Available online at www.sciencedirect.com
SCIENCE DIRECT®
Journal of Sound and Vibration 297 (2006) 414–419
JOURNAL OF SOUND AND VIBRATION
www.elsevier.com/locate/jsvi

Short Communication

Dynamic stability of viscoelastic pipes on elastic foundations of variable modulus

Vassil M. Vassilev*, Peter A. Djondjorov

Institute of Mechanics, Bulgarian Academy of Sciences, Acad. G. Bonchev St., Block 4, 1113 Sofia, Bulgaria

Received 17 October 2005; received in revised form 6 March 2006; accepted 16 March 2006
Available online 2 June 2006

Abstract

The dynamic stability of straight cantilevered viscoelastic pipes conveying inviscid fluid and lying on an elastic foundation of variable modulus is studied. The corresponding eigenvalue problem is solved using both Galerkin and shooting methods. It is found that certain combinations of the pipe parameters (the elastic foundation modulus, mass ratio and internal damping coefficient) can destabilize the pipe.

1. Introduction

Fluid-conveying pipes are widely used in various industrial branches. Sometimes, their role is simply to transport fluids. In other cases, they function as basic structural components as well. In both cases, however, the dynamic stability of the pipes is crucial for the proper operation of the entire equipment.

Being acknowledged to be of such a significant importance, the dynamic stability of fluid-conveying pipes has been extensively studied in the past 40 years (see, e.g., the comprehensive book [1] by Païdoussis). In general, it has been established that if an initially straight pipe conveys inviscid fluid with a relatively low velocity, then each disturbance applied to that pipe causes vibration diminishing with the time. In this case, the initial equilibrium state of the pipe is referred to as a stable one. However, for fluid velocities higher than a certain value (called critical flow velocity) even small disturbances could result in non-diminishing vibration. Under these circumstances, the pipe equilibrium state is referred to as an unstable one.

Usually the pipes are supported at the ends but, for different reasons, they are often supported along the span too. From mathematical point of view, these internal supports could be described as a continuous foundation the pipe is resting on. Surprisingly, in spite of the intuitive expectation, it turns out that a foundation does not always stabilize a pipe. The same holds true with respect to the internal damping as well.

In 1978, Becker et al. [2] considered the dynamic stability of cantilevered viscoelastic pipes on foundations of constant modulus for several small mass ratios. Later, Lottati and Kornecki [3] studied the same problem but for all admissible values of the mass ratio and several different values of the internal damping coefficient. In these works, it has been established that Winkler foundations of constant modulus have a stabilizing effect,

*Corresponding author. Tel.: +359 2 979 6478; fax: +359 2 870 7498.
E-mail address: vasilvas@imbm.bas.bg (V.M. Vassilev).

0022-460X/$ - see front matter
doi:10.1016/j.jsv.2006.03.025

as expected (see also Ref. [4] by Doare and de Langre). However, the internal damping has been found to have either destabilizing or stabilizing effect on the pipe depending on the mass ratio. Elishakoff and Impolonia [5] and Djondjorov [6] have studied the dynamic stability of cantilevered pipes on foundations of constant modulus that support only a part of the pipe span. They have found that such foundations could either destabilize or stabilize the pipe depending on the position and length of the foundations. Djondjorov et al. [7] and Djondjorov [8] have examined cantilevered pipes on Winkler foundations whose modulus is a certain sixth-, second- or first-order polynomial. They have concluded that all such foundations stabilize the pipe.

In Refs. [2,3,5], the authors have determined critical flow velocities on the basis of the dispersion equations resulting from the respective differential equations. Such an approach is possible when the differential equations considered are of constant coefficients but one could not make use of this method with equations of variable coefficients. For this reason, numerical methods (mostly Galerkin method) have been applied in the aforementioned papers concerning pipes on foundations of variable modulus. For further details on the methods of analysis and some other features of the influence of the elastic and viscoelastic foundations on the dynamic stability of fluid-conveying pipes we refer to the basic book [1] and recent review [9].

