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Short communication

Fast frequency response analysis of partially damped structures with non-proportional viscous damping

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Abstract

This paper presents a new method for the modal frequency response analysis of partially damped structural system with non-proportional viscous damping. The basic idea is to handle the modal viscous damping matrix by noting that the rank of the viscous damping matrix is typically very low for problems of interest. Then, the Sherman–Morrison–Woodbury formula provides a convenient expression for the inverse of equation which includes the low-rank matrix. The new method, fast frequency response analysis (FFRA) algorithm, dramatically improves the performance of the modal frequency response analysis compared to conventional methods in industry with the same accuracy.

1. Introduction

The frequency response analysis (FRA) for large structures in terms of all of the finite element (FE) degrees of freedom in the millions has been prohibitive. Instead, industry has mainly used the modal FRA. Some of the most challenging aspects of performing the modal FRA arise from damping [1,2]. For an undamped system or a system with proportional viscous damping, the modal frequency response problem becomes uncoupled, so that it is inexpensive to solve the modal frequency response problem. However, non-proportional damping, which describes the realistic damping of structures, results in a fully populated coefficient matrix in the modal formulation [3]. One of the most commonly used types of non-proportional damping for representing energy dissipation is viscous damping that assumes the existence of dissipative forces that are a function of velocity.

With non-proportional viscous damping, the coupled modal frequency response problem has been solved with either direct methods or iterative methods [3]. Although direct methods are the most straightforward and accurate, they are expensive due to the factorization cost, $O(m^3)$ operations [4], where m is the number of modes used to represent the response and is usually in the thousands for large structures. Iterative methods have more advantages than direct methods in terms of speed. However, the disadvantages are that the convergence rate of iterative methods depends on spectral properties of the coefficient matrix, and the cost

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increases in proportion to the number of right hands [5]. Alternatively, the coupled modal frequency response problem can be uncoupled with the quadratic eigensolutions [6]. Generally, in order to obtain the quadratic eigensolutions, one needs to linearize the quadratic eigenvalue problem into a $2m \times 2m$ generalized eigenvalue problem. This approach, however, is too expensive for large-scale FE models that require more than thousands of modes to represent the responses. Unfortunately, there are still disadvantages with traditional methods to solve the modal frequency response problem with many modes, thousands of modes.

This paper presents an efficient algorithm, the fast frequency response analysis (FFRA) algorithm, for solving the modal frequency response problem with non-proportional viscous damping. The focus of FFRA algorithm is the partially damped structural system that has a low-rank viscous damping matrix. With most structures, a relatively small amount of viscous damping mechanism provides a large reduction in stress and deflection by dissipating energy from the structure. For example, with an automobile suspension, a few shock absorbers, which are using viscous dampers, are used to control the motion of the springs that support the vehicle. The FFRA algorithm can dramatically improve the performance of FRA compared to conventional methods in the industry with the same accuracy.

2. Modal frequency response problem with non-proportional viscous damping

A system of equations for the direct FRA in the FE dimension can be represented as

$$[-\omega^2 M + \mathrm{i}\omega B + K]\mathbf{X}(\omega) = \mathbf{P}(\omega), \tag{1}$$

where M, B, and $K \in \mathbb{R}^{n\times n}$ are the FE mass, non-proportional viscous damping, and stiffness matrix, respectively, and n represents the number of FE degrees of freedom. For excitations $\mathbf{P}(\omega) \in \mathbb{C}^{n\times nf}$, the frequency responses $\mathbf{X}(\omega) \in \mathbb{C}^{n\times nf}$ are calculated at each excitation frequency ω by solving a set of complex linear equations (1), where,where nf is the number of load cases.

