbb{R}^{p,1}, \mathbf{\Psi}\in\mathbb{R}^{i,j}, \mathbf{\Theta}\in\mathbb{R}^{i,p}$ are the generalized coordinates vector and the static and dynamic basis, respectively, which are a function of the interface configurations type [29,30].

In the case of the Craig and Bampton (CB) method, the static basis $\mathbf{\Psi} = -\mathbf{K}_{0ii}^{-1}\mathbf{K}_{0ij}$ represents the constraints modes and the dynamic basis $\mathbf{\Theta}$ corresponds to the p first normal modes at blocked interfaces.

The physical coordinates transformation consists of eliminating the modal coordinates $\mathbf{c}$ in the transformation (22). The internal coordinates vector $\mathbf{U}_i$ is decomposed in a sub-vector with coordinates that are conserved $\mathbf{U}_i^m$ and a sub-vector with coordinates that are eliminated $\mathbf{U}_i^s$.

Therefore, the second set of the row of the Eq. (22) is given by:

$$\mathbf{U}_i = \begin{Bmatrix} \mathbf{U}_i^m \\ \mathbf{U}_i^s \end{Bmatrix} = \begin{bmatrix} \boldsymbol{\Psi}_m & \boldsymbol{\Theta}_m \\ \boldsymbol{\Psi}_s & \boldsymbol{\Theta}_s \end{bmatrix} \begin{Bmatrix} \mathbf{U}_j \\ \mathbf{c} \end{Bmatrix}. \tag{23}$$

The choice of the internal coordinates $\mathbf{U}_i^m$ is effected in such a way that the squared matrix $\boldsymbol{\Theta}_m \in \mathbb{R}^{m,m}$ will be regular ($m = p$, number of normal modes) and conducts to the modal coordinates vector $\mathbf{c}$ by the relationship:

$$\mathbf{c} = \boldsymbol{\Theta}_m^{-1}\mathbf{U}_i^m - \boldsymbol{\Theta}_m^{-1}\boldsymbol{\Psi}_m\mathbf{U}_j. \tag{24}$$

The transformation (see Eq. (22)) that is expressed in physical coordinates takes the following form:

$$\mathbf{U} = \begin{Bmatrix} \mathbf{U}_j \\ \mathbf{U}_i^m \\ \mathbf{U}_i^s \end{Bmatrix} = \begin{bmatrix} \mathbf{I}_j & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_{i_m} \\ \boldsymbol{\Psi}_s - \boldsymbol{\Theta}_s\boldsymbol{\Theta}_m^{-1}\boldsymbol{\Psi}_m & \boldsymbol{\Theta}_s\boldsymbol{\Theta}_m^{-1} \end{bmatrix} \begin{Bmatrix} \mathbf{U}_j \\ \mathbf{U}_i^m \end{Bmatrix} = \mathbf{T}_0\mathbf{U}^c. \tag{25}$$

An alternative for making this reduction basis more robust and for making the stochastic dynamic models less costly is to extend the nominal transformation $\mathbf{T}_0$ in the form defined in the previous section (see Eq. (18)): $\mathbf{T} = [\mathbf{T}_0 \,\vdots\, \Delta\mathbf{T}]$.

The correction $\Delta\mathbf{T} = \mathrm{E}\{\mathbf{R}_\Delta(\theta)\}$ is the variation due to the stochastic terms $\Delta\mathbf{Z}$ (E is the first moment). It should be noted that when the uncertainty level of the uncertain parameters is weak, $\mathbf{R}_\Delta(\theta)$ is calculated for one sample.

In the case of the CB method, the robust extended basis will be written as

$$\mathbf{U} = \mathbf{T}\,\mathbf{U}^c = \left[\begin{array}{cc:c} \mathbf{I}_j & \mathbf{0} & \mathbf{0} \\ \boldsymbol{\Psi} & \boldsymbol{\Theta} & \mathbf{R}_\Delta \end{array}\right] \begin{Bmatrix} \mathbf{U}_j \\ \mathbf{c} \\ \mathbf{f}_\Delta \end{Bmatrix} \tag{26}$$

The condensed stochastic model at the first level allows calculation of the stochastic response of the model from Eq. (9) by using the enriched CB (CBE) method. In the predicted calculation, the case of weakly damped linear structure can be considered, which is submitted to the deterministic excitation $\mathbf{f}_e$.

The stochastic responses that are obtained by using the first-level condensation will be followed by a second-level condensation to obtain a rapid synthesis, a considerable reduction of the calculation costs and a good predictivity of the model.

3.3. Second-level condensation of the stochastic model

Two spectral strategies for second-level condensation can be proposed by exploiting the MP method or the projection on the PC method.

3.3.1. Second-level condensation by the mp method

The MP method is based on a Taylor series development of the model. The first-order decomposition of the response vector $\mathbf{U}(\omega, \theta)$ can be written as

$$\mathbf{U}(\omega, \theta) = \mathbf{U}_0(\omega) + \sum_{r=1}^{q} \frac{\partial\mathbf{U}(\omega)}{\partial\xi_r}\xi_r(\theta), \tag{27}$$

with $\mathbf{U}_0(\omega)$ being the deterministic response vector of the mean structure and $\partial\mathbf{U}(\omega)/\partial\xi_r$ being the first-order sensitivities response vector with respect to the random variables ξ_r.

The generalized displacements (of initial or condensed model) and their sensitivities are calculated for each excitation frequency by solving the following linear systems:

Zeroth order: (deterministic case):

$$\left(-\omega^2\mathbf{M}_0 + (1+\mathrm{j}\eta)\mathbf{K}_0\right)\mathbf{U}_0(\omega) = \mathbf{f}_e(\omega). \tag{28}$$

The response is $\mathbf{U}_0(\omega) = \mathbf{Z}_0^{-1}(\omega)\mathbf{f}_e(\omega)$, with $\mathbf{Z}_0(\omega) = \left(-\omega^2\mathbf{M}_0 + (1+\mathrm{j}\eta)\mathbf{K}_0\right)$.

First order ($r = 1, ..., q$):

$$\left(-\omega^2\mathbf{M}_0 + (1+\mathrm{j}\eta)\mathbf{K}_0\right)\frac{\partial\mathbf{U}(\omega)}{\partial\xi_r} = -\left(-\omega^2\frac{\partial\mathbf{M}}{\partial\xi_r} + (1+\mathrm{j}\eta)\frac{\partial K}{\partial\xi_r}\right)\mathbf{U}_0(\omega). \tag{29}$$

The sensitivity of the response is given by

$$\frac{\partial\mathbf{U}(\omega)}{\partial\xi_r} = -\mathbf{Z}_0^{-1}(\omega)\mathbf{Z}_r(\omega)\mathbf{U}_0(\omega) \text{ with } \mathbf{Z}_r(\omega) = \left(-\omega^2\frac{\partial M}{\partial\xi_r} + (1+\mathrm{j}\eta)\frac{\partial K}{\partial\xi_r}\right).$$

Therefore, the first-order Taylor series development of the random response vector $\mathbf{U}(\omega, \theta)$ can be obtained. This development is valid if the following condition is satisfied: $\left\|\mathbf{Z}_0^{-1}(\omega).\mathbf{Z}_r(\omega)\right\| \ll 1$.

Of note is the poor numerical conditioning of the matrix $\mathbf{Z}_0(\omega)$ (or $\mathbf{Z}_0^c(\omega)$ for the condensed model) around the resonances of the deterministic model, which affects the quality of the responses envelopes with this method.