The aim of the present note is to analyse the effect of the internal damping and the magnitude of the foundation modulus on the dynamic stability of cantilevered viscoelastic pipes lying on elastic foundations of Winkler type with variable modulus. For that purpose, a computational procedure based on the Galerkin method is developed for determination of the eigenfrequencies of the pipes and the critical flow velocities, the results obtained being then verified by the shooting method.

2. Basic problem

The small transverse vibration of an initially straight viscoelastic pipe conveying inviscid fluid and lying on an elastic foundation of Winkler type is governed by the partial differential equation (see, e.g., Refs. [1–3,5])

$$EI\left(\frac{\partial^4 u}{\partial z^4}+\lambda\frac{\partial^5 u}{\partial z^4\,\partial\tau}\right)+MU^2\frac{\partial^2 u}{\partial z^2}+2MU\frac{\partial^2 u}{\partial z\,\partial\tau}+(m+M)\frac{\partial^2 u}{\partial\tau^2}+c(z)u=0, \tag{1}$$

where $u(z,\tau)$ denotes the transverse displacement of the pipe axis, z is the coordinate along this axis, τ is the time, E is Young's modulus of the pipe material, I is the inertia moment of the pipe cross-section, λ is the internal damping coefficient related to the viscosity of the pipe material, m and M are the masses per unit length of the pipe and the fluid, respectively, U is the flow velocity, and $c(z)$ is the variable foundation modulus.

Let L be the pipe length. Upon introducing the dimensionless parameters

$$x=\frac{z}{L},\quad t=\frac{\tau}{L^2}\sqrt{\frac{EI}{m+M}},\quad w=\frac{u}{L},\quad k(x)=\frac{L^4}{EI}c\left(\frac{z}{L}\right),$$
$$\eta=\frac{\lambda}{L^2}\sqrt{\frac{EI}{m+M}},\quad \beta=\frac{M}{m+M},\quad v=UL\sqrt{\frac{M}{EI}},$$

Eq. (1) takes the form

$$\frac{\partial^4 w}{\partial x^4}+\eta\frac{\partial^5 w}{\partial x^4\,\partial t}+v^2\frac{\partial^2 w}{\partial x^2}+2v\sqrt{\beta}\frac{\partial^2 w}{\partial x\,\partial t}+\frac{\partial^2 w}{\partial t^2}+k(x)w=0. \tag{2}$$

Let the pipe under consideration be of cantilevered type, i.e., its end $x=0$ is fixed and the other one, $x=1$, is free. Then, the boundary conditions read

$$w|_{x=0}=\left.\frac{\partial w}{\partial x}\right|_{x=0}=0,\quad \left(1+\eta\frac{\partial}{\partial t}\right)\left.\frac{\partial^2 w}{\partial x^2}\right|_{x=1}=\left(1+\eta\frac{\partial}{\partial t}\right)\left.\frac{\partial^3 w}{\partial x^3}\right|_{x=1}=0. \tag{3}$$

In this study, solutions to the boundary value problem (2), (3) of the form

$$w(x,t)=y(x)\exp(\omega t) \tag{4}$$

are sought. Substituting expression (4) for the function $w(x,t)$ in Eq. (2) and boundary conditions (3), one obtains the two-point boundary value problem

$$(1+\eta\omega)y'''' + v^2 y'' + 2v\sqrt{\beta}\omega y' + \omega^2 y + k(x)y = 0, \tag{5}$$

$$y|_{x=0} = 0, \quad y'|_{x=0} = 0, \quad y''|_{x=1} = 0, \quad y'''|_{x=1} = 0, \tag{6}$$

where the prime indicates differentiation with respect to x. Actually, this constitutes a non-self-adjoint eigenvalue problem, the eigenvalue parameter being the frequency ω.