The frequency response problem in Eq. (1) is projected onto the space spanned by eigenvectors in $\Phi \in \mathbb{R}^{n\times m}$ of a partial eigensolution of the generalized eigenvalue problem $K\Phi = M\Phi\Lambda$, in which $\Lambda \in \mathbb{R}^{m\times m}$ is a eigenvalue matrix and m is the number of modes obtained up to cutoff frequency ($m \ll n$). By making the substitution $\mathbf{X}(\omega) = \Phi\mathbf{Z}(\omega)$ and premultiplying by Φ^{T}, the modal frequency response problem is obtained in the form

$$[-\omega^2 I + \mathrm{i}\omega\bar{B} + \Lambda]\mathbf{Z}(\omega) = \mathbf{F}(\omega), \tag{2}$$

where the mass and stiffness matrices are diagonalized as a result of the mode orthogonality and mass normalization, and $\mathbf{F}(\omega) = \Phi^{\mathrm{T}}\mathbf{P}(\omega) \in \mathbb{R}^{m\times nf}$. Note that the modal viscous damping matrix $\bar{B} = \Phi^{\mathrm{T}}B\Phi \in \mathbb{R}^{m\times m}$ is not diagonal. Although there is an enormous reduction in dimension from the original problem in Eq. (1), Eq. (2) is still expensive for large-scale structures with many modes, thousands of modes, due to $O(m^3)$ operations to factor the coefficient matrix.

3. Fast frequency response analysis algorithm

When viscous damping exists, Eq. (2) can be rewritten as

$$[D(\omega) + \mathrm{i}\omega\bar{B}]\mathbf{Z}(\omega) = \mathbf{F}(\omega), \tag{3}$$

where $D(\omega) = (-\omega^2 I + \Lambda)$ is a frequency-dependent diagonal matrix.

The new algorithm FFRA for viscous damping handles the modal viscous damping matrix by noting that the rank of the viscous damping matrix B is typically very low for problems of interest in the automobile industry. This is because of the small number of viscous damping elements such as shock absorbers and engine mounts. Conventionally, the rank of matrix can be identified with singular value decomposition (SVD). Since B is symmetric, the eigenvalue decomposition provides the same results as the SVD method.

First, in order to reduce the cost of eigenvalue decomposition for the modal viscous damping matrix $\bar{B} = \Phi^{\mathrm{T}}B\Phi$, the $\bar{B}$ matrix is decomposed efficiently into

$$\bar{B} = \Phi_b^{\mathrm{T}} B_b \Phi_b \tag{4}$$

by noting that B is very sparse matrix. The condensed viscous damping matrix $B_b \in \mathbb{R}^{b\times b}$ contains only non-zero rows and columns of the finite-element matrix B. $\Phi_b \in \mathbb{R}^{b\times m}$ contains rows of Φ which correspond to non-zero elements in B. Generally b is much smaller than n since B is very sparse. b is typically tens of degrees of freedom in automobile structures.

Then, the eigenvalue decomposition for the condensed viscous damping matrix B_b, which is also symmetric, is performed as

$$B_b = U\Sigma U^{\mathrm{T}} \tag{5}$$

in which $\Sigma \in \mathbb{R}^{r\times r}$ is the diagonal matrix of singular values, and $U \in \mathbb{R}^{b\times r}$ is orthogonal matrix [4]. The r is the rank of B_b. Generally $r \leqslant b$ is much smaller than m.

By substituting Eq. (5) into Eq. (4), $\bar{B}$ can be represented in the form

$$\bar{B} = \Phi_b^{\mathrm{T}}(U\Sigma U^{\mathrm{T}})\Phi_b. \tag{6}$$

Combining Eq. (3) with Eq. (6) results in

$$[D(\omega) + \mathrm{i}\omega\bar{U}\Sigma\bar{U}^{\mathrm{T}}]\mathbf{Z}(\omega) = \mathbf{F}(\omega), \tag{7}$$

where $\bar{U} = \Phi_b^{\mathrm{T}}U \in \mathbb{R}^{m\times r}$. Eq. (7) can be rewritten in the form of a diagonal matrix, $D(\omega)$, plus low rank matrices, $\bar{U}$ and $Q(\omega)$, in the form

$$[D(\omega) + \bar{U}Q(\omega)\bar{U}^{\mathrm{T}}]\mathbf{Z}(\omega) = \mathbf{F}(\omega), \tag{8}$$

