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Harmonic balance approach to limit cycles for nonlinear jerk equations

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Abstract

The method of harmonic balance (HB) is employed to estimate the attributes of limit cycles of some nonlinear third-order (jerk) differential equations which are parity-invariant but not time-reversal-invariant. Two examples with cubic nonlinearities show that the HB method can give good values for the frequency and both the velocity and displacement amplitudes of a period one limit cycle.

1. Introduction

In a previous paper [1], the author applied the method of harmonic balance (HB) (see e.g. [2] for second-order ordinary differential equations (ODEs)) to some nonlinear jerk equations, which are ODEs expressing the third time-derivative of displacement x in terms of x and its first and second derivatives. Several examples exhibiting oscillations were analysed and results were compared with numerical solutions, exhibiting good agreement over a range of parameter values. The method is limited by its nature to zero initial acceleration (when started from zero initial displacement), but was rather successful for that situation. The terms in the jerk equations as considered in [1], as well as being parity-invariant, were also time-reversal invariant. For such equations, having ordinary periodic solutions (a centre), the initial velocity amplitude may be assigned and then the phase space trajectory is a closed loop passing through that point.

If the restriction to time-reversal invariance is dropped, there results the possibility of obtaining limit cycles (LCs). As in the case of second-order, i.e. acceleration, differential equations [2, pp. 154–155], the HB method may be able to detect LCs: it then involves the coefficients of both cosine and sine terms, and leads to two simultaneous approximate equations for both the angular frequency Ω and an amplitude. The study of such LCs for nonlinear jerk equations is the subject of this paper. The results of some numerical experiments are reported and compared with the HB estimates.

It should be stressed that this paper deals with isolated LCs, i.e. of period one (single loop), away from any possible chaotic regions. Third (and higher) dimension systems commonly have a series of period-doubling LCs that involve multiple loops which successively split further towards chaos as a parameter is varied (cf. e.g.

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Refs. [3, p. 83 and p. 94]). Evidently, an HB approach as outlined above would not describe such cascading situations.

In the present case, the initial velocity amplitude does not persist (unless evolution happens to start exactly on a LC): for LCs, path parameters cannot be set a priori. Some questions arise for these third-order oscillators. Does HB give a good estimate of the actual LC period? Does the HB approximation B^{HB} give a good estimate for the velocity amplitude B on the LC, and does the corresponding A^{HB} give a good estimate of its displacement amplitude A? These questions will be treated below.

2. Harmonic balance and limit cycles

The harmonic balance method for jerk equations [1] sets

$$x = \frac{B}{\Omega}\sin \Omega t \equiv A \sin \Omega t, \tag{2.1}$$

so

$$\dot{x} = B \cos \Omega t, \tag{2.2a}$$

$$\ddot{x} = -\Omega B \sin \Omega t \equiv C \sin \Omega t, \tag{2.2b}$$

$$\dddot{x} = -\Omega^2 B \cos \Omega t. \tag{2.2c}$$

A number of linear and nonlinear terms may contribute to the jerk function

$$\dddot{x} = j(x, \dot{x}, \ddot{x}). \tag{2.3}$$

Consideration here is limited to odd-parity terms in j, suitable for the conventional approach via HB, and to polynomial terms not higher than third order. There are four categories of terms: linear (L) or nonlinear (NL), odd (OT) or even (ET) under time reversal, $\dddot{x}$ itself being (L, OT):

L, OT :$\dot{x}$;
L, ET :$x, \ddot{x}$;
NL, OT :$\dot{x}^3$, $x^2\dot{x}$, $x\dot{x}\ddot{x}$, $\dot{x}\ddot{x}^2$;
NL, ET :x^3, $x^2\ddot{x}$, $x\dot{x}^2$, $x\ddot{x}^2$, $\ddot{x}^3$, $\dot{x}^2\ddot{x}$.

The OT terms alone were dealt with in the previous paper on periodic (centre) solutions [1]. For a LC, there must be two relations. One must come from "balancing" the jerk term $\dddot{x}$ (L, OT) with at least one term from L, OT and/or NL, OT. Another relation must come from the ET terms, at least two of which must therefore be present. The HB results may then be compared with results obtained by numerical integration of the jerk d.e.s. In all cases, the initial conditions are taken as

$$x(0) = 0 = \ddot{x}(0), \tag{2.4}$$

following from Eqs. (2.1) and (2.2b), as discussed at length in Ref. [1], together with a value for $\dot{x}(0)$. For numerical purposes, this last value is chosen so that the phase space trajectory "decays" to the LC.