Here, the above eigenvalue problem is solved by a standard Galerkin method (see, e.g., Ref. [10]), an N-term approximate solution to it being expressed as a linear combination of the first N well-known eigenfunctions of a cantilevered elastic pipe without flow and foundation, i.e., $\eta = 0$, $v = 0$, $k(x) = 0$ (see, e.g., Refs. [1,7]). Consequently, the eigenfrequencies are determined as the roots ω_i $(i = 1, 2, \ldots, 2N)$ of a $2N$th-order polynomial whose coefficients depend on η, β, v and some other parameters describing the foundation considered. The critical flow velocities v_{cr} are determined as the lowest values of v at which this polynomial has a root with non-negative real part, the rest of the pipe parameters being kept fixed. Once the values of a critical flow velocity and the corresponding eigenfrequency are obtained for a given number N, a Maple implementation of the shooting method (package *shoot*[1]) is applied to check the existence of a sufficiently accurate approximate solution to the respective two-point boundary value problem (5), (6). The results presented below are achieved using 10 terms in the Galerkin approximation of the considered eigenvalue problems, i.e. $N = 10$. The values of the critical flow velocities and the corresponding eigenfrequencies computed at this level of Galerkin approximation turned out to provide an excellent accuracy of the approximate solutions obtained then by the shooting method, namely: each such solution whose maximal norm is about one satisfies the equation and boundary conditions within an absolute error of order less than 10^{-10}.

3. Numerical results

First, in order to test the aforementioned computational procedure, the critical flow velocities of several well-known problems concerning dynamic stability of cantilevered pipes without foundation have been determined. The results of our computations, shown in Fig. 1, are in an excellent agreement with the earlier results presented in Refs. [1, Fig. 3.30] and [3, Fig. 8] up to the limiting case $\beta \to 0$, $\eta = 0$ to be discussed below.

Let us first note that in the vicinity of $\beta = 1$, for $0.919 \leqslant \beta \leqslant 0.994$, the 10-term Galerkin approximation, verified by the shooting method, predicts that the v_{cr} curve corresponding to $\eta = 0$ (the thick curve in Fig. 1(a)) contains a new S-shaped domain in addition to the ones presented in Refs. [1, Fig. 3.30] and [3, Fig. 8]. This observation is in accordance with the remark in the Païdoussis' book [1], "As $\beta \to 1$, more and more S-shaped jumps are encountered". Let us recall that the so-called S-shaped domains are associated with an instability-restabilization-instability sequence (see Ref. [1]) and that Mukhin [12] has shown that at $\beta = 1$ the critical flow velocity tends to infinity.

As for the vicinity of $\beta = 0$, the results of our computations shown in Fig. 1 confirm the curves for $\eta = 0.001$, 0.01 and 0.1 given in Ref. [3, Fig. 8]. However, the curve corresponding to $\eta = 0$ is not confirmed. The matter is that for $\sqrt{\beta} < 0.1$ this curve in Ref. [3] is a straight horizontal line at $v_{cr} = 4.18$ but in the vicinity of $\beta = 0$ it turns rapidly upward and smoothly goes to $v_{cr} = 4.48$ (see also formulae (16) in Ref. [3]). Our computations show that when $\sqrt{\beta}$ approaches zero with positive values, the critical flow velocity is $v_{cr} = 4.19$ and the respective curve in Fig. 1 never turns upward for β down to 10^{-24}. Therefore, we can conclude that at $\eta = 0$, the limit value of the critical flow velocity when $\beta \to 0$, $\beta > 0$ is $v_{cr} = 4.19$, whereas at $\beta = 0$ it is known to be $v_{cr} = 4.48$ (see Refs. [1,3]). For pipes without foundation, the critical flow velocity depends only on the parameters β and η, that is $v_{cr} = v_{cr}(\beta, \eta)$, and hence the above conclusion means that the function $v_{cr}(\beta, 0)$ is

[1]This package can be downloaded from the website of the first author of Ref. [11] (Douglas B. Meade) at http://www.math.sc.edu/meade/maple/Shoot9/Shoot9.zip.