where $Q(\omega) = \mathrm{i}\omega\Sigma \in \mathbb{C}^{r\times r}$. To solve Eq. (8), instead of factoring the coefficient matrix with $O(m^3)$ operations, the Sherman–Morrison–Woodbury (SMW) formula [4] gives a convenient expression for the inverse of Eq. (8) as

$$\begin{aligned}\mathbf{Z}(\omega) &= [D + \bar{U}Q\bar{U}^{\mathrm{T}}]^{-1}\mathbf{F}(\omega)\\ &= [D^{-1} - D^{-1}\bar{U}Q^{1/2}(I + Q^{1/2}\bar{U}^{\mathrm{T}}D^{-1}\bar{U}Q^{1/2})^{-1}Q^{1/2}\bar{U}^{\mathrm{T}}D^{-1}]\mathbf{F}(\omega)\\ &= [D^{-1} - D^{-1}W(I + W^{\mathrm{T}}D^{-1}W)^{-1}W^{\mathrm{T}}D^{-1}]\mathbf{F}(\omega)\\ &= [D^{-1} - D^{-1}WR^{-1}W^{\mathrm{T}}D^{-1}]\mathbf{F}(\omega),\end{aligned} \tag{9}$$

where $W = \bar{U}Q^{1/2} \in \mathbb{C}^{m\times r}$ and $R = I + W^{\mathrm{T}}D^{-1}W \in \mathbb{C}^{r\times r}$. The general form of the SMW formula is

$$(A + BC^{\mathrm{T}})^{-1} = A^{-1} - A^{-1}B(I + C^{\mathrm{T}}A^{-1}B)^{-1}C^{\mathrm{T}}A^{-1}. \tag{10}$$

Note that the cost for solving Eq. (9) involves only $O(r^3)$ operations to invert R and some matrix multiplications, which are inexpensive because r is small. In addition, the FFRA algorithm is an efficient modal frequency response problem *reformulation, not an approximation* approach.

Table 1
The cost of operations for the FFRA algorithm for non-proportional viscous damping

Step	Task	Cost
(1.1)	$B_b = U\Sigma U^{\mathrm{T}}$	$O(b^3)$
(1.2)	$\bar{U} = \Phi_b^{\mathrm{T}}U$	$O(m * b * r)$
	for $i = 1, \mathrm{nfreq}$	
(2.1)	$D^{-1}\mathbf{F}$	$O(m * nf)$
(2.2)	$W = \bar{U}Q^{1/2}$	$O(m * r^2)$
(2.3)	$W^{\mathrm{T}}(D^{-1}\mathbf{F})$	$O(r * m * nf)$
(2.4)	R^{-1}	$O(r^3)$
(2.5)	$R^{-1}(W^{\mathrm{T}}D^{-1}\mathbf{F})$	$O(r^2 * nf)$
(2.6)	$W(R^{-1}W^{\mathrm{T}}D^{-1}\mathbf{F})$	$O(m * r * nf)$
(2.7)	$D^{-1}(WR^{-1}W^{\mathrm{T}}D^{-1}\mathbf{F})$	$O(m * nf)$
	end	

Table 1 shows the detailed cost of the FFRA algorithm. In step (1.1), the cost of eigenvalue decomposition for the symmetric B_b is inexpensive because the dimension of matrix B_b is b which is relatively very small compared to m. At each excitation frequency, the main cost is $O(m * r^2)$ operations per load case. So, the FFRA algorithm results in a dramatic reduction in the cost of operations compared to conventional method that factors the coefficient matrix at each frequency with $O(m^3)$ operations per load case.

4. Numerical examples

As numerical examples, two industry automobile FE models are selected to evaluate the FFRA algorithm. The performance and accuracy of the FFRA algorithm are compared to those of a commercial FE software NASTRAN modal solution (SOL 111) [7] and ZSYSV in LAPACK [8], in which the coefficient matrix of a complex linear system for the partially damped modal frequency response problem is factored at each frequency. An HP rx5670 with 900 MHz Itanium II processor is used for evaluating the performance of the algorithm.

