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Discussion

On the energy transfer at boundaries of translating continua

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1. Introduction

In 1997, Lee and Mote [1,2] investigated the energetics of one-dimensional translating continua. The term 'translating' was used in a generalized sense implying that there is a mass transfer along the continua. A string and a beam in axial motion, as well as a pipe conveying fluid, were considered as the examples. The main emphasis in the papers of Lee and Mote was placed on the energy exchange at a boundary of the continua. As an approach to analysis of this energy exchange, they proposed a very elegant 'travelling wave method' based on comparison of the energy of an incident wave to that of the wave reflected by the boundary. Employing this method, a conclusion may be drawn on whether the energy is lost or gained at a boundary without explicit identification of the forces acting on this boundary.

Studying the wave reflection at a boundary, Lee and Mote considered an incident harmonic wave and stated that 'the energy ΔW transferred into the continuum span over one period by wave reflection' (cited from [1, p. 724]) is given as

$$\Delta W = E_{\lambda,r} - E_{\lambda,i} = (R-1)E_{\lambda,i},$$

where $E_{\lambda,r}$ and $E_{\lambda,i}$ are the energies contained in one wavelength of the reflected and incident waves, respectively, and $R = E_{\lambda,r}/E_{\lambda,i}$ is the 'energy reflection coefficient'. According to Lee and Mote, if this coefficient is larger than unity then it is concluded that the energy is gained at the boundary and vice versa.

In this paper it is shown that the energy exchange at a boundary of a dispersive translating continuum cannot be analysed using the coefficient introduced by Lee and Mote but the following expression for the 'true' energy reflection coefficient, R_ω, should be used

$$R_\omega = \frac{E_{\lambda,r}}{E_{\lambda,i}}\frac{|k_r c_{\mathrm{gr},r}|}{|k_i c_{\mathrm{gr},i}|} = R\frac{|c_{\mathrm{gr},r}|}{|c_{\mathrm{ph},r}|}\frac{|c_{\mathrm{ph},i}|}{|c_{\mathrm{gr},i}|},$$

where $c_{\mathrm{ph},r}$ and $c_{\mathrm{gr},r}$ are the phase and group velocities of the reflected wave, whereas $c_{\mathrm{ph},i}$ and $c_{\mathrm{gr},i}$ are those of the incident wave. The true reflection coefficient represents the ratio of the energy that is contained in a differential frequency bandwidth of the reflected pulse to that contained in the same bandwidth of the incident

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pulse. The expression for R_ω shows, in particular, that the energy reflection coefficient proposed by Lee and Mote cannot be used for translating dispersive continua. It is applicable, however, if the continuum is not translating ($|c_{\mathrm{gr},r}| = |c_{\mathrm{gr},i}|$ and $|c_{\mathrm{ph},r}| = |c_{\mathrm{ph},i}|$) or it is translating but not dispersive ($|c_{\mathrm{ph},i}| = |c_{\mathrm{gr},i}|$ and $|c_{\mathrm{ph},r}| = |c_{\mathrm{gr},r}|$).

To obtain the correct expression for the energy reflection coefficient, a pulse of a finite bandwidth is considered in this paper. Calculating the total energy of this pulse, an expression is derived for the spectral energy density in the translating continuum. The ratio of this density in the reflected and incident pulses gives the energy reflection coefficient for a differential frequency band of the pulse. This coefficient could be derived by considering reflection of a harmonic wave instead of the pulse. In this case, however, one should additionally account for the difference in the range of wavenumbers, which correspond to the same differential frequency band in the incident and reflected waves. This has not been done by Lee and Mote, which limited applicability of their results to non-dispersive systems only.