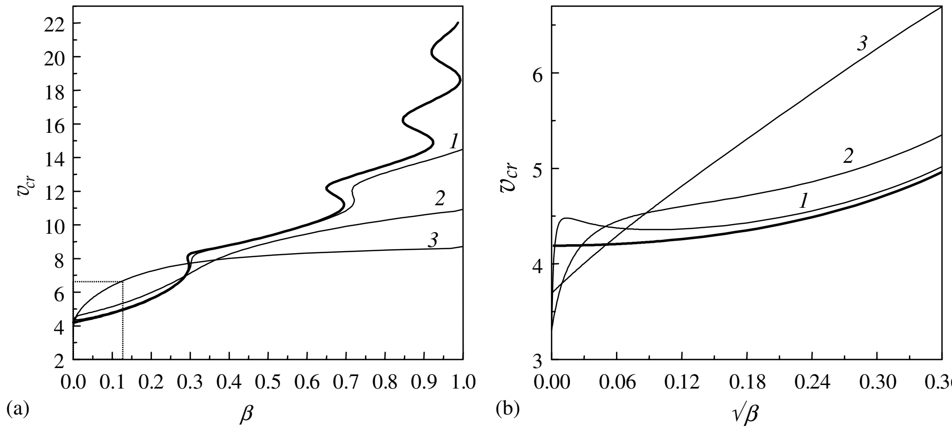

Fig. 1. Critical flow velocity v_{cr} of a cantilevered pipe without foundation ($k = 0$) as a function of the mass ratio β at the following four values of the internal damping coefficient η: (a) $\eta = 0$ (thick curve), $\eta = 0.001$ (curve 1), $\eta = 0.01$ (curve 2), $\eta = 0.1$ (curve 3); (b) magnification of the domain marked by the dashed rectangle in (a).

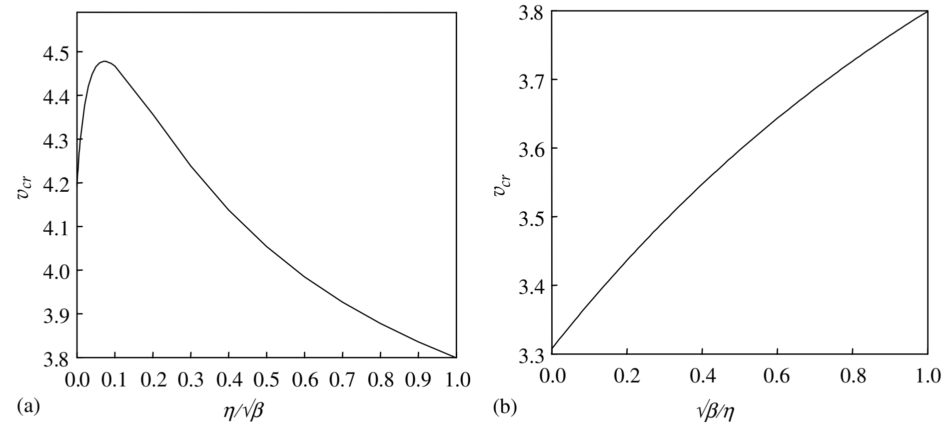

Fig. 2. Variation of the critical flow velocity v_{cr} of a cantilevered pipe without foundation ($k = 0$): (a) for $\eta \leqslant \sqrt{\beta} \leqslant 10^{-3}$ as a function of the ratio $\eta/\sqrt{\beta}$; (b) for $\sqrt{\beta} \leqslant \eta \leqslant 10^{-3}$ as a function of the ratio $\sqrt{\beta}/\eta$.

discontinuous at $\beta = 0$ and the jump is

$$v_{cr}(0,0) - \lim_{\beta \to 0, \beta > 0} v_{cr}(\beta, 0) = 4.48 - 4.19 = 0.29.$$

This observation contradicts the idea that the critical flow velocity smoothly tends to $v_{cr} = 4.48$ when $\beta \to 0, \beta > 0$.