3. Jerk equation involving Euclidean norm nonlinearity

Some time ago, Mulholland [4] considered the nonlinear oscillations of a third-order (i.e. jerk) equation whose nonlinearity depended on the Euclidean norm of the associated phase space (sum of squares of dynamical quantities), and analysed the LC solution. His equation may be written in the form

$$\dddot{x} = -x - (1-\eta)(\dot{x} + \ddot{x}) - \eta(\dot{x} + \ddot{x})(x^2 + \dot{x}^2 + \ddot{x}^2), \tag{3.1}$$

where η is a positive parameter. This has as nonlinearities three terms from the class (NL, OT) and three terms from (NL, ET) as categorized in Section 2 above.

The origin of the three-dimensional phase space is the sole equilibrium point, and the corresponding linearized equation is

$$\dddot{x} + (1-\eta)\ddot{x} + (1-\eta)\dot{x} + x = 0. \tag{3.2}$$

The roots of the characteristic equation (cf. appendix) corresponding to Eq. (3.2) are explicitly given (cf. [4]) by $\Lambda_1 = -1$, $\Lambda_{2,3} = (1/2)\ [\eta \pm \sqrt{(\eta^2-4)}]$, so the real parts of the second and third roots are positive for any positive value of η, so the origin is unstable. We may note that, in the notation of the appendix with a,b,c as the coefficients, respectively, of the second-, first- and zeroth-order derivatives in Eq. (3.2), $ab = (1-\eta)^2$ and $c = 1$. The necessary and sufficient condition for stability of Eq. (3.2), namely $a, b, c > 0$ and $c < ab$ (see Eq. (A.2)) is therefore never satisfied, confirming the afore mentioned instability. As mentioned in Ref. [4], this is necessary for the existence of a nearby LC, i.e. one encircling the origin.

We are here interested in whether the HB procedure can give good estimates of the LC parameters compared with the "exact" values found by numerical integration. The HB procedure of Section 2 utilises identities to re-express powers and products of trigonometric functions. Equating first the coefficients of the $\cos(\Omega t)$ term yields

$$B^2 = \frac{4}{\eta}\frac{\Omega^2(\Omega^2+\eta-1)}{\Omega^4+3\Omega^2+1}. \tag{3.3}$$

Secondly, the coefficients of the $\sin(\Omega t)$ term yield

$$B^2 = \frac{4}{\eta}\frac{1-(1-\eta)\Omega^2}{3\Omega^4+\Omega^2+3}. \tag{3.4}$$

Together, these give a quartic equation in Ω^2:

$$3(\Omega^2)^4 + (2\eta-1)(\Omega^2)^3 + (4-2\eta)(\Omega^2)^2 + (2\eta-5)(\Omega^2) - 1 = 0. \tag{3.5}$$

For given value of parameter η, this can be solved numerically for the positive solution for Ω^2 to give the approximate angular frequency Ω^{HB} and hence the harmonic balance period value

$$T^{\text{HB}} \equiv 2\pi/\Omega^{\text{HB}}. \tag{3.6}$$

Then the approximate velocity amplitude B^{HB} can be found by substitution from either above equation (3.3), (3.4) for B^2; and then the approximate displacement amplitude is given by Eq. (2.1) as

$$A^{\text{HB}} = B^{\text{HB}}/\Omega^{\text{HB}}. \tag{3.7}$$

The harmonic balance approximate displacement is then given in terms of the above by

$$x^{\text{HB}} = \left(B^{\text{HB}}/\Omega^{\text{HB}}\right)\ \sin(\Omega^{\text{HB}}t). \tag{3.8}$$

If $\eta = 0$, then $\Omega = 1$ is the solution, and $x = A \sin t$ with A arbitrary is the exact solution (for $x(0) = 0 = \ddot{x}(0)$) of the resulting d.e., which is linear. For η small, make a perturbation: $\Omega = 1 + \Delta$. The result, to first order in η, assuming Δ is also small, is found from Eq. (3.5) to be $\Delta = -(1/12)\eta \approx -0.08333\,\eta$. This may be compared with Mulholland, who also obtained Δ = —(1/12) η by an averaging method; he also obtained $\Delta = -0.084\eta$ by a perturbation method on the d.e.. The approximate angular frequency, to first order (FO) in η, is therefore $\Omega^{\text{HB}}_{\text{FO}_\eta} = 1 - \eta/12$. Then the approximate period, to first order in η, is