4.1. Example 1

For the FE model with 114,219 degrees of freedom shown in Fig. 1, 1381 global modes are obtained from the partial eigensolution. Only two viscous damping elements are used. From the eigenvalue decomposition in Eq. (5), in which DSYEV in LAPACK [8] is used, the rank of the modal viscous damping matrix is identified as 4. The modal FRA is performed up to 700 Hz. Table 2 shows the analysis time from the FFRA algorithm. The FFRA algorithm is 37.9 times faster than NASTRAN modal solution (SOL 111). For accuracy evaluation, the solution of the FFRA algorithm is compared with that of NASTRAN as shown in Fig. 2, in which both the FFRA and NASTRAN solution yield the same result.

4.2. Example 2

The performance of FFRA algorithm is evaluated for a trim body car FE model, which has 1.8 million degrees of freedom. The number of load cases is 3. The frequency range of interest is from 1 to 500 Hz with a 1 Hz increment. The number of global modes is 7570. The rank of modal viscous damping matrix is 4.

Table 3 shows the analysis time of FFRA and ZSYSV in LAPACK. The FFRA algorithm is almost 224 times faster than ZSYSV algorithm. For accuracy evaluation, the solution of the FFRA algorithm is compared with that of ZSYSV solution. Fig. 3(a) and (b) show the magnitude of the response in the Z direction at driving point and cross-point. As revealed in the figures, the results from both the FFRA and

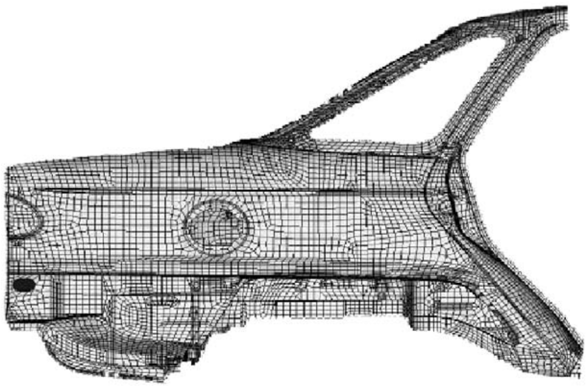


Fig. 1. Finite element model with two viscous damping element.

Table 2
Elapsed time of the modal frequency response analysis for FE model 1

	FFRA	NASTRAN modal FRA (SOL 111)
Elapsed time	29 s	18 min 21 s

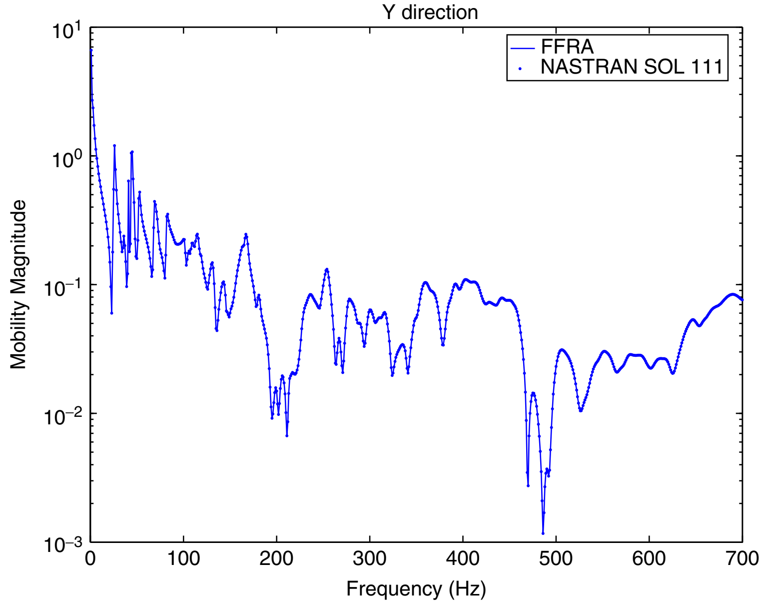


Fig. 2. Comparison of responses from the FFRA and NASTRAN modal solution (SOL 111) for FE model 1.