From theoretical point of view, it seems natural to study also the continuity of the function $v_{cr}(\beta, \eta)$ at $\beta = 0, \eta \to 0$, $\eta > 0$. Similarly to the previous case, this function turns out to be discontinuous. Indeed, at $\beta = 0$, $\eta \to 0$, $\eta > 0$ the critical flow velocity tends to $v_{cr} = 3.30$, whereas at $\eta = 0$ it is $v_{cr} = 4.48$. This jump

$$v_{cr}(0,0) - \lim_{\eta \to 0, \eta > 0} v_{cr}(0, \eta) = 4.48 - 3.30 = 1.18$$

is even bigger than the previous one.

Finally, in order to clarify the behaviour of the function $v_{cr} = v_{cr}(\beta, \eta)$ in the close neighbourhood of the point $\beta = 0$, $\eta = 0$, critical flow velocities for $0 < \sqrt{\beta} \leqslant 10^{-3}$ and $0 < \eta \leqslant 10^{-3}$ are computed and the results are shown in Fig. 2. Surprisingly, it turned out that for any pair of such small values of the parameters η and β, the critical flow velocity v_{cr} depends only on the ratio of these parameters. Fig. 2(a) and (b) present the variation of the critical flow velocity for $\eta \leqslant \sqrt{\beta} \leqslant 10^{-3}$ as a function of the ratio $\eta/\sqrt{\beta}$ and for $\sqrt{\beta} \leqslant \eta \leqslant 10^{-3}$ as a function of the ratio $\sqrt{\beta}/\eta$, respectively. It is seen that for such small values of the parameters η and β, the critical flow velocities vary over the relatively large range 3.30–4.48.

The computations show that when the ratio of η and β is such that $v_{\mathrm{cr}} < 4.48$, for flow velocities between v_{cr} and 4.48, the real part of the corresponding eigenfrequency being positive is of vary small magnitude. For instance, for $\eta = 10^{-6}$, $\sqrt{\beta} = 10^{-8}$, the ratio is $\sqrt{\beta}/\eta = 0.01$, the critical flow velocity is $v_{\mathrm{cr}} = 3.32$ and for flow velocities up to 4.47, the real part of the corresponding eigenfrequency is less than 10^{-3}, whereas at 4.48 it is 0.3 and rapidly grows beyond 4.48. This means that although the flow velocities up to 4.47 are all critical, a substantial growth of the vibration amplitude indicating the pipe instability can be observed after a long time, which in this case exceeds 10^3.

On account of the results presented in Fig. 2, one can conclude that each value between 3.30 and 4.48 may be considered as a limit value of the critical flow velocity as $\eta \to 0$, $\beta \to 0$. Indeed, let v_0 be such that $3.30 \leqslant v_0 \leqslant 4.48$. Then, it corresponds to a certain ratio $r_0 = \eta/\sqrt{\beta}$ in Fig. 2(a) if $v_0 \leqslant 3.8$ or $r_0 = \sqrt{\beta}/\eta$ in Fig. 2(b) if $v_0 \geqslant 3.8$. Hence, considering any two sequences of values $\eta_1, \eta_2, \ldots$ and $\beta_1, \beta_2, \ldots$, of the parameters η and β both tending to zero and such that the ratio between η_i and $\sqrt{\beta_i}$ is r_0, the corresponding sequence of critical flow velocities $v_{\mathrm{cr}}(\beta_i, \eta_i)$ tends to v_0. This conclusion casts doubt on the applicability of the model of a fluid-conveying viscoelastic cantilevered pipe based on Eq. (1) for values of the mass ratio and damping parameters β and η smaller than 10^{-3}.