To solve the problem of poor predictability of the stochastic responses around the resonances, the MP method can be used. First, the generalized eigenproblem $\mathbf{K}_0\bar{\mathbf{Y}}_j = \bar{\omega}_j^2\mathbf{M}_0\bar{\mathbf{Y}}_j$ is solved to identify the eigenmodes, eigenfrequencies $\bar{\omega}_j$ of the mean structure and their associated eigenvectors $\bar{\mathbf{Y}}_j (j = 1, \ldots, N)$.

To evaluate the random response $\mathbf{U}(\omega, \theta)$ of the model, the truncated random eigenmodes ($N' \ll N$) can be used:

$$\mathbf{U}(\omega, \theta) = \sum_{j=1}^{N'} \alpha_j(\omega, \theta)\mathbf{Y}_j(\theta). \tag{30}$$

The normal coordinates $\alpha_j(\omega, \theta)$ are expressed by

$$\alpha_j(\omega, \theta) = \frac{\left(\mathbf{Y}_j(\theta)\right)^{\mathrm{T}}\mathbf{f}_e(\omega)}{-\omega^2 + (1+\mathrm{j}\eta)\left(\omega_j(\theta)\right)^2}, \tag{31}$$

where

$$\omega_j(\theta) = \bar{\omega}_j + \sum_{r=1}^{q}\frac{\partial\omega_j}{\partial\xi_r}\xi_r(\theta); \quad \mathbf{Y}_j(\theta) = \bar{\mathbf{Y}}_j + \sum_{r=1}^{q}\frac{\partial\mathbf{Y}_j}{\partial\xi_r}\xi_r(\theta). \tag{32}$$

The calculation of first-order sensitivities can be obtained by using previous methods that were proposed by Adelman and Hoftka [21]; these methods are based on the classic approaches of Fox and Kapoor [23] and of Nelson [22]. The eigenvalues sensitivity is given by

$$\frac{\partial\lambda_j}{\partial\xi_r} = \bar{\mathbf{Y}}_j^T.\left(\frac{\partial\mathbf{K}}{\partial\xi_r} - \bar{\lambda}_j\frac{\partial\mathbf{M}}{\partial\xi_r}\right).\bar{\mathbf{Y}}_j \quad \text{and} \quad \partial\omega_j/\partial\xi_r = \left(1/2\bar{\omega}_j\right).\left(\partial\lambda_j/\partial\xi_r\right). \tag{33}$$

The eigenvectors sensitivity is expressed by the following relationship:

$$\frac{\partial\mathbf{Y}_j}{\partial\xi_r} = \mathbf{V} + c\bar{\mathbf{Y}}_j, \tag{34}$$

where $\mathbf{V}$ is the solution of a modified system that is obtained by applying the penalization method at the kth row and column of $\left(\mathbf{K}_0 - \bar{\lambda}_j\mathbf{M}_0\right)$ to obtain $\mathbf{V}_k = 0$:

$$\left(\mathbf{K}_0 - \bar{\lambda}_j\mathbf{M}_0\right)_{\mathrm{mod}}.\mathbf{V} = -\left(\frac{\partial\mathbf{K}}{\partial\xi_r} - \frac{\partial\lambda_j}{\partial\xi_r}\mathbf{M}_0 - \bar{\lambda}_j\frac{\partial\mathbf{M}}{\partial\xi_r}\right).\bar{\mathbf{Y}}_j. \tag{35}$$

The value of k is generally chosen in such a way that it equals the index of the element of $\bar{\mathbf{Y}}_j$.

The expression of the unknown scalar c is given by the following relationship:

$$c = -\left(\frac{1}{2}\bar{\mathbf{Y}}_j^{\mathrm{T}}\frac{\partial \mathbf{M}}{\partial \xi_r}\bar{\mathbf{Y}}_j + \bar{\mathbf{Y}}_j^{\mathrm{T}}\mathbf{M}_0\mathbf{V}\right). \tag{36}$$

The derivative of the mass and stiffness global matrices are, respectively, expressed by

$$\frac{\partial \mathbf{M}}{\partial \xi_r} = \bigcup_e \left[\int_{\Omega_e}\sum_{r=1}^{q} H_r(x)\mathbf{N}^{\mathrm{T}}\mathbf{N}\,\mathrm{d}\Omega_e\right]; \quad \frac{\partial \mathbf{K}}{\partial \xi_r} = \bigcup_e \left[\int_{\Omega_e}\sum_{r=1}^{q} H_r(x)\mathbf{B}^{\mathrm{T}}\mathbf{D}\mathbf{B}\,\mathrm{d}\Omega_e\right]. \tag{37}$$

This procedure is applied to the first-level condensed model.

3.3.2. Second-level condensation by projection on the pc method

The expansion of the random response $\mathbf{U}(\theta)$ in the PC method can be written in the following truncated form:

$$\mathbf{U}(\theta) = \sum_{n=0}^{P} u_n \Psi_n, \tag{38}$$

The SFEM method consists of representing each component $U^n(\theta)$ (random variable of unknown statistic law) by a polynomial development in standard normal random variables:

$$\mathbf{U}(\theta) = \sum_{n=0}^{P} \mathbf{u}_n \Psi_n\left(\left\{\xi_r(\theta)\right\}_{r=1}^{q}\right), \tag{39}$$

where $\left\{\xi_r(\theta), r = 1, \ldots, q\right\}$ are the variables that are used to discretize the random field describing the data, $\Psi_n\left(\left\{\xi_r(\theta)\right\}_{r=1}^{q}\right)$, are the multidimensional Hermite polynomials that are defined from a set of q random variables ξ_r; u_n and P, which are the coefficients and the development order of the expansion ($P = (q+p)!/(q! \times p!)$).

The expansion of the condensed displacement $\mathbf{U}^c(\omega, \theta)$ in the PC method can be written in the following form:

$$\mathbf{U}^c(\theta) = \sum_{n=0}^{P} \mathbf{u}_n^c \Psi_n\left(\left\{\xi_r(\theta)\right\}_{r=1}^{q}\right). \tag{40}$$

Similarly, the vector $\mathbf{f}_\Delta^c(\omega, \theta)$ can be written as

$$\mathbf{f}_\Delta^c(\omega, \theta) = \sum_{n=0}^{P} \Delta\mathbf{Z}^c\mathbf{u}_n^c \Psi_n\left(\left\{\xi_r(\theta)\right\}_{r=1}^{q}\right). \tag{41}$$

The insertion of the relationships (40, 41) in Eq. (21) of the condensed model by the transformation $\mathbf{T}$ leads to the following equation:

$$\sum_{n=0}^{P} \mathbf{Z}_0^c\mathbf{u}_n^c \Psi_n + \sum_{n=0}^{P}\sum_{r=1}^{q}\left(-\omega^2\mathbf{M}_r^c + (1+\mathrm{j}\eta)\mathbf{K}_r^c\right)\xi_r\mathbf{u}_n^c\Psi_n = \mathbf{f}_e^c(\omega). \tag{42}$$

The projection of Eq. (42) on the polynomials $\Psi_m (m = 0, \ldots, P)$ leads to the following linear system:

$$\sum_{n=0}^{P} \mathbf{Z}_0^c\mathbf{u}_n^c\langle\Psi_n\Psi_m\rangle + \sum_{n=0}^{P}\sum_{r=1}^{q}\left(-\omega^2\mathbf{M}_r^c + (1+\mathrm{j}\eta)\mathbf{K}_r^c\right)\mathbf{u}_n^c\left\langle\xi_r\Psi_n\Psi_m\right\rangle = \mathbf{f}_e^c\langle\Psi_m\rangle. \tag{43}$$

The relationship (see Eq. (43)) can be written in the following form:

$$\sum_{n=0}^{P} \mathbf{Z}_0^c\mathbf{u}_n^c\left\langle\Psi_n^2\right\rangle + \sum_{n=0}^{P}\sum_{r=1}^{q}\mathbf{Z}_r^c\mathbf{u}_n^c\left\langle\xi_r\Psi_n\Psi_m\right\rangle = \mathbf{f}_e^c\langle\Psi_m\rangle. \tag{44}$$

It is important to note that the vectors of $\langle \Psi_n^2 \rangle$ and $\langle \xi_r \Psi_n \Psi_m \rangle$ must be calculated only once and kept in memory for all the calculations using this method.