$$T^{\text{HB}}_{\text{FO}_\eta} = 2\pi[1 + (\eta/12)]. \tag{3.9}$$

The HB values (3.6), (3.3), (3.4), (3.7), obtained via Eq. (3.5), can be compared with the exact (EX) values for period, velocity amplitude and displacement amplitude obtained by numerically integrating the nonlinear ODE (3.1) (using ODE Workbench [5]) and letting the system settle onto its LC in the x, $\dot{x}$ plane before determining T, B and A. For this paper, the stability of the limit cycle was confirmed numerically, in that the values remained unchanged over many cycles, after several hundred initial circuits. As a check, results for the $\ddot{x}$, $\dot{x}$ phase plane (which has the same initial conditions) were also computed, and found to agree.

Some results for several values of the parameter η are displayed in Table 1. The results for periods are also compared with the harmonic balance first-order (in η) (HB,FOη) expansion result (3.9) for small η.

Table 1
Period T, velocity amplitude B, and displacement amplitude A, for the Mulholland equation (3.1) with parameter η

η	T			B		A		$\lvert \ddot{x}\rvert_{x=0}\rvert_{LC}$
	HB	LC	HB,FO	HB	LC	HB	LC	
0.1	6.3358	6.3369	6.3355	0.8170	0.8176	0.8239	0.8259	0.0055
0.5	6.5534	6.5799	6.5450	0.8184	0.8223	0.8536	0.8671	0.030
1	6.8460	6.9473		0.8185	0.8291	0.8918	0.9260	0.064
2	7.5457	7.8427		0.8121	0.8471	0.9752	1.0506	0.155

Harmonic balance (HB) results (3.5), (3.6), (3.3), (3.4), (3.7)) are compared with the exact results obtained by numerically integrating the d.e. for the limit cycle (LC). The period is also compared with the HB result obtained by expanding to first order (FO) in η, Eq. (3.9), for $\eta < 1$. The last column shows the value of the second derivative when $x = 0$, on the computed LC.

In addition, the value $\ddot{x}_{LC0}$ of the second derivative $\ddot{x}$ on the computed LC when $x = 0$ was noted, for comparison with the harmonic balance value of zero corresponding to equations (2.2b) and (2.1). It can be seen that the HB values are in good agreement with the exact values, to within just a few percent. The results improve as the value of $\ddot{x}_{LC0}$ more nearly approaches zero, which is consistent with the conditions discussed for the HB method in this application. For this d.e., this corresponds to decreasing value of the parameter η.

4. A modified simple dissipative jerk equation

Consider the following jerk equation, which is a modification of a simple dissipative jerk equation due to Malsoma [6] extended by the inclusion of a term linear in velocity:

$$\dddot{x} = -\lambda\ddot{x} + x\dot{x}^2 - \gamma\dot{x} - x; \quad \lambda > 0, \gamma > 0. \tag{4.1}$$

There are overall two terms of type (L, OT) of Section 2, and two of type (L, ET); the nonlinearity in Eq. (4.1) is of the type (NL, ET).

The HB approach for this d.e. as described in Section 2 leads to

$$\Omega^{HB} = \sqrt{\gamma}; \; T^{HB} = 2\pi/\sqrt{\gamma}, \tag{4.2}$$

$$B^{HB} = 2\sqrt{1 - \lambda\gamma} \tag{4.3}$$

with

$$A^{HB} \equiv B^{HB}/\Omega^{HB} = 2\sqrt{1 - \lambda\gamma}/\sqrt{\gamma} \tag{4.4}$$

and the approximate acceleration amplitude is given by

$$C^{HB} \equiv -B^{HB}\Omega^{HB} = -2\sqrt{1 - \lambda\gamma}\sqrt{\gamma}. \tag{4.5}$$

Then the HB approximate displacement is given explicitly by

$$x^{HB} = 2\left[\sqrt{1 - \lambda\gamma}/\sqrt{\gamma}\right] \; \sin\left(\sqrt{\gamma}t\right). \tag{4.6}$$

(Unlike the complexity of the example of Section 3 above, here the formulae for Ω^{HB} and B^{HB}, etc., are simple and explicit.)