Consider now elastic foundations whose modulus is a second-order polynomial of the form

$$k(x) = 4hx(1-x), \quad h = \mathrm{const}, \quad h > 0,$$

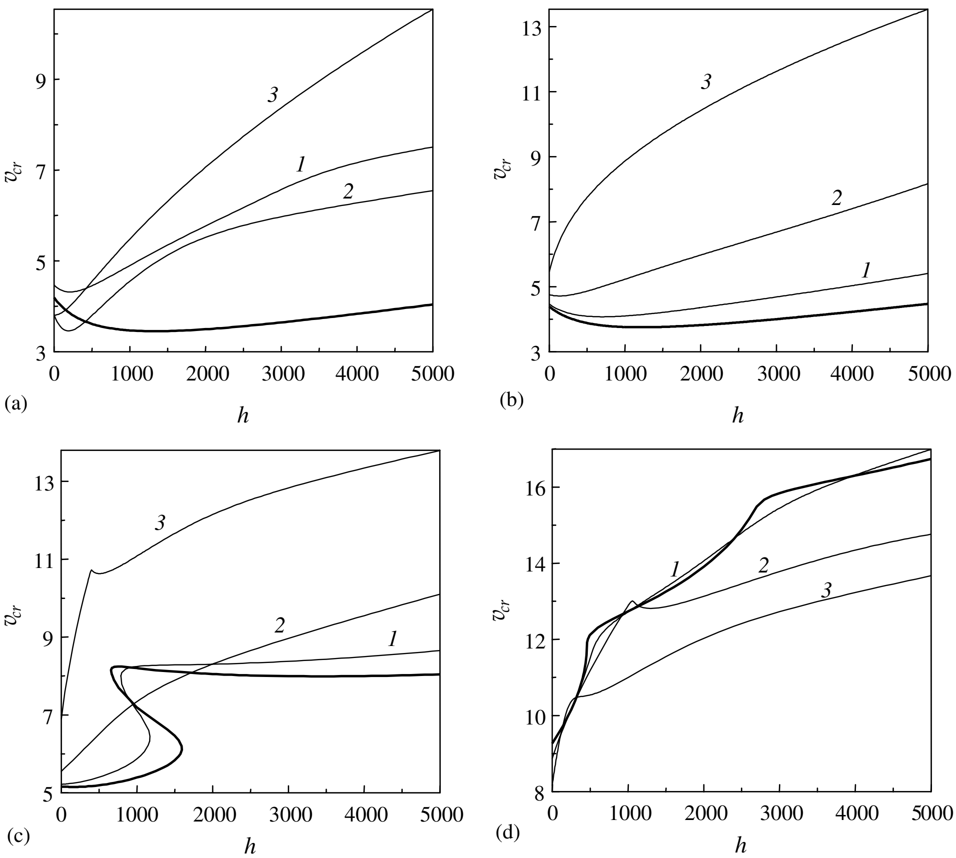

Fig. 3. Critical flow velocity v_{cr} of a cantilevered pipe as a function of the foundation parameter h at values $\eta = 0$ (thick curve), $\eta = 0.001$ (curve 1), $\eta = 0.01$ (curve 2), $\eta = 0.1$ (curve 3) of the internal damping coefficient at mass ratios: (a) $\beta = 0.0001$, (b) $\beta = 0.04$, (c) $\beta = 0.1296$, (d) $\beta = 0.49$.

i.e., it is a concave function with a maximal value h at the middle of the pipe span vanishing at the pipe ends. This foundation modulus differs from those considered earlier (see Ref. [7]) in that it depends only on one parameter h. Since the purpose here is to study the influence of an elastic foundation on the dynamic stability of elastic and viscoelastic pipes, the consideration of such a one-parameter class of foundation moduli makes it easier to deduce whether hardening of the foundation stabilizes the pipe.