Table 3
Elapsed time of the modal frequency response analysis for FE model 2

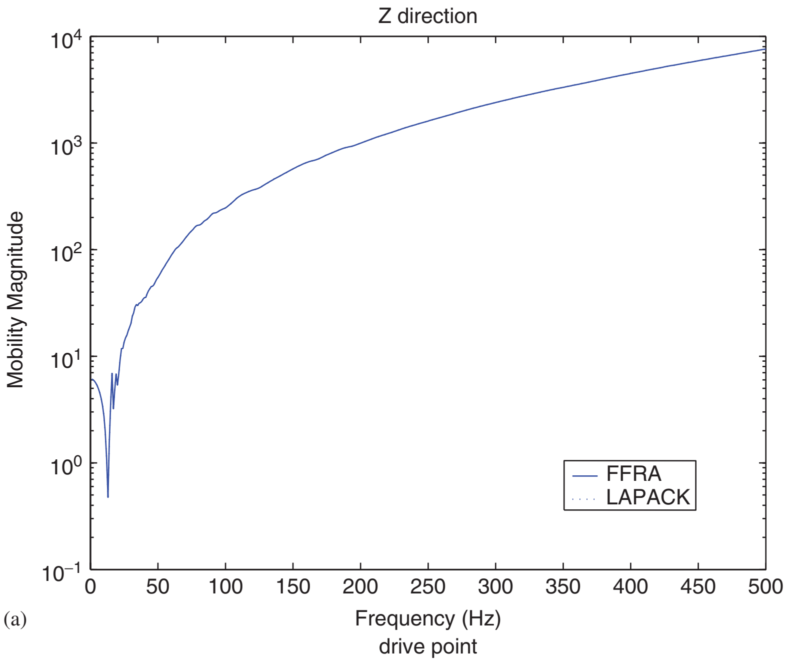
	FFRA	ZSYSV in LAPACK
Elapsed time	7 min 5 s	26 h 30 min

ZSYSV are indistinguishable. The figures and tables show outstanding performance of the FFRA algorithm with good accuracy for a FE model with partially damped viscous damping.

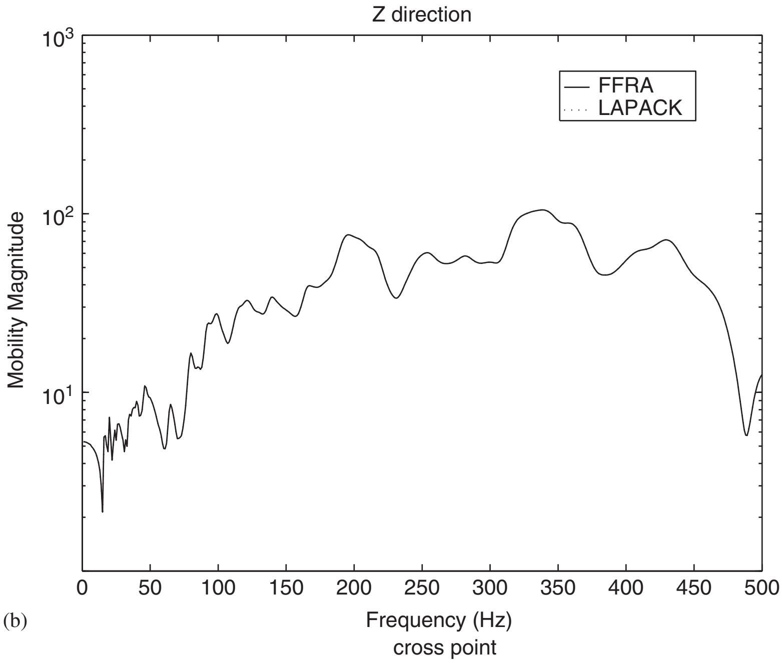
5. Conclusion

For a partially damped structural system with non-proportional viscous damping, the FFRA algorithm is developed to solve the modal frequency response problem efficiently. While observing the same accuracy as the conventional methods, which require $O(m^3)$ operations at each frequency, the FFRA algorithm dramatically improves the performance of the modal frequency response analysis with $O(m * r^2)$ operations.

Remarks: Some contents of this paper is submitted for patent approval.



(a)



(b)

Fig. 3. Comparison of responses from the FFRA and ZSYSV in LAPACK for FE model 2.

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