For convenience, the parameters λ and γ are here chosen such that

$$\lambda\gamma = 3/4 \tag{4.7a}$$

so

$$B^{HB} = 1; \; A^{HB} = 1/\sqrt{\gamma}, \; T^{HB} = 2\pi/\sqrt{\gamma}, \; C^{HB} = -\sqrt{\gamma}. \tag{4.7b}$$

The characteristic equation of the linearized equation corresponding to Eq. (4.1), with Eq. (4.7a), is

$$\Lambda^3 + \lambda\Lambda^2 + [3/(4\lambda)]\Lambda + 1 = 0. \tag{4.8}$$

In Eq. (4.8) all coefficients are positive and, in the notation of the Appendix for the coefficients of a cubic, $c = 1 > 3/4 = ab$. Thus by Eq. (A.2), the equilibrium point at the origin is unstable. This suggests the possibility of a nearby LC, encircling the origin. (Note that, for the cubic characteristic equation resulting from the linearized equation corresponding to Eq. (4.1) with general coefficients, instability of the origin requires $ab < c$, i.e. $\lambda\gamma < 1$ (satisfied by Eq. (4.7a). This is consistent with requiring Eq. (4.3) to be real, so that HB approximations exist.)

Table 2 compares HB values with numerically determined exact values (obtained using the software [5]), for several parameter combinations satisfying Eq. (4.7a). It can be seen that the HB approximations give good estimates of the LC values. As expected, the accuracy again improves as $\ddot{x}_{LC0}$ gets closer to zero, which in this case corresponds to decreasing λ, i.e. here, increasing γ.

Table 2
Harmonic balance (HB) values of velocity amplitude B (4.3), displacement amplitude A (4.4), and period T (4.2), for the modified Malasoma nonlinear jerk equation (4.1), for several values of the parameters λ and γ, chosen such that $\lambda\gamma = 3/4$ so, by (4.7), $B^{HB} = 1$, $A^{HB} = 1/\sqrt{\gamma}$, $T^{HB} = 2\pi/\sqrt{\gamma}$

B		A		T		$\lvert \ddot{x}\rvert_{x=0}\rvert_{LC}$
HB	LC	HB	LC	HB	LC	
$\gamma = 1$, $\lambda = 3/4$						
1	0.9701	1	0.9742	6.2832	6.2493	0.071
$\gamma = 4$, $\lambda = 3/16$						
1	0.99945	0.5	0.4998	3.14159	3.14127	0.021
$\gamma = 9$, $\lambda = 1/12$						
1	0.99995	0.3333	0.3333	2.09440	2.09438	0.009

These are compared with the exact limit cycle (LC) values, obtained by numerical integration of the d.e.

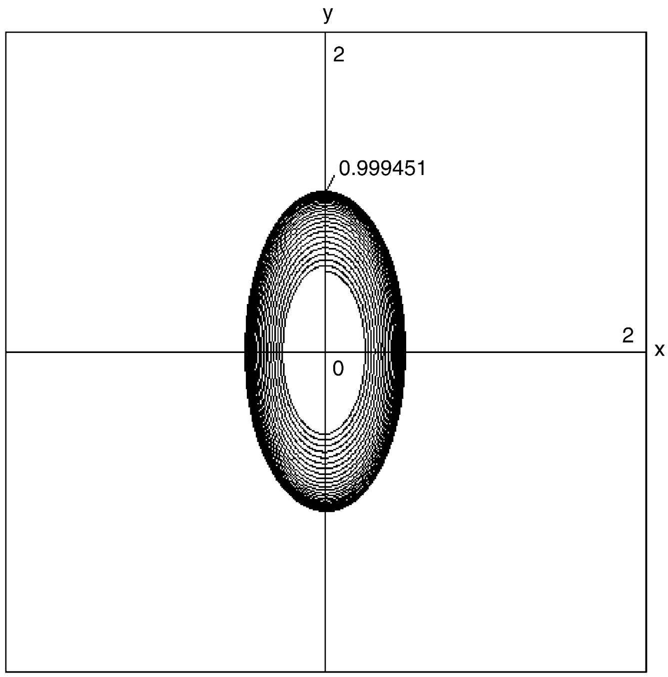

Fig. 1. Jerk equation (4.1), $y = \dot{x}$: parameters $\gamma = 4$, $\lambda = 3/16$; $\dot{x}(0) = 0.5$.