It is important to note that the energy transport is conventionally studied using the energy flux. For example, to calculate the total energy of a pulse that propagates in one-dimensional, infinitely long system, the energy flux through an arbitrary cross section is integrated over time. In this paper, as well as in the papers of Lee and Mote, the energy is calculated by integrating the energy density of the continuum over the spatial coordinate. If a continuum is not translating, both approaches give, obviously, the same expression for the energy. In a translating continuum, however, the energy calculated using the energy flux differs from that calculated using the energy density. More precisely, this difference exists only in such translating continua, whose translation velocity is prescribed kinematically. Such kinematic formulation implicitly implies that there is an external force that maintains the prescribed velocity. This force can act along the whole extension of the continuum and its work is not necessarily zero. As a result, the energy conservation equation

$$\partial E/\partial t + \mathrm{div}(F) = 0,$$

where E is the energy density and F is the energy flux does not hold anymore. Instead, the energy flux is coupled by an equation of the same form to a so-called pseudo-energy G [3]:

$$\partial G/\partial t + \mathrm{div}(F) = 0.$$

The fact that there exists an equation of this form has been used extensively in acoustics for deriving expressions for the energy flux in various approximations [4–12].

From the short discussion above, it is clear that the energy flux cannot be used straightforwardly for finding the correct expression for the energy of a travelling pulse in a translating continuum, the velocity of which is prescribed a priori. That is why integration of the energy density over space is adopted in this paper.

Concluding this introduction, it must be noted that the modification of the reflection coefficient proposed in this letter is not aimed at undermining the usefulness of the 'travelling wave method' but at a correct description of the energy transfer at a solitary boundary of a dispersive translating continuum.

2. Energy of a pulse travelling in one-dimensional translating continuum

Consider a generalized, uniform, one-dimensional continuum, translating at a constant speed v and conveying fluid at a constant speed u, both in the positive x-direction. In accordance with [1,2], the total energy per unit length $e(x,t)$ of such a continuum can be written as

$$e(x,t) = \frac{1}{2}\left(m_s\left(\frac{\partial w}{\partial t} + v\frac{\partial w}{\partial x}\right)^2 + m_f\left(\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x}\right)^2 + T\left(\frac{\partial w}{\partial x}\right)^2 + EI\left(\frac{\partial^2 w}{\partial x^2}\right)^2 \right), \tag{1}$$

where m_s is the mass per unit length of the 'solid part' of the continuum (for example, the mass of steel per unit length of a pipe), m_f is the mass per unit length of the fluid, EI is the bending stiffness, and T is the axial tension. This energy can be reduced to that of a translating beam by setting $m_f = 0$ or to the energy of a tensioned pipe conveying fluid by setting $v = 0$.

The equation of the transverse motion of the continuum, which corresponds to the energy density given by Eq. (1), reads

$$EI\frac{\partial^4 w}{\partial x^4} - (T - m_f u^2 - m_s v^2)\frac{\partial^2 w}{\partial x^2} + 2(m_f u + m_s v)\frac{\partial^2 w}{\partial x \partial t} + (m_f + m_s)\frac{\partial^2 w}{\partial t^2} = 0. \tag{2}$$

To find the correct expression for the energy transfer at a boundary of the continuum, a propagating pulse with a finite frequency bandwidth is considered in this development. Assuming that this pulse propagates in the positive x-direction, the transverse displacement of the continuum corresponding to this pulse can be represented as

$$w(x,t) = \int_{-\infty}^{\infty} \tilde{w}(\omega)\exp(\mathrm{i}(\omega t - k(\omega)x))\,\mathrm{d}\omega, \tag{3}$$

where $\tilde{w}(\omega)$ is the displacement of the continuum in the frequency domain, ω is the radial frequency and $k(\omega)$ is the wavenumber, which is real to ensure propagation of all harmonics of the pulse. Substitution of Eq. (3) into the equation of motion, Eq. (2), yields the following dispersion equation:

$$k^4(\omega)EI + k^2(\omega)(T - m_f u^2 - m_s v^2) + 2\omega k(\omega)(m_f u + m_s v) - \omega^2(m_f + m_s) = 0 \tag{4}$$

from which $k(\omega)$ can be derived.