Eq. (44) is considered for $m = 0, \ldots, P$, and leads to a system of $(P+1)$ linear matrix equations, the solution of which corresponds to vectors $\mathbf{u}_n^c$.

$$(\mathbf{D} + \mathbf{A})\mathbf{U} = \mathbf{b}, \tag{45}$$

where $\mathbf{D}$ is a diagonal matrix by blocks and $\mathbf{A}$ is a hollow matrix, such that:

$$\mathbf{D}_{ii} = \mathbf{Z}_0^c \langle \Psi_i^2 \rangle; \quad \mathbf{A}_{ij} = \sum_{r=1}^{q} \mathbf{Z}_r^c \langle \xi_r \Psi_i \Psi_j \rangle \tag{46}$$

It is necessary to note that, because of the orthogonality of the polynomials, most of the expressions $\langle \xi_r \Psi_i \Psi_j \rangle$ are null. The vector $\mathbf{U}$ is built from the sub-vectors $\mathbf{u}_n^c$, and $\mathbf{b}$ contains only the sub-vector $\mathbf{f}_e^c$, in the case where $\mathbf{f}_e^c$ is not correlated to $\mathbf{Z}_r^c$.

$$\left(\begin{bmatrix} \mathbf{D}_{0,0} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{D}_{1,1} & & \vdots \\ & & \ddots & \\ \mathbf{0} & & & \mathbf{D}_{P-1,P-1} \end{bmatrix} + \begin{bmatrix} \mathbf{A}_{0,0} & \cdots & \cdots & \mathbf{A}_{0,P-1} \\ \vdots & & & \vdots \\ \vdots & & & \vdots \\ \mathbf{A}_{P-1,0} & \cdots & \cdots & \mathbf{A}_{P-1,P-1} \end{bmatrix} \right) \begin{bmatrix} \mathbf{U}_0 \\ \mathbf{U}_1 \\ \vdots \\ \mathbf{U}_{P-1} \end{bmatrix} = \begin{bmatrix} \mathbf{f}_e^c \\ \mathbf{0} \\ \vdots \\ \mathbf{0} \end{bmatrix}. \tag{47}$$

The first statistical moments (mean and standard deviation) can be found by

$$\langle \mathbf{U}_{ij} \rangle = \mathbf{u}_{ij}^0; \quad \sigma_{U_{ij}} = \sqrt{\sum_{n=1}^{P} \left(\mathbf{u}_{ij}^n \right)^2 \langle \Psi_n^2 \rangle}. \tag{48}$$

The projection of the PC method is considered as a model condensation step; a double model condensation ($\mathbf{T} \times \mathbf{T}_{\mathrm{PC}}$) can therefore be obtained, which allows the calculation of the first moments (mean, standard deviation and extreme statistics).

To calculate the condensed eigenmodes of the model associated with Eq. (43), the technique described in Ref. [32] can be exploited to identify the problem of the orthogonality conditions of the random eigenmodes with regards to the random stiffness and mass matrices.

In the simulation examples, the random eigenmodes that are calculated by the condensed model are compared to the reference random eigenmodes that result from the complete model by the following criterion:

$$\varepsilon_f\% = \frac{\left| f_j^{\mathrm{ref}} - f_j^{\mathrm{cal}} \right|}{\left| f_j^{\mathrm{ref}} \right|} \times 100, \quad \varepsilon_U\% = \frac{\left\| \mathbf{Y}_j^{\mathrm{ref}} - \mathbf{Y}_j^{\mathrm{cal}} \right\|}{\left\| \mathbf{Y}_j^{\mathrm{ref}} \right\|} \times 100. \tag{49}$$

4. Numerical examples

4.1. SFEM of a rotor

For the first simulation, the proposed method will be illustrated in the case of component mode synthesis of a SFEM. The presented example (Fig. 2) concerns a rotor structure [33], which is modelled as follows:

- The axis modelled by a beam in flexion in the two planes (yz) and (yx) and discretized by a two-dimensional beam element (four dofs per node: $U_x, U_z, \theta_z, \theta_x$). The model uses 13 finite elements, 14 nodes and 56 dofs that are distributed in 32 internal dofs of SS1, 20 internal dofs of SS2 and four junction dofs. The mechanical and geometric characteristics of the axis are given in Table 1.
- The disks localized on nodes 3, 6 and 11 and characterized by the following properties (Table 2).
- The bearings are modelled by identical stiffness and damping localized on nodes 1 and 14 in the two flexion planes, according to the translation dofs (Table 3).

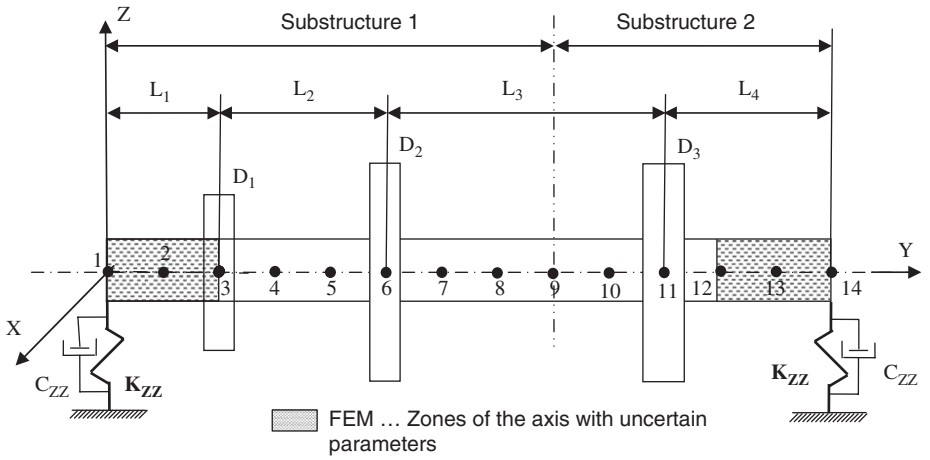

Fig. 2. FEM of the rotor.

Table 1
Mechanical and geometric characteristics of the axis

Mechanical characteristics	Geometrical characteristics
$E_0 = 2.1 \times 10^{11}\,\mathrm{N/m^2}$; $\rho_0 = 7800\,\mathrm{kg/m^3}$; $v = 0.3$	Axle radius (m) $R = 0.05$ Length between disks (m) : $L_1 = 0.2$; $L_2 = 0.3$; $L_3 = 0.5$; $L_4 = 0.3$

Table 2
Mechanical and geometric characteristics of the disks

Mechanical characteristics	Geometrical characteristics			
	Disks data	D_1	D_2	D_3
$E_0 = 2.1 \times 10^{11}\,\mathrm{N/m^2}$; $\rho_0 = 7800\,\mathrm{kg/m^3}$	Thickness (m)	0.05	0.05	0.06
	Inner radius (m)	0.05	0.05	0.05
	Outer radius (m)	0.12	0.20	0.20

Table 3
Bearing characteristics

Stiffness (N/m)			Damping (N/m/s)		
$K_{XX} = 5 \times 10^7$	$K_{ZZ} = 7 \times 10^7$	$K_{XZ} = K_{ZX} = 0$	$C_{XX} = 5 \times 10^2$	$C_{ZZ} = 7 \times 10^2$	$C_{XZ} = C_{ZX} = 0$

The rotor contains two stochastic zones: both ends of the beam axis and the two bearings. The Young modulus of the beam axis and the stiffness of the bearings constitute the uncertain parameters of these zones.