It may be mentioned that when the $\ddot{x}$, $\dot{x}$ phase plane was investigated, the "B" value of the limit cycle, i.e. $\dot{x}$-axis intercept when $\ddot{x}$ is zero, was slightly different from the B value found in the usual $x, \dot{x}$ phase plane. The difference was less than 0.3% for the first parameter set in Table 2, and much less for the other two cases. This

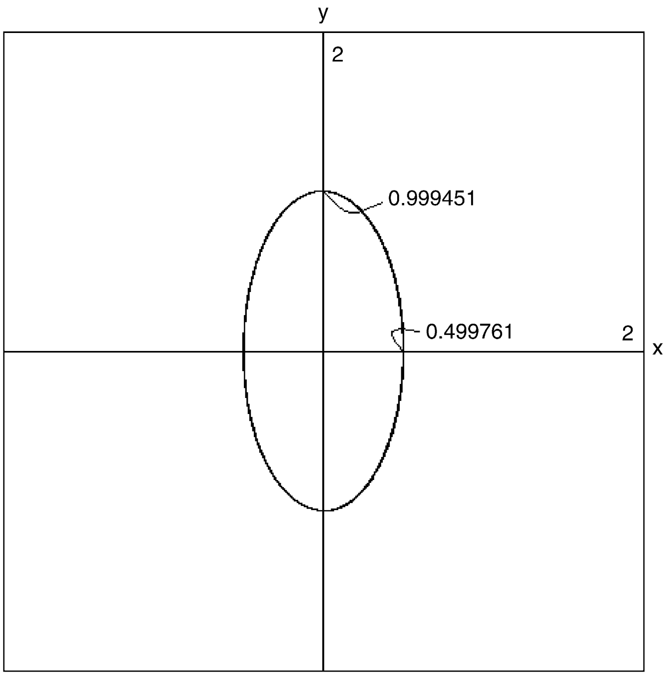

Fig. 2. Jerk equation (4.1), $y = \dot{x}$: $\gamma = 4$, $\lambda = 3/16$; $\dot{x}(0) = 1$.

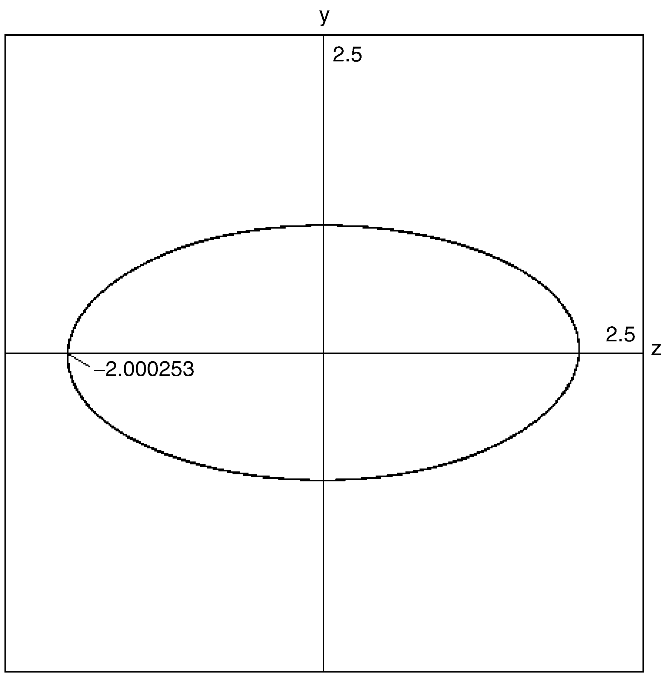

Fig. 3. Jerk equation (4.1), $y = \dot{x}$, $z = \ddot{x}$: parameters $\gamma = 4$, $\lambda = 3/16$; $\dot{x}(0) = 1$.