Considering a time moment when the pulse is located so far from the boundaries that it is not disturbed by their presence, the energy of the pulse E can be computed by integrating the energy density $e(x,t)$ over space from the minus to plus infinity:

$$E = \int_{-\infty}^{\infty} e(x,t)\,\mathrm{d}x. \tag{5}$$

By inserting the energy density of the continuum Eq. (1) into this expression, making use of the representation of the pulse displacement, Eq. (3), and taking into account that this displacement can also be represented by its complex conjugate:

$$w(x,t) = \int_{-\infty}^{\infty} \tilde{w}(\omega)\exp(\mathrm{i}(\omega t - k(\omega)x))\,\mathrm{d}\omega = \int_{-\infty}^{\infty} \tilde{w}^*(\omega_1)\,\exp(-\mathrm{i}(\omega_1 t - k(\omega_1)x))\,\mathrm{d}\omega_1 \tag{6}$$

the following expression can be obtained:

$$E = \frac{1}{2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\{(m_f + m_s)\omega\omega_1 + (T + m_f u^2 + m_s v^2)k(\omega)k(\omega_1) - 2(m_f u + m_s v)\omega k(\omega_1)$$
$$+ EIk^2(\omega)k_1^2(\omega_1)\}\tilde{w}(\omega)\mathrm{e}^{\mathrm{i}(\omega t - k(\omega)x)}\tilde{w}^*(\omega_1)\mathrm{e}^{-\mathrm{i}(\omega_1 t - k(\omega_1)x)}\,\mathrm{d}\omega\,\mathrm{d}\omega_1\,\mathrm{d}x, \tag{7}$$

The integral over x in Eq. (7) can be evaluated using the following representation of the Dirac delta-function [13]:

$$\int_{-\infty}^{\infty} \exp(\pm\mathrm{i}\alpha x)\,\mathrm{d}x = 2\pi\delta(\pm\alpha) \tag{8}$$

to give

$$E = \pi\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\{(m_f + m_s)\omega\omega_1 + (T + m_f u^2 + m_s v^2)k(\omega)k(\omega_1) - 2(m_f u + m_s v)\omega k(\omega_1)$$
$$+ \mathrm{EI}k^2(\omega)k_1^2(\omega_1)\}\tilde{w}(\omega)\tilde{w}^*(\omega_1)\mathrm{e}^{\mathrm{i}(\omega t - \omega_1 t)}\delta(k(\omega) - k(\omega_1))\,\mathrm{d}\omega\,\mathrm{d}\omega_1. \tag{9}$$

To calculate the integral over ω_1 in Eq. (9), the following property of the delta-function can be used [14]:

$$\int_{-\infty}^{\infty} \delta(k(\omega_1) - k(\omega))f(\omega_1)\,\mathrm{d}\omega_1 = \left.\frac{f(\omega_1)}{|\mathrm{d}k(\omega_1)/\mathrm{d}\omega_1|}\right|_{k(\omega_1)=k(\omega)}. \tag{10}$$

Employing Eq. (10), the expression for the energy of the pulse can be reduced to

$$E = \pi \int_{-\infty}^{\infty} \{(m_f + m_s)\omega^2 + (T + m_f u^2 + m_s v^2)k^2(\omega) \\ -2(m_f u + m_s v)\omega k(\omega) + EIk^4(\omega)\} |c_{\mathrm{gr}}(\omega)||\tilde{w}(\omega)|^2 \, \mathrm{d}\omega, \tag{11}$$

where the group velocity $c_{\mathrm{gr}}(\omega) = \mathrm{d}\omega/\mathrm{d}k(\omega)$ is introduced.

Eq. (11) can be simplified further by making use of the dispersion equation, Eq. (4), and by noting that the resulting integrand is an even function of the frequency. This simplification yields

$$E = 4\pi \int_0^{\infty} k(\omega)^2 (T + EIk^2(\omega))|c_{\mathrm{gr}}(\omega)||\tilde{w}(\omega)|^2 \, \mathrm{d}\omega = \int_0^{\infty} E_\omega(\omega) \, \mathrm{d}\omega, \tag{12}$$

where E_ω is the spectral energy-density of the pulse.

Since the continuum under consideration is linear, the energy exchange at a boundary during wave reflection can be analysed considering differential bandwidths $\mathrm{d}\omega$ of the pulse separately. For each bandwidth with the central frequency ω, the energy transferred into the continuum during wave reflection is given as

$$\Delta W_\omega = E_{\omega,r} - E_{\omega,i} = (R_\omega - 1)E_{\omega,i}, \tag{13}$$

where $R_\omega = E_{\omega,r}/E_{\omega,i}$ is the 'true' energy reflection coefficient. Employing Eq. (12), this coefficient can be expressed as

$$R_\omega = \frac{k_r^2 (T + EIk_r^2)|c_{\mathrm{gr},r}|}{k_i^2 (T + EIk_i^2)|c_{\mathrm{gr},i}|} r^2, \tag{14}$$

where the subscripts r and i stand for the quantities associated with the reflected and incident waves, respectively and

$$r = |\tilde{w}_r / \tilde{w}_i| \tag{15}$$

is the amplitude reflection coefficient, which can be found by considering either reflection of a spectral component of the pulse or that of a harmonic wave to give the same result.

If the reflection coefficient R_ω is larger (smaller) than unity throughout the complete frequency band, then one should conclude that the energy is gained (lost) upon reflection of any pulse from the considered boundary. If, on the contrary, $R_\omega - 1$ is not a sign-definite function of frequency, the energy transfer at the boundary depends on the amplitude spectrum of a particular pulse. It is important to realize, however, that a single reflection cannot be considered as a measure of the energy exchange at a boundary. Every one-dimensional continuum has two boundaries, which reflect any pulse many times. Each reflection in a dispersive continuum is accompanied by a change of the spectrum of the pulse. The spectral components of the pulse for which $R_\omega > 1$ increase their contribution to the total energy with each reflection. This implies that if at a boundary of an undamped continuum the energy reflection R_ω is greater than unity at a frequency band that is present at the initial motion of the continuum, then the energy of the continuum will increase because of interaction with this boundary (after a sufficiently long time). If the continuum is damped along its length, the energy gain at a boundary can be overruled by the energy lost during the time that the pulse needs to return to the boundary. Since the energy loss associated with the distributed damping is usually frequency dependent, it is very important to predict the frequency bands at which the energy is gained at the boundaries correctly.

In the papers of Lee and Mote [1,2], the energy reflection coefficient R is defined as the ratio of the energy $E_{\lambda,r}$ contained in one wavelength of the reflected wave to the energy $E_{\lambda,i}$ contained in one wavelength of the incident wave. For the continuum, whose energy density is defined by Eq. (1), the coefficient R defined by Lee and Mote reads

$$R = \frac{k_r (T + EIk_r^2)}{k_i (T + EIk_i^2)} r^2. \tag{16}$$

Comparing Eq. (16) with Eq. (14), the following relation can be found between R_ω and R:

$$R_\omega = R\frac{|k_r c_{\mathrm{gr},r}|}{|k_i c_{\mathrm{gr},i}|} = R\frac{|c_{\mathrm{gr},r}|}{|c_{\mathrm{ph},r}|}\frac{|c_{\mathrm{ph},i}|}{|c_{\mathrm{gr},i}|}, \tag{17}$$

where $c_{\mathrm{ph},r} = \omega/k_r$ and $c_{\mathrm{ph},i} = \omega/k_i$ are the phase velocities of the reflected and incident waves, respectively. As discussed above, the conclusion as to whether the energy is gained or lost at a boundary is drawn comparing the energy reflection coefficient to unity. Since the factor $|c_{\mathrm{gr},r}/c_{\mathrm{ph},r}||c_{\mathrm{gr},i}/c_{\mathrm{ph},i}|$ is not necessarily equal to unity, Eq. (17) clearly shows that this conclusion may be wrong if the coefficient R is employed. In the next section, an example is presented, in which the use of R leads to a wrong conclusion as to the energy exchange at a certain frequency band.

3. Energy exchange at a downstream end of a pipe conveying fluid

As an example, the energy transfer at a downstream end of a pipe conveying fluid is considered. The term 'downstream end' specifies the side of the pipe where the fluid leaves the pipe. A rotational dashpot is connected to the downstream end, as sketched in Fig. 1. The equation of motion for a pipe conveying fluid can be obtained by setting $v = 0$ in the generalized equation of motion, Eq. (2).

At a downstream end connected to a rotational dashpot, the following two boundary conditions should be satisfied:

$$w(0,t) = 0 \text{ and } EI\frac{\partial^2 w}{\partial x^2} = -C_{rd}\frac{\partial^2 w}{\partial x\,\partial t}, \tag{18}$$

where C_{rd} is the rotational dashpot coefficient.

To find the energy reflection coefficient, a harmonic incident wave can be considered of the following form:

$$w_i(x,t) = w_i \mathrm{e}^{\mathrm{i}(\omega t - k_i x)}, \tag{19}$$

where w_i is the complex amplitude of this wave and k_i is the real positive root of the dispersion equation, Eq. (4), in which $v = 0$. Impinging on the boundary, this wave gives rise to a propagating reflected wave, $w_r^{\mathrm{pr}}(x,t)$, and to an evanescent reflected wave, $w_r^{\mathrm{ev}}(x,t)$. Together with the incident wave these waves form the following pattern of the transverse deflection of the pipe:

$$w(x,t) = w_i(x,t) + w_r^{\mathrm{pr}}(x,t) + w_r^{\mathrm{ev}}(x,t) = w_i \mathrm{e}^{\mathrm{i}(\omega t - k_i x)} + w_r^{\mathrm{pr}} \mathrm{e}^{\mathrm{i}(\omega t - k_r^{\mathrm{pr}} x)} + w_r^{\mathrm{ev}} \mathrm{e}^{\mathrm{i}(\omega t - k_r^{\mathrm{ev}} x)}, \tag{20}$$

where w_r^{pr} and w_r^{ev} are the complex amplitudes of the reflected propagating and reflected evanescent waves, k_r^{pr} is the real negative root of the dispersion equation, and k_r^{ev} is the complex root of this equation with a positive imaginary part.

Substitution of representation (20) into the two boundary conditions, Eq. (18), results in a system of two algebraic equations, from which the following expression for the amplitude reflection coefficient r can be found

$$r = \left|\frac{w_r}{w_i}\right| = \left|\frac{EI\left((k_i)^2 - \left(k_r^{\mathrm{ev}}\right)^2\right) + C_{\mathrm{rd}}\omega\left(k_i^{\mathrm{pr}} - k_r^{\mathrm{ev}}\right)}{EI\left(\left(k_r^{\mathrm{pr}}\right)^2 - \left(k_r^{\mathrm{ev}}\right)^2\right) + C_{\mathrm{rd}}\omega\left(k_r^{\mathrm{pr}} - k_r^{\mathrm{ev}}\right)}\right|. \tag{21}$$

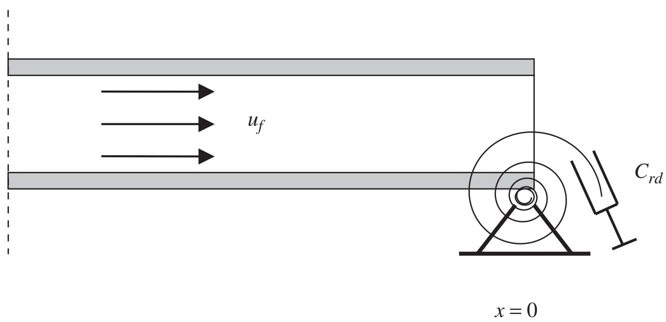

Fig. 1. Sketch of a downstream end connected to a rotational dashpot.

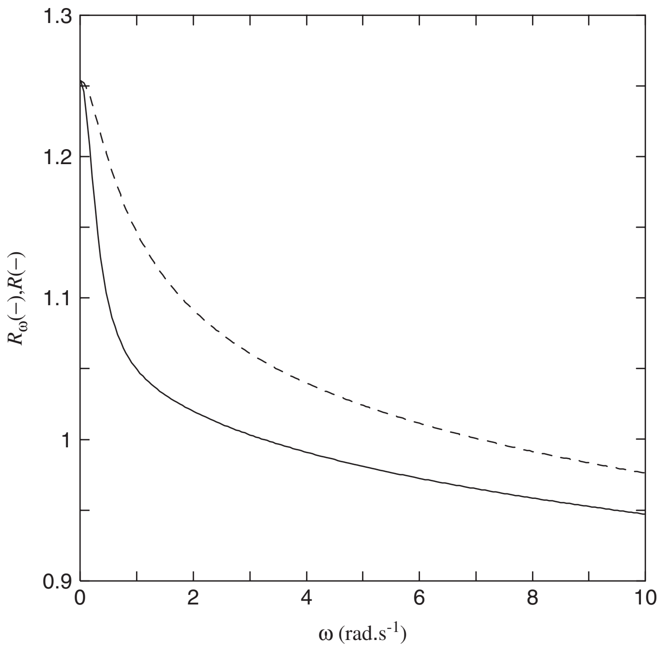

Fig. 2. Comparison of the energy reflection coefficients R_ω (----) and R (———) at the downstream end connected to a rotational dashpot.

This amplitude reflection coefficient can be substituted into the expression for the energy reflection coefficient R_ω derived in this paper, Eq. (14), and into the expression for the energy reflection coefficient R suggested by Lee and Mote, Eq. (16). Both R_ω and R obtained in this manner are depicted in Fig. 2 as functions of the radial frequency ω. To plot Fig. 2, the following physical parameters of the system were used:

$$EI = 1.0 \times 10^9\,\mathrm{N\,m^2}, T = 1.0 \times 10^6\,\mathrm{N},\ m_f = 1.0 \times 10^3\,\mathrm{kg/m},$$

$$m_s = 1.0 \times 10^3\,\mathrm{kg/m},\ u_f = 5.0\,\mathrm{m/s},\ C_{\mathrm{rd}} = 5.0 \times 10^5\,\mathrm{kg\,m^2\,s^{-1}\,rad^{-1}}.$$

Fig. 2 shows that the two energy reflection coefficients are not equal. Moreover, in the frequency band from approximately 3–8 rad/s, the use of these coefficients would lead to opposite predictions as to the energy exchange at the boundary. According to R_ω, which is greater than unity in this band, energy is gained, whereas according to R, which is smaller than unity, energy is lost.

4. Conclusions

The expression for the energy reflection coefficient at a boundary of a translating continuum as introduced by Lee and Mote [1,2], has been corrected in this contribution. This correction does not influence the prediction of the stability of a finite-length translating continuum in absence of distributed damping. This is because the stability of such a continuum is characterized by the multiplication of the energy reflection coefficients at the downstream and upstream ends, and at these ends the correction ratios $|c_{\mathrm{gr},r}/c_{\mathrm{ph},r}||c_{\mathrm{gr},i}/c_{\mathrm{ph},i}|$ obtained in this paper (see Eq. (17)) are precisely inverse to each other. If distributed damping is present, the proposed correction can influence the stability prediction. This is because the energy contained in each spectral component of a pulse that travels along the continuum changes (decreases) during propagation. Therefore, the energy variation in the continuum cannot be characterized by the multiplication of the energy reflection coefficients at the two boundaries and accurate prediction of both the magnitude and the frequency dependence of the energy reflection coefficient at each boundary gains importance.

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