The corresponding dynamic problem can be written in the following form:

$$\left[\left(-\Omega^2\mathbf{M}_0 + \mathrm{j}\Omega(\mathbf{B}_0 + \mathbf{B}_G) + \mathbf{K}_0\right) + \Delta\mathbf{Z}(\Omega,\theta)\right]\mathbf{U}(\Omega,\theta) = \mathbf{f}_e(\Omega). \tag{50}$$

where $\mathbf{M}_0$, $\mathbf{K}_0$ and $\mathbf{B}_0$ are the mean assembly mass matrix, the stiffness matrix and the damping matrix, respectively. $\mathbf{B}_\mathrm{G}$ is the gyroscopic matrix function of the speed of rotation Ω (rad/s) of the axis, and $\Delta\mathbf{Z}(\Omega,\theta) = \sum_{r=1}^{q}\mathbf{K}_r\xi_r$ is the stochastic dynamic stiffness matrix.

The unbalance force, which is localized at node 6, is considered with the amplitude $F_b = m_b \times d \times \Omega^2 = 0.2 \times 10^{-3} \times \Omega^2$.

The observation point is localized at node 6, according to the translation dof U_x. The unbalance response is calculated for the rotation speed $\Omega \in [0 - 30\,000]$ rev/min.When the CBE method is applied, the problem of 56 dofs that was seen when using the SFEM method, is reduced to 14 dofs.

Figs. 3–6 illustrate the evolution of the mean and the standard deviation of the unbalance random response, as a function of the rotor rotation speed. The dispersions of the parameter E, K_{XX} and K_{ZZ} are $\delta_E = 5\%$; $\delta_{K_{XX}} = \delta_{K_{ZZ}} = 5\%$.

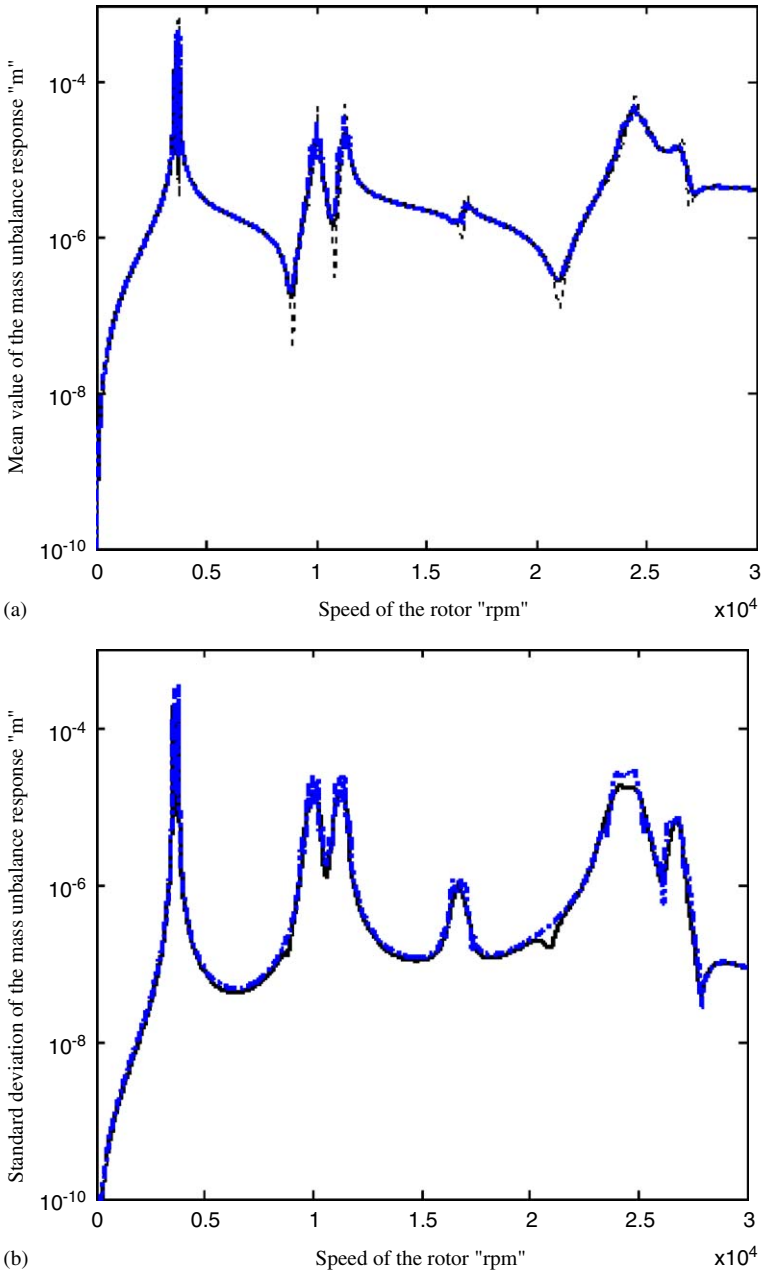

Fig. 3. Mean (a) and standard deviation (b) o the random response. PC order 4 and MC simulation, reference model. Keys: ------, Deterministic; ——, MC; —·—, PC order 4.

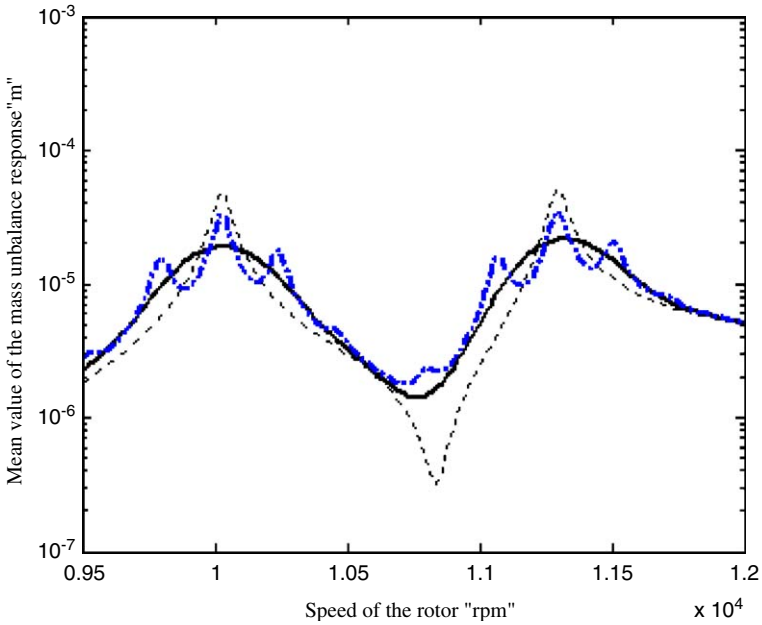

Fig. 4. Zoom of the mean random response. PC order 4 and MC simulation, reference model. Same keys of Fig. 3.

In Figs. 3 and 4, the results of the PC method (with order 4) for 5000 samples are compared with those of the MC method for 1000 samples. The examination of these results shows that the projection on the PC method allows the prediction of the reference solution obtained by using the MC method.

Fig. 5 shows the prediction quality of the reduced model when CBE is used. Fig. 6 illustrates the close relationship between the mean and the standard deviation plots of the responses that were obtained by the projection on the PC (of order 4) for 5000 samples and by the MC method for 5000 samples for the CBE model.

The numerical performances, as shown by the different CPU times that are summarized in Table 4, show that the proposed method is based on a double condensation, standard reduction basis, as physical coordinates are enriched by vectors of the stochastic modification (CBE method) and projection on the PC.

4.2. SFEM of a frame structure

In the SFEM, the proposed method is illustrated in the context of the dynamic substructuring with two uncertain parameters and two levels of uncertainties. The proposed example concerns a frame structure (Fig. 7), which is discretized by a two-dimensional beam element (three dofs per node: U_x, U_y, θ_z). The FEM contains 162 dofs that are distributed in 57 internal dofs of SS_1, 102 internal dofs of SS_2 and 3 junction dofs. The mechanical and geometrical characteristics are given by: $b = 5 \times 10^{-3}$ m; $h = 10^{-2}$ m; Area $= b \times h$; $E_0 = 2.1 \times 10^{11}$ N/m^2; $\rho_0 = 7800$ kg/m^3; $\nu = 0.3$.

The dynamic analysis is realized in the frequency band 0–700 Hz, including the first 10 global eigenmodes. The structure is submitted to a localized excitation force at the node N_f, according to the dof U_x. The observation point is considered at node N_0, according to the direction U_x.

The structure can therefore be thought of as two uncertain parameters per zone (modulus of elasticity at the foot of the vertical beams and the flexion rigidity throughout the horizontal beam). Note that uncertainties are introduced in each zone. In addition, the SFEM is applied only in these zones and the rest of the structure is deterministic. The uncertainties on the flexion rigidity (EI) are introduced by considering a decoupling of the membrane effects ($\mathbf{K}_m$) and the flexion effects ($\mathbf{K}_f$) of the stiffness matrix of a modified zone: $\mathbf{K}^{\text{zone}} = EA\mathbf{K}_m^{\text{zone}} + EI\mathbf{K}_f^{\text{zone}}$.

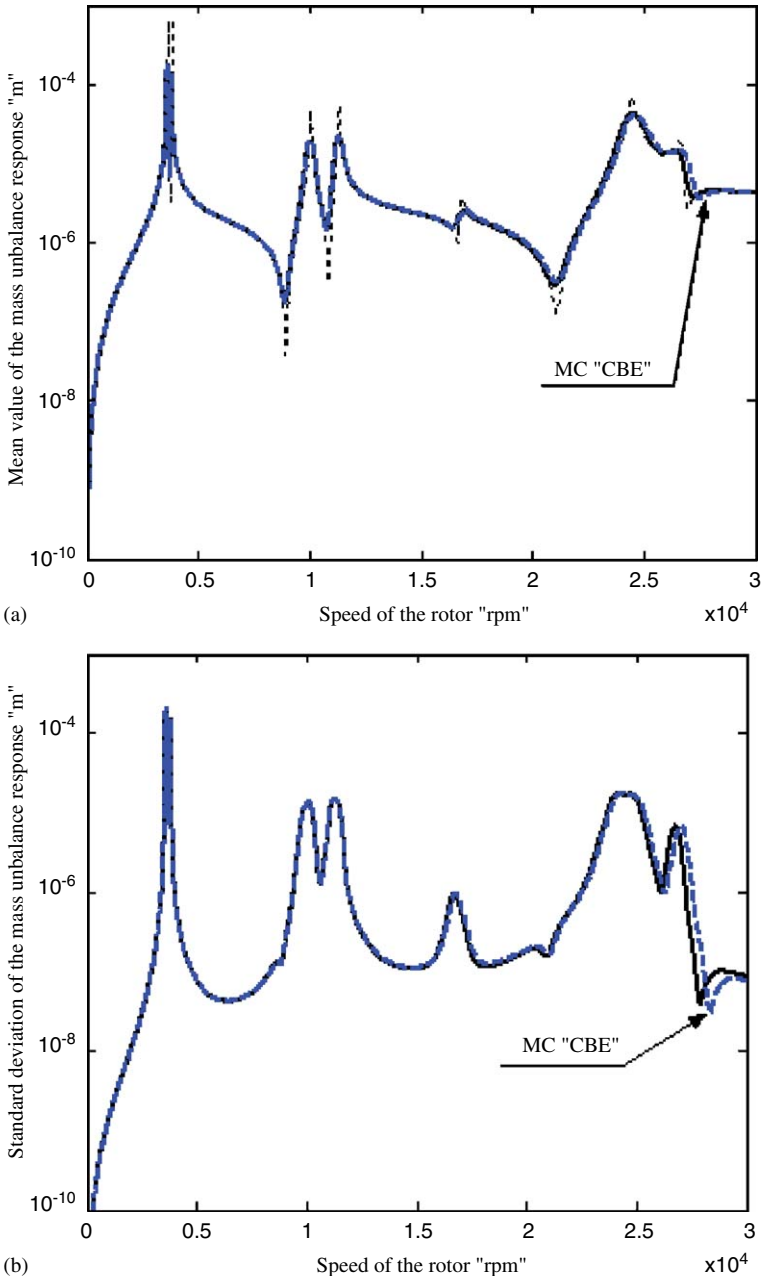

Fig. 5. Mean (a) and standard deviation (b) of the random response. MC simulation, reference and condensed model. Keys: ------, Deterministic; ——, REF model; — —, CBE model.

The CB method is first applied and the initial model with 162 dofs is reduced to a condensed one with 21 dofs (six dofs for SS_1, 12 dofs for SS_2 and three junction dofs). The CB method is then enriched by nine random static residual vectors, resulting in a new robust condensed model with 30 dofs (CBE). This can be compared to the data in Table 5, in which the first 10 random eigenmodes are calculated by the two substructuring methods (CB and CBE) and those of the reference model. It is shown that the CBE method gives good accuracy when predicting the first 10 eigenmodes, in comparison with the standard CB method, which gives more limited prediction level.

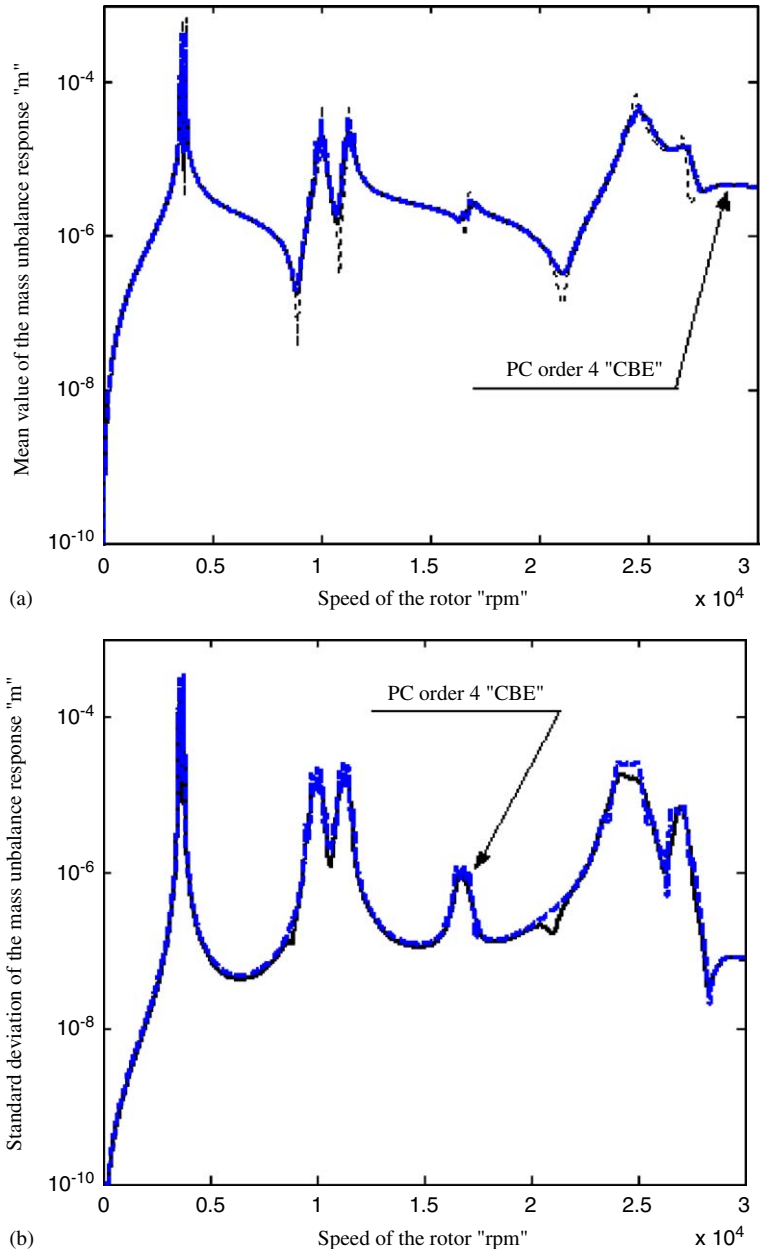

Fig. 6. Mean (a) and standard deviation (b) of the random response. PC order 4 and MC simulation, condensed model. Keys: ------, Deterministic; ——, MC; — —, PC order 4.

Table 4
CPU time for reference model and condensed model

Dispersion level of uncertain parameters		CPU Time (min)			
E	K_{XX}, K_{ZZ}	MC "REF" with 1000 samples	MC "CBE" with 5000 samples	PC order 4 "REF" with 5000 samples	PC order 4 "CBE" with 5000 samples
5%	5%	35	22	24	10

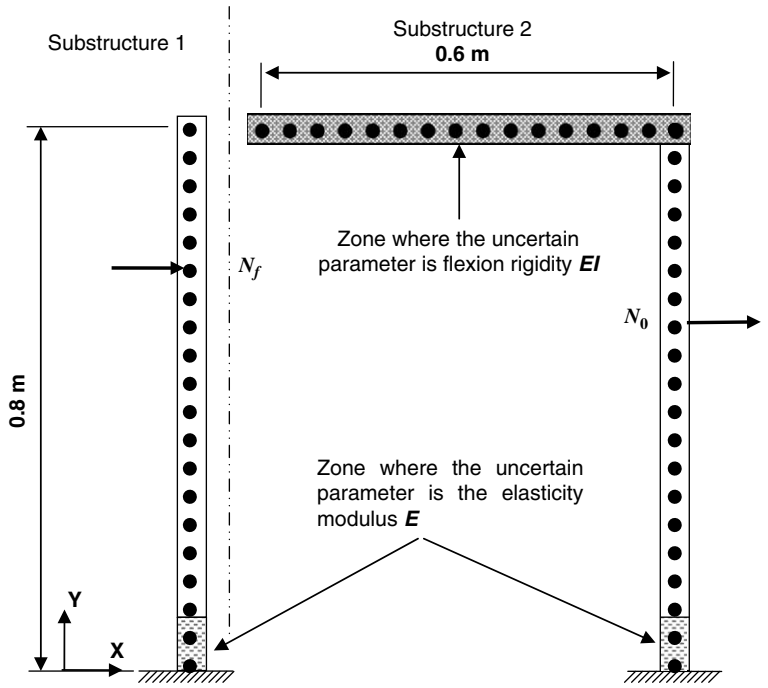

Fig. 7. FEM of a planar frame structure.

Table 5
Precision of the eigenfrequencies (ε_f) and the eigenvectors (ε_u)

	Mean eigenfrequency (Hz)	Random eigenfrequency (Hz) (first moment)	CB		CB		CBE	
			ε_f (%)	ε_u (%)	ε_f (%)	ε_u (%)	ε_f (%)	ε_u (%)
	162 dofs	162 dofs	21 dofs		30 dofs		30 dofs	
Case a: $\delta_E = 5\%$; $\delta_{EI} = 5\%$								
1	11.70	11.67	0.006	0.014	0.004	0.010	0.000	0.003
2	64.65	64.56	0.035	0.075	0.030	0.056	0.000	0.007
3	71.36	71.30	0.010	0.015	0.007	0.012	0.000	0.009
4	113.90	113.75	0.040	0.116	0.031	0.072	0.000	0.026
5	198.15	198.01	0.011	0.089	0.007	0.078	0.000	0.025
6	228.05	227.90	0.104	0.526	0.097	0.453	0.000	0.051
7	321.00	320.39	0.187	0.427	0.142	0.386	0.000	0.069
8	425.20	424.96	0.145	1.026	0.135	0.864	0.001	0.062
9	439.73	439.40	0.073	0.364	0.060	0.291	0.000	0.040
10	634.11	633.04	0.219	1.418	0.124	0.934	0.003	0.272
Case b: $\delta_E = 10\%$; $\delta_{EI} = 10\%$								
1	11.70	11.661	0.007	0.016	0.006	0.012	0.000	0.006
2	64.65	64.482	0.037	0.095	0.032	0.077	0.000	0.008
3	71.36	71.245	0.012	0.026	0.009	0.017	0.000	0.018
4	113.90	113.59	0.042	0.224	0.037	0.139	0.000	0.037
5	198.15	197.87	0.016	0.121	0.010	0.091	0.000	0.028
6	228.05	227.75	0.194	0.656	0.111	0.491	0.000	0.059
7	321.00	319.77	0.429	0.468	0.342	0.426	0.000	0.072
8	425.20	424.71	0.222	1.183	0.185	0.964	0.001	0.066
9	439.73	439.07	0.172	0.734	0.095	0.591	0.000	0.059
10	634.11	631.97	0.432	1.712	0.382	0.951	0.004	0.273

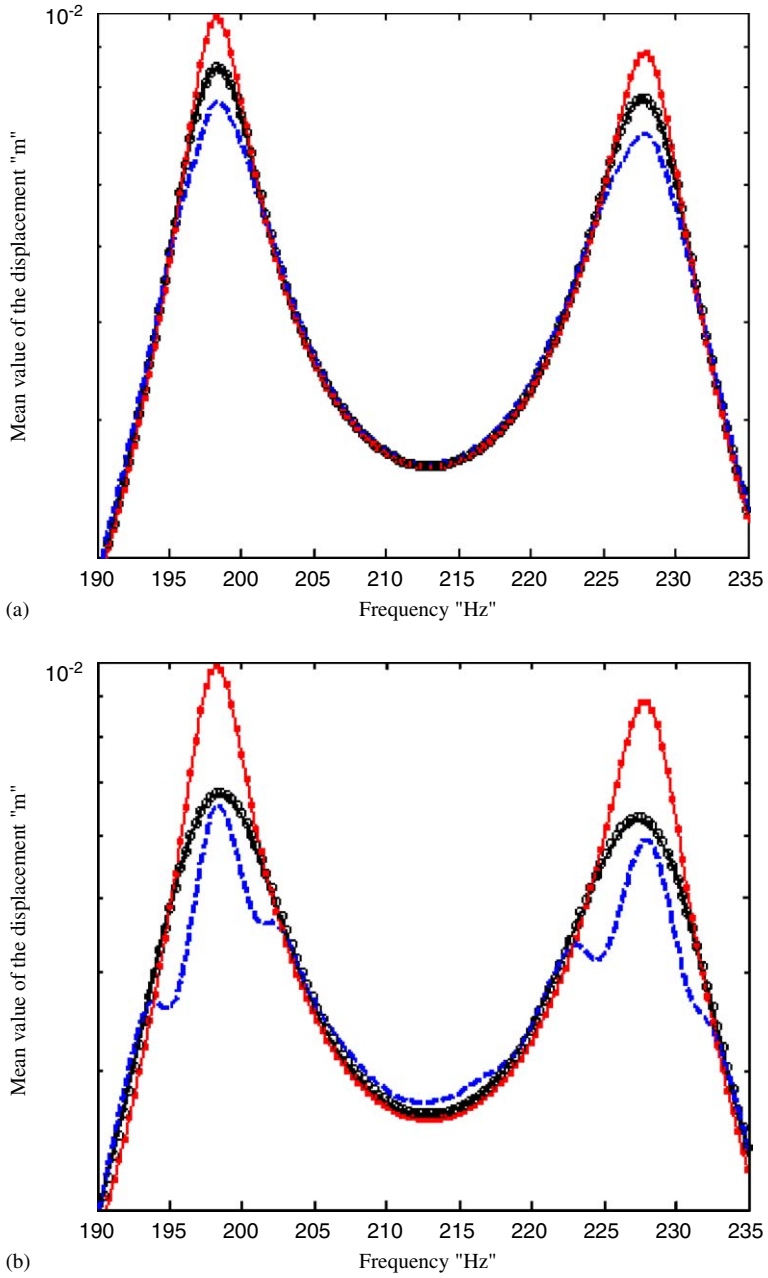

Fig. 8. Mean of the random response at the position $\ll N_O \gg$. MC simulation, PC order 4, direct perturbation, and modal perturbation, reference model. Case of the dispersion: (a) $\delta_E = 5\%$; $\delta_{EI} = 5\%$; (b) $\delta_E = 10\%$; $\delta_{EI} = 10\%$. Keys: ——, MC; — —, PC order 4; ■, direct pert; ○ modal pert.

Figs. 8–11 illustrate the evolution of the mean and the extreme statistics of the random response as a function of the excitation frequency for the following dispersion levels: $\delta_E = 5\%$; $\delta_{EI} = 5\%$; and $\delta_E = 10\%$; $\delta_{EI} = 10\%$.

In Fig. 8, the results of the different perturbation method of order 1 (direct method, modal method) and the PC method (of order 4) are compared with those of the reference MC simulation for 1000 samples.

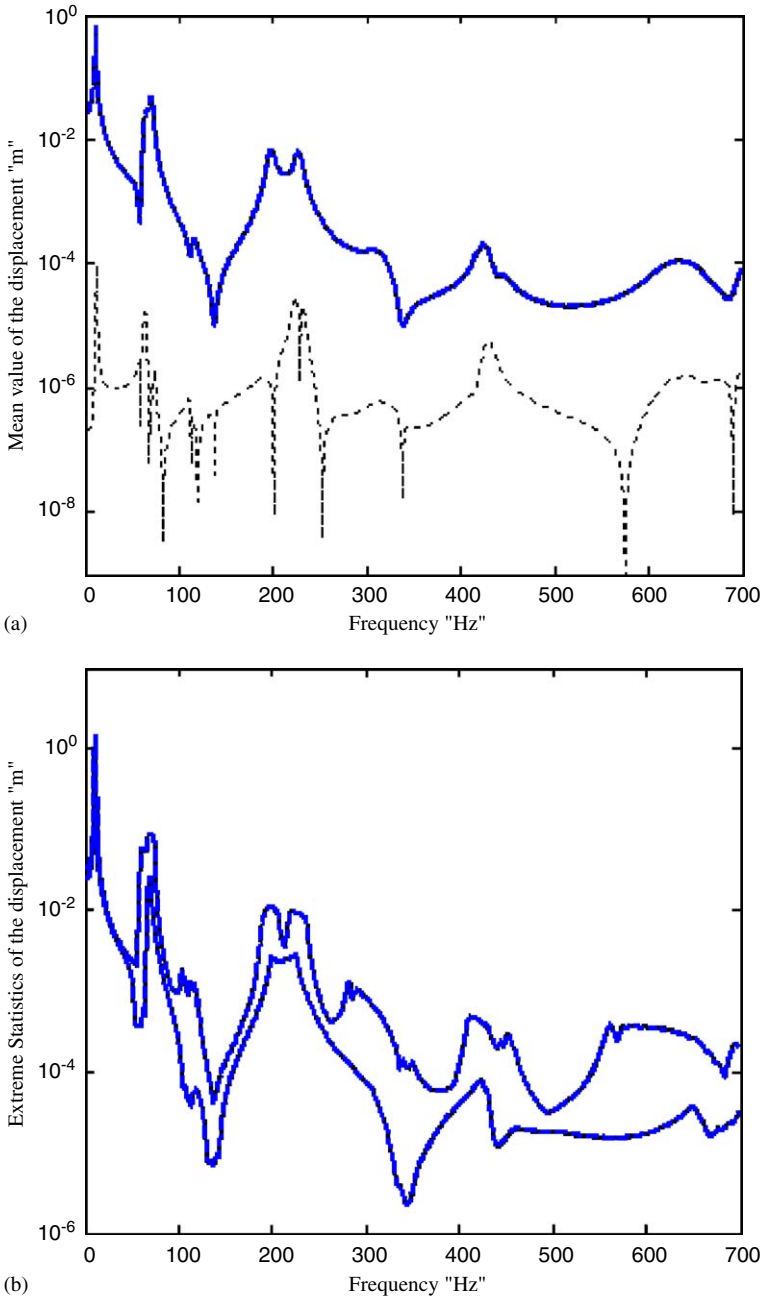

Fig. 9. Mean (a) and extreme statistics (b) of the random response at the position $\ll N_O \gg$. MC simulation, reference and condensed models. Case of the dispersion: $\delta_E = 10\%$; $\delta_{EI} = 10\%$. Keys: ——, REF model; — —, CBE model; ------, Diff: REF/CBE.

Examination of the results shows that the projection on the PC method and the MP method allow prediction of the reference solution obtained by the MC simulation.

In Fig. 9, the results of MC simulation on the CBE method are compared to those of the reference MC simulation on the complete model. These results show that the use of the CBE method allows a good dynamic representation throughout the frequency band 0–700 Hz. The quality of the reduced model, CBE, compared with the reference model is equally highlighted by the differences in response, as illustrated on the same figure.

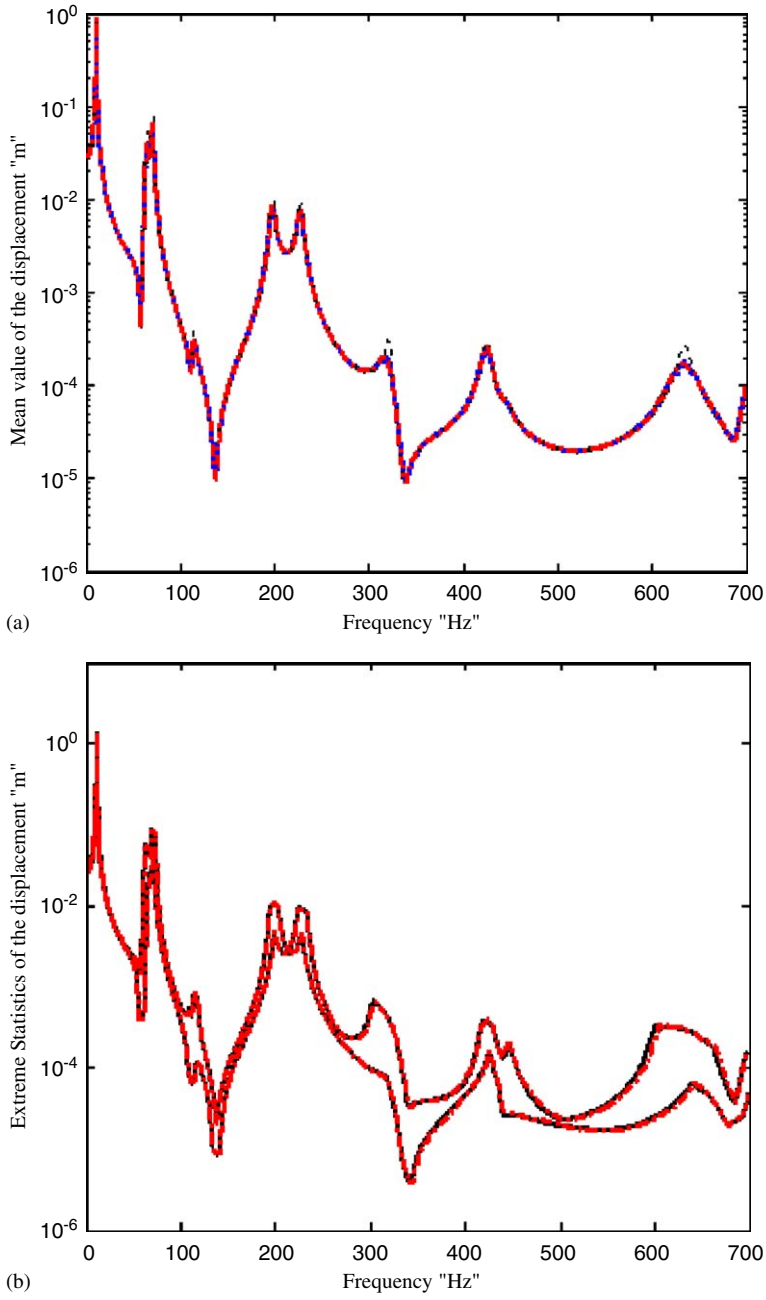

Fig. 10. Mean (a) and extreme statistics (b) of the random response at the position $\ll N_O \gg$. PC order 4, Modal Perturbation and MC simulation, condensed model. Case of the dispersion: $\delta_E = 5\%$; $\delta_{EI} = 5\%$. Keys: ------, deterministic; ——, MC; —·—, PC order 4; —·—, MP.

Figs. 10 and 11 show that a close relationship is obtained between the calculated random responses by the PC and MP methods and by the MC simulation for 5000 samples.

Note that, when the dispersion factor increases, the order of PC directly influences the reconstitution quality of the response and must, therefore, be increased.

In order to highlight the performances of the reduced models in terms of calculation costs, the CPU time between the condensed and reference models can be compared. Examination of Table 6 shows the good

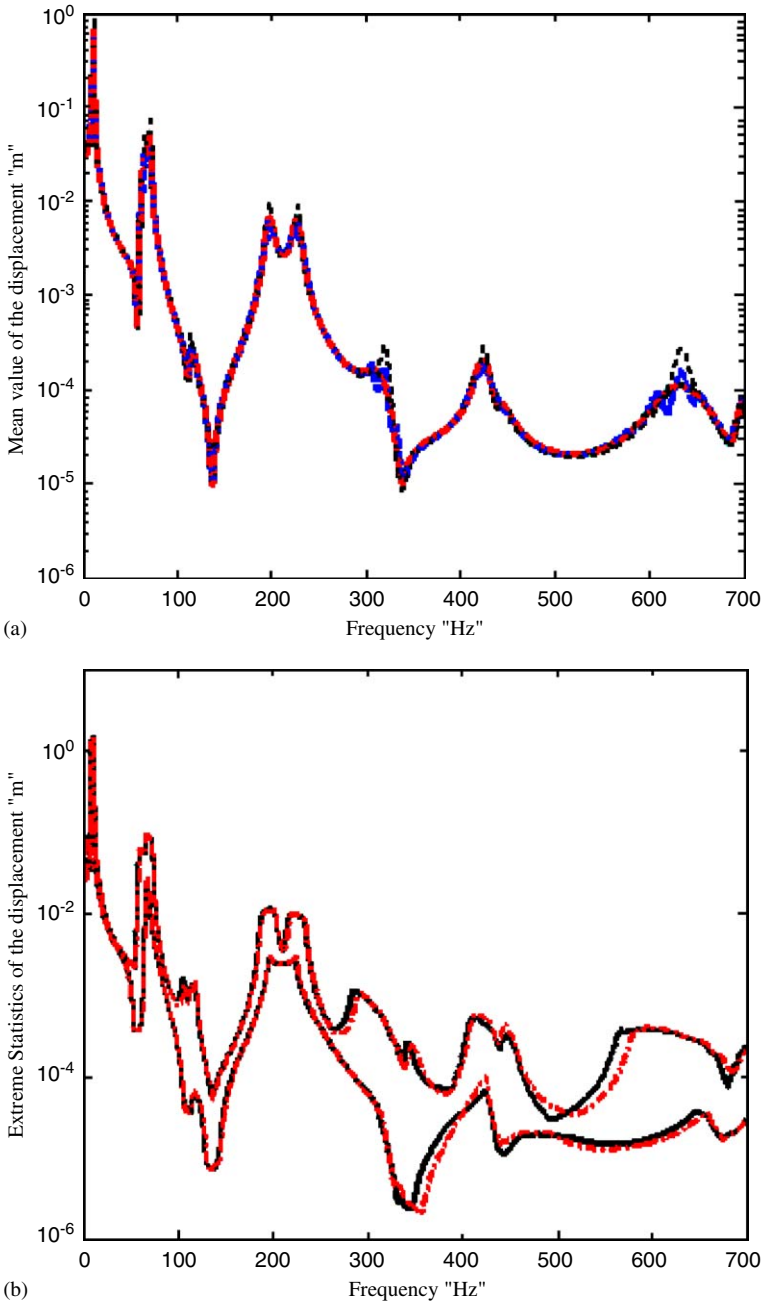

Fig. 11. Mean (a) and extreme statistics (b) of the random response at the position $\ll N_O \gg$. PC order 4, Modal perturbation and MC simulation, condensed model. Case of the dispersion: $\delta_E = 10\%$; $\delta_{EI} = 10\%$. Same keys of Fig. 10.

performances of the proposed method, which exploits a double condensation—CBE followed by a projection on the PC method (CBE + PC) or by the MP method (CBE + MP).

5. Concluding remarks

In this article, a new strategy is proposed in view of optimizing the dynamic behaviour of structures with local uncertainties. It consists of coupling the SFEM and a robust condensation method. The Ritz

Table 6
CPU time for reference model and condensed model

	Dispersion level of uncertain parameters		CPU time (min)			
	E	*EI*	MC "REF" with 1000 samples	MC "CBE" with 5000 samples	PC order 4 "CBE" with 5000 samples	MP method "CBE" with 5000 samples
Case a	5%	5%	210	201	59	61
Case b	10%	10%				

condensation basis is enriched by additional vectors. These vectors are obtained from the random static loadings, which are representative of the modification forces. This enriched basis allows the construction of reduced models, which are robust towards uncertain structural modifications.

The analysis of simulation results showed that this method constitutes an interesting alternative to the classic reduction methods that are maladjusted to the condensation of the SFEMs.

In addition, comparison of the calculation costs highlights the performances of the double condensation by the CBE method followed by a projection on exploiting the same enriched basis. Note that the PC and MP methods give comparable results in the second-level condensation. This methodology can be generalized to many types of condensation or dynamic substructuring method, as proposed in the literature.

The presented approach reduces the size of the model and increases the predictivity. This is in contrast to the dynamic behaviour of SFEMs of large size, which often require reanalysis or updating of design parameters in order to optimize procedures.

Current work is analysing:

- the study of the statistic link between the proposed method, which uses a physical approach, in which the uncertainties of the conception parameters are included directly in the model, and the non-parametric approach, which includes uncertainties of the finite element matrices without a direct physical link with the parameters;
- a more fine-tuned (precision and calculation time) comparison of the performances of the two approaches—PC and MP—according to the number of the uncertain design parameters and the uncertainty level.

One of the perspectives of the current work concerns the integration of the proposed methodology in a stochastic multi-objective optimization procedure of complex mechanical structures.

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