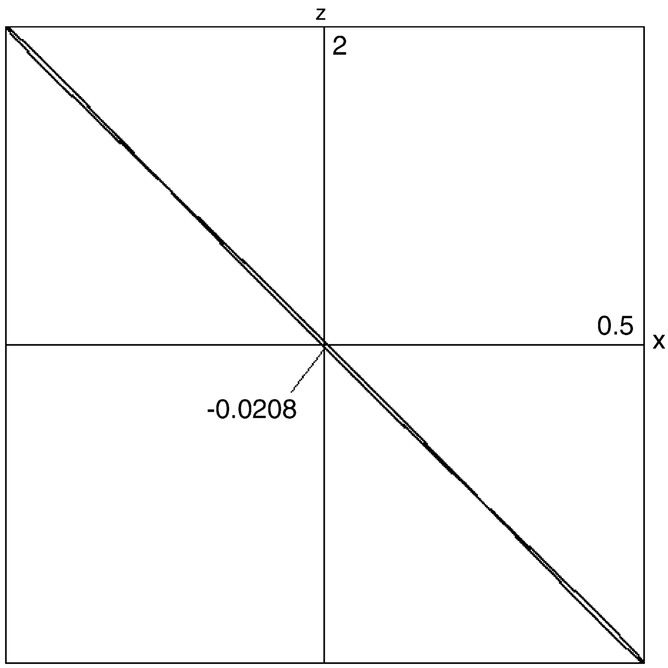

Fig. 4. Jerk equation (4.1), $z = \ddot{x}$: parameters $\gamma = 4$, $\lambda = 3/16$; $\dot{x}(0) = 1$. (Transients up to ~100 T omitted.)

is due to the fact that the orbit actually moves in the three-dimensional phase space, so there would be slightly different projections onto the $\dot{x}$-axis in the two planes. Such details lie beyond the precision of the HB approach.

Figs. 1–4 show some phase plane trajectories for the nonlinear jerk equation (4.1) for the case (see Table 2) $\gamma = 4$, $\lambda = 3/16 = 0.1875$. Fig. 1, for $\dot{x} \equiv y$ vs. x with $\dot{x}(0) = 0.5$ (and $x(0) = 0 = \ddot{x}(0)$), shows the spiralling out from the unstable origin to the stable LC. (For a choice such as $\dot{x}(0) = 0.01$, the path uncoils from much nearer the origin, much more densely.) Figs. 2–4 start with $\dot{x}(0) = B^{\mathrm{HB}} = 1$ (and $x(0) = 0 = \ddot{x}(0)$), for the phase planes indicated, where $y = \dot{x}$, $z = \ddot{x}$. The pictures for the x,y and z,y planes (which have the same initial conditions) look qualitatively similar, although with different oblateness. It should be noted that $B^{\mathrm{HB}} = 1$, $A^{\mathrm{HB}} = 0.5$ and $C^{\mathrm{HB}} = -2$ here, and these B and A HB values are very close to their LC values as tabulated in the second row of Table 2. Furthermore, C^{HB} is also very close to the LC value which was found by computation (see Fig. 3) to be $C^{\mathrm{LC}} = -2.000253$. In Figs. 2 and 3, the trajectories spiral inwards towards the LC, as indicated by the final value designators. The picture in Fig. 4 (in which initial transients have been omitted) for the x,z plane is quite different, and shows the slight departure from $z \propto -x$ which would pertain if the HB Eqs. (2.1), (2.2b) were exact. The value of $\ddot{x}|_{x=0}$ is evidently not exactly zero, but for these parameters is nevertheless small on the LC, which is why the HB approximation does give good results here.

5. Conclusion

For limit cycles, the amplitudes are determined by the dynamical system and are not assigned as initial conditions, unlike the case of ordinary periodic solutions (centres). In particular, the acceleration on the limit cycle might not be zero when the displacement is zero. Thus, HB for limit cycles of jerk equations might be expected to be less successful than for centres as studied in Ref. [1]. Nevertheless, the above examples show that, for a range of equation coefficients, the HB method can give good results. This is borne out by the two jerk equations having very different cubic nonlinearity terms.

The method will only be appropriate for jerk equations with (at least) one term in the jerk expression of the type OT (odd under time-reversal) and (at least) two terms of the type ET (even under time reversal), but this represents a sufficiently large class of equations to be of importance.

Appendix

A general cubic equation may be written in the form

$$f(\Lambda) \equiv \Lambda^3 + a\Lambda^2 + b\Lambda + c = 0; \; a, b, c \text{ real.} \tag{A.1}$$

One approach to examining the stability of a third-order linearized system whose characteristic equation is a cubic equation such as Eq. (A.1) is to note that for stability it is necessary that none of the roots has a positive real part. For all roots real and negative, this corresponds to a node; for one root real and negative and the other two complex conjugate with negative real part, to a spiral node (see e.g. Ref. [3, pp. 81–82]). This requires that all three coefficients in Eq. (A.1) be positive (see Ref. [7, pp. 242–244]). Further analysis as in Ref. [7] yields the necessary and sufficient conditions for stability associated with Eq. (A.1) as

$$a > 0, \; b > 0, \; c > 0; \; c < ab. \tag{A.2}$$

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