First, in order to study this influence for small β, the cases $\beta = 0.0001$ and 0.04 are considered. The results for four different values of the internal damping coefficient are shown in Fig. 3(a) and (b), respectively. Apparently, in the case $\eta = 0.1$ each foundation stabilizes the pipe, but in the other cases a foundation of small h destabilizes the pipe whereas foundations of larger h are stabilizing ones. For instance, when $\beta = 0.04$, the elastic cantilever ($\eta = 0$) is destabilized for foundations with $h < 4680$, the maximal destabilization effect being achieved at $h = 1220$ where $v_{cr} = 3.75$ that is about 85% of the critical flow velocity $v_{cr} = 4.39$ at $h = 0$. Thus, due to the influence of an elastic foundation of variable modulus, the critical flow velocity of an elastic cantilevered pipe can be reduced by approximately 15%.

Next, pipes of comparatively large mass ratio are considered. The results for pipes with $\beta = 0.1296$ are displayed in Fig. 3(c). It is seen that at the largest value of the internal damping coefficient $\eta = 0.1$ considered here, all foundations have a strong stabilizing effect except for the fold in the vicinity of $h = 500$. Stabilizing effect is observed for $\eta = 0.01$ as well. It should be noted also that the curves corresponding to $\eta = 0$ and 0.001 contain S-shaped domains in the intervals $680 \leqslant h \leqslant 1600$ and $790 \leqslant h \leqslant 1170$, respectively.

Finally, the critical flow velocities for a pipe with $\beta = 0.49$ are displayed in Fig. 3(d). It is seen that in the cases $\eta = 0$, 0.001 and 0.1 all foundations considered have a strong stabilizing effect. As for the case $\eta = 0.01$, only foundations such that $1060 < h < 1800$ destabilize the pipe in the sense that the critical flow velocities for such values of the foundation parameter h are less than the critical flow velocity for $h = 1060$.

Acknowledgement

This research is supported by the National Science Fund, Bulgaria, Contract no. TH 1518/05.

References

[1] M. Païdoussis, *Fluid–Structure Interactions: Slender Structures and Axial Flow*, vol. 1, Academic Press, London, 1998.
[2] M. Becker, W. Haugher, W. Winzen, Exact stability analysis of uniform cantilevered pipes conveying fluid or gas, *Archives of Mechanics* 30 (1978) 757–768.
[3] I. Lottati, A. Kornecki, The effect of an elastic foundation and of dissipative forces on the stability of fluid-conveying pipes, *Journal of Sound and Vibration* 109 (1986) 327–338.
[4] O. Doare, E. de Langre, Local and global instability of fluid-conveying pipes on elastic foundations, *Journal of Fluids and Structures* 16 (1) (2002) 1–14.
[5] I. Elishakoff, N. Impolonia, Does a partial elastic foundation increase the flutter velocity of a pipe conveying fluid?, *Journal of Applied Mechanics—Transactions of the ASME* 68 (2001) 206–212.
[6] P. Djondjorov, Dynamic stability of pipes partly resting on Winkler foundation, *Theoretical and Applied Mechanics* 31 (3) (2001) 101–112.
[7] P. Djondjorov, V. Vassilev, V. Dzhupanov, Dynamic stability of fluid conveying cantilevered pipes on elastic foundations, *Journal of Sound and Vibration* 247 (3) (2001) 537–546.
[8] P. Djondjorov, On the critical velocities of pipes on variable elastic foundation, *Theoretical and Applied Mechanics* 31 (4) (2001) 73–81.
[9] I. Elishakoff, Controversy associated with the so-called "Follower Forces": Critical overview, *Applied Mechanics Reviews* 58 (2005) 117–142.
[10] C.A.J. Fletcher, *Computational Galerkin Methods*, Springer, New York, 1984.
[11] D.B. Meade, B.S. Haran, E.R. White, The shooting technique for the solution of two-point boundary value problems, *MapleTech* 3 (1) (1996) 85–93.
[12] O.N. Mukhin, Stability of a pipeline and some methods in nonconservative problems, *Vestnik Moskovskogo Universiteta, Series I (Mathematics)* 2 (1965) 76–87 (in